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3-Rainbow index and forbidden subgraphs

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Abstract A tree in an edge-colored connected graph G is called *a rainbow tree* if no two edges of it are assigned the same color. For a vertex subset $S \subseteq V(G)$, a tree is called an *S-tree* if it connects S in G. A *k-rainbow coloring* of G is an edge-coloring of G having the property that for every set S of k vertices of G, there exists a rainbow S-tree in G. The minimum number of colors that are needed in a k-rainbow coloring of G is the *k-rainbow index* of G, denoted by $\mathrm{rx}_k(G)$. The *Steiner distance* $d(S)$ of a set S of vertices of G is the minimum size of an S-tree T. The *k-Steiner diameter* $\mathrm{sdiam}_k(G)$ of G is defined as the maximum Steiner distance of S among all sets S with k vertices of G. In this paper, we focus on the 3-rainbow index of graphs and find all finite families $\mathcal{F}$ of connected graphs, for which there is a constant $C_{\mathcal{F}}$ such that, for every connected $\mathcal{F}$-free graph G, $\mathrm{rx}_3(G) \leq \mathrm{sdiam}_3(G) + C_{\mathcal{F}}$.

Keywords rainbow tree, k-rainbow index, 3-rainbow index, forbidden subgraphs.

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1 Introduction

All graphs considered in this paper are simple, finite, undirected and connected. We follow the terminology and notation of Bondy and Murty [1] for those not defined here.

Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \to \{1, 2, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored with the same color. A path in G is called *a rainbow path* if no two edges of the path are colored with the same color. The graph G is called *rainbow connected* if for any two distinct vertices of G, there is a rainbow path connecting them. For a connected graph G, the *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$, is defined as the minimum number of colors that are needed to make G rainbow connected. These concepts were first introduced by Chartrand et al. in [4] and have been well-studied since then. For further details, we refer the reader to a survey paper [9] and a book [10].

In [5], Chartrand et al. generalized the concept of a rainbow path to a rainbow tree. A tree in an edge-colored graph G is called *a rainbow tree* if no two edges of it are assigned the same color. For a vertex subset $S \subseteq V(G)$, a tree is called an *S-tree* if it connects S in G. Let G be a connected graph of order n. For a fixed integer k with $2 \le k \le n$, a *k-rainbow coloring* of G is an edge-coloring of G having the property that for every k-subset S of G, there exists a rainbow S-tree in G, and in this case, the graph G is called *k-rainbow connected*. The minimum number of colors that are needed in a k-rainbow coloring of G is the *k-rainbow index* of G, denoted by $\mathrm{rx}_k(G)$. Clearly, $\mathrm{rx}_2(G)$ is just the rainbow connection number $\mathrm{rc}(G)$ of G. In the sequel, we assume that $k \ge 3$. It is easy to see that $\mathrm{rx}_2(G) \le \mathrm{rx}_3(G) \le \cdots \le \mathrm{rx}_n(G)$. Recently, some results on the k-rainbow index have been published, especially on the 3-rainbow index. We refer to [3,6] for more details.

The *Steiner distance* $d(S)$ of a set S of vertices in G is the minimum size of a tree in G containing S. Such a tree is called a *Steiner S-tree* or simply a *Steiner tree*. The *k-Steiner diameter* $\mathrm{sdiam}_k(G)$ of G is defined as the maximum Steiner distance of S among all k-subsets S of G. Then the following observation is immediate.

Observation 1 *[5] For every connected graph G of order $n \ge 3$ and each integer k with $3 \le k \le n$,*

$$k - 1 \le \mathrm{sdiam}_k(G) \le \mathrm{rx}_k(G) \le n - 1.$$

The authors of [5] showed that the k-rainbow index of trees can achieve the upper bound.

Proposition 1 *[5] Let T be a tree of order $n \ge 3$. For each integer k with $3 \le k \le n$,*

$$\mathrm{rx}_k(T) = n - 1.$$

From above, we notice that for a fixed integer k with $k \geq 3$, the difference $\mathrm{rx}_k(G) - \mathrm{sdiam}_k(G)$ can be arbitrarily large. In fact, if G is a star $K_{1,n}$, then we have $\mathrm{rx}_k(G) - \mathrm{sdiam}_k(G) = n - k$.

They also determined the precise values for the k-rainbow index of the cycle C_n and the 3-rainbow index of the complete graph K_n.

Theorem 1 *[5] For integers k and n with $3 \leq k \leq n$,*

$$\mathrm{rx}_k(C_n) = \begin{cases} n-2 & \text{if } k = 3 \text{ and } n \geq 4 \\ n-1 & \text{if } k = n = 3 \text{ or } 4 \leq k \leq n. \end{cases}$$

Theorem 2 *[5]*

$$\mathrm{rx}_3(K_n) = \begin{cases} 2 & \text{if } 3 \leq n \leq 5 \\ 3 & \text{if } n \geq 6. \end{cases}$$

Let $\mathcal{F}$ be a family of connected graphs. We say that a graph G is *$\mathcal{F}$-free* if G does not contain any induced subgraph isomorphic to a graph from $\mathcal{F}$. Specifically, for $\mathcal{F} = \{X\}$ we say that G is *X-free*, for $\mathcal{F} = \{X, Y\}$ we say that G is *(X,Y)-free*, and for $\mathcal{F} = \{X, Y, Z\}$ we say that G is *(X,Y,Z)-free*. The members of $\mathcal{F}$ will be referred to as *forbidden induced subgraphs* in this context. If $\mathcal{F} = \{X_1, X_2, \ldots, X_k\}$, we also refer to the graphs $X_1, X_2, \ldots, X_k$ as a *forbidden k-tuple*, and for $|\mathcal{F}| = 2$ and 3 we say a *forbidden pair* and a *forbidden triple*, respectively.

In [7], Holub et al. considered the question: For which families $\mathcal{F}$ of connected graphs, a connected $\mathcal{F}$-free graph G satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + C_{\mathcal{F}}$, where $C_{\mathcal{F}}$ is a constant (depending on $\mathcal{F}$). They gave a complete answer for $|\mathcal{F}| \in \{1, 2\}$ in the following two results (where N denotes the *net*, a graph obtained by attaching a pendant edge to each vertex of a triangle).

Theorem 3 *[7] Let X be a connected graph. Then there is a constant C_X such that every connected X-free graph G satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + C_X$, if and only if $X = P_3$.*

Theorem 4 *[7] Let X, Y be connected graphs such that $X, Y \neq P_3$. Then there is a constant C_{XY} such that every connected (X, Y)-free graph G satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + C_{XY}$, if and only if (up to symmetry) either $X = K_{1,r}$ $(r \geq 4)$ and $Y = P_4$, or $X = K_{1,3}$ and Y is an induced subgraph of N.*

Surprisingly, Brousek et al. [2] then gave a complete answer for all finite families $\mathcal{F}$. For the rainbow vertex-connection number, Li et al. [8] recently gave a complete answer for $|\mathcal{F}| \in \{1, 2\}$. Now we consider an analogous question concerning the k-rainbow index of graphs, where $k \geq 3$ is a positive integer. From Observation 1, we know that the k-Steiner diameter is a lower bound for the k-rainbow index. In this paper, we will consider the following question.

For which families $\mathcal{F}$ of connected graphs, there is a constant $C_{\mathcal{F}}$ such that $\mathrm{rx}_k(G) \leq \mathrm{sdiam}_k(G) + C_{\mathcal{F}}$ if a connected graph G is $\mathcal{F}$-free?

In general, it is very difficult to give answers to the above question, even if one considers the case $k = 4$. So, in this paper, we pay our attention only to the case $k = 3$. In Sections 3, 4 and 5, we give complete answers for the 3-rainbow index when $|\mathcal{F}| = 1, 2$ and 3, respectively. Finally, we give a complete characterization for an arbitrary finite family $\mathcal{F}$.

2 Preliminaries

In this section, we introduce some further terminology and notation that will be used in the sequel. Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers.

Let G be a graph. We use $V(G)$, $E(G)$, and $|G|$ to denote the vertex set, edge set, and the order of G, respectively. For $A \subseteq V(G)$, $|A|$ denotes the number of vertices in A, and $G[A]$ denotes the subgraph of G induced by the vertex set A. For two disjoint subsets X and Y of $V(G)$, we use $E[X, Y]$ to denote the set of edges of G between X and Y. For graphs X and G, we write $X \subseteq G$ if X is a subgraph of G, $X \overset{\text{IND}}{\subseteq} G$ if X is an induced subgraph of G, and $X \cong G$ if X is isomorphic to G. In an edge-colored graph G, we use $c(uv)$ to denote the color assigned to an edge $uv \in E(G)$.

Let G be a connected graph. For $u, v \in V(G)$, a path in G from u to v will be referred to as a *(u, v)-path*, and, whenever necessary, it will be considered with orientation from u to v. The *distance* between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. The *eccentricity* of a vertex v is $\mathrm{ecc}(v) := \max_{x\in V(G)} d_G(v, x)$. The *diameter* of G is $\mathrm{diam}(G) := \max_{x\in V(G)}\mathrm{ecc}(x)$, and the *radius* of G is $\mathrm{rad}(G) := \min_{x\in V(G)}\mathrm{ecc}(x)$. One can easily check that $\mathrm{rad}(G) \le \mathrm{diam}(G) \le 2\mathrm{rad}(G)$. A vertex x is *central* in G if $\mathrm{ecc}(x) = \mathrm{rad}(G)$. Let $D \subseteq V(G)$ and $x \in V(G) \setminus D$. Then we call a path $P = v_0v_1 \ldots v_k$ a v-D path if $v_0 = v$ and $V(P) \cap D = v_k$, and we set $d_G(v, D) := \min_{w\in D} d_G(v, w)$.

For a set $S \subseteq V(G)$ and $k \in \mathbb{N}$, we use $N^k_G(S)$ to denote the *neighborhood at distance k* of S, i.e., the set of all vertices of G at distance k from S. In the special case when $k = 1$, we simply write $N_G(S)$ for $N^1_G(S)$ and if $|S| = 1$ with $x \in S$, we write $N_G(x)$ for $N_G(\{x\})$. For a set $M \subseteq V(G)$, we set $N_M(S) = N_G(S) \cap M$ and $N_M(x) = N_G(x) \cap M$. Finally, we will also use the *closed neighborhood* of a vertex $x \in V(G)$ defined by $N^k_G[x] = (\cup^k_{i=1} N^i_G(x)) \cup \{x\}$.

A set $D \subseteq V(G)$ is called *dominating* if every vertex in $V(G) \setminus D$ has a neighbor in D. In addition, if $G[D]$ is connected, then we call D a *connected dominating set.* A *clique* of a graph G is a subset $Q \subseteq V(G)$ such that $G[Q]$ is complete. A clique is *maximum* if G has no clique Q' with $|Q'| > |Q|$. For a graph G, a subset $I \subseteq V(G)$ is called an *independent set* of G if no two vertices of I are adjacent in G. An independent set is *maximum* if G has no independent set I' with $|I'| > |I|$.

For two positive integers a and b, the *Ramsey number* $R(a, b)$ is the smallest integer n such that in any two-coloring of the edges of a complete graph on n vertices K_n by red and blue, either there is a red K_a (i.e., a complete

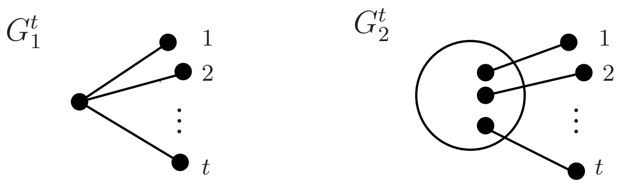

Fig. 1 The graphs G_1^t and G_2^t.

subgraph on a vertices all of whose edges are colored red) or there is a blue K_b. Ramsey [11] showed that $R(a, b)$ is finite for any a and b.

Finally, we will use P_n to denote the path on n vertices. An edge is called a *pendant edge* if one of its end vertices has degree one.

3 Families with one forbidden subgraph

In this section, we characterize all possible connected graphs X such that every connected X-free graph G satisfies $\text{rx}_3(G) \leq \text{sdiam}_3(G) + C_X$, where C_X is a constant.

Theorem 5 *Let X be a connected graph. Then there is a constant C_X such that every connected X-free graph G satisfies* $\text{rx}_3(G) \leq \text{sdiam}_3(G) + C_X$*, if and only if $X = P_3$.*

Proof If G is a connected P_3-free graph, then G is complete, and by Theorem 2, we have $\text{rx}_3(G) \leq 3 = \text{sdiam}_3(G) + 1$.

Conversely, let t be an arbitrarily large integer, set $G_1^t = K_{1,t}$, and let G_2^t denote the graph obtained by attaching a pendant edge to each vertex of the complete graph K_t (see Fig 1). We also use K_t^h to denote G_2^t. Since $\text{rx}_3(G_1^t) = t$ but $\text{sdiam}_3(G_1^t) = 3$, X is an induced subgraph of G_1^t. Clearly, $\text{rx}_3(G_2^t) \geq t+2$ but $\text{sdiam}_3(G_2^t) = 5$, and G_2^t is $K_{1,3}$-free. Hence, $X = K_{1,2} = P_3$. The proof is thus complete.

4 Forbidden pairs

The following statement, which is the main result of this section, characterizes all possible forbidden pairs X, Y for which there is a constant C_{XY} such that $\text{rx}_3(G) \leq \text{sdiam}_3(G) + C_{XY}$ if G is (X, Y)-free. Since any P_3-free graph is a complete graph, we exclude the case that one of X, Y is P_3.

Theorem 6 *Let $X, Y \neq P_3$ be a pair of connected graphs. Then there is a constant C_{XY} such that every connected (X, Y)-free graph G satisfies* $\text{rx}_3(G) \leq \text{sdiam}_3(G) + C_{XY}$*, if and only if (up to symmetry) $X = K_{1,r}, r \geq 3$ and $Y = P_4$.*

The proof of Theorem 6 will be divided into two parts. We prove the necessity in Proposition 2, and then we establish the sufficiency in Theorem 7.

Proposition 2 *Let $X, Y \neq P_3$ be a pair of connected graphs for which there is a constant C_{XY} such that every connected (X,Y)-free graph G satisfies* $\mathrm{rx}_3(G) \leq \mathrm{sdiam}_3(G) + C_{XY}$. *Then, (up to symmetry) $X = K_{1,r}, r \geq 3$ and $Y = P_4$.*

Proof Let t be an arbitrarily large integer, and set $G_3^t = C_t$. We will also use the graphs G_1^t and G_2^t shown in Figure 1.

Consider the graph G_1^t. Since $\mathrm{sdiam}_3(G_1^t) = 3$ but $\mathrm{rx}_3(G_1^t) = t$, we have, up to symmetry, $X = K_{1,r}, r \geq 3$. Then we consider the graphs G_2^t and G_3^t. It is easy to verify that $\mathrm{sdiam}_3(G_2^t) = 5$ but $\mathrm{rx}_3(G_2^t) \geq t+2$, and $\mathrm{sdiam}_3(G_3^t) = \lceil \frac{2}{3}t \rceil$ while $\mathrm{rx}_3(G_3^t) \geq t-2 \geq \frac{3}{2}(\mathrm{sdiam}_3(G_3^t) - 1) - 2$, respectively. Clearly, G_2^t and G_3^t are both $K_{1,3}$-free, so neither of them contains X, implying that both G_2^t and G_3^t contain Y. Since the maximum common induced subgraph of them is P_4, we get that $Y = P_4$. This completes the proof.

Next, we can prove that the converse of Proposition 2 is true.

Theorem 7 *Let G be a connected $(P_4, K_{1,r})$-free graph for some $r \geq 3$. Then* $\mathrm{rx}_3(G) \leq \mathrm{sdiam}_3(G) + r + 3$.

Proof. Let G be a connected $(P_4, K_{1,r})$-free graph ($r \geq 3$). Then, $\mathrm{sdiam}_3(G) \geq 2$. For simplicity, we set $V = V(G)$. Let $S \subseteq V$ be a maximum clique of G.

Claim 1: S is a dominating set.

Proof Assume that there is a vertex y at distance 2 from S. Let yxu be a shortest path from y to S, where $u \in S$. Because S is a maximum clique, there is some $v \in S$ such that $vx \notin E(G)$. Thus the path $vuxy \cong P_4$, a contradiction. So S is a dominating set.

Let X be a maximum independent set of $G[V \setminus S]$ and $Y = V \setminus (S \cup X)$. Then for any vertex $y \in Y$, y is adjacent to some $x \in X$. Furthermore, for any independent set W of graph $G[Y]$, $|N_X(W)| \geq |W|$ since X is maximum.

Claim 2: There is a vertex $v \in S$ such that v is adjacent to all the vertices in X.

Proof Suppose that the claim fails. Let u be a vertex of S with the largest number of neighbors in X. Set $X_1 = N_X(u)$, $X_2 = X \setminus X_1$. Then, $X_2 \neq \emptyset$ according to our assumption. Pick a vertex w in X_2. Then, $uw \notin E(G)$. Let v be a neighbor of w in S. For any vertex z in X_1, $G[w, v, u, z]$ cannot be an induced P_4, so vz must be an edge of G. Thus, $N_X(v) \supseteq N_X(u) \cup \{w\}$, contradicting the maximality of u.

Let z be a vertex in S which is adjacent to all the vertices of X. Set $X = \{x_1, x_2, \ldots, x_\ell\}$. Then, $0 \leq \ell \leq r-1$ since G is $K_{1,r}$-free. Now we demonstrate a 3-rainbow coloring of G using at most $\ell + 6$ colors. Assign color i to the edge zx_i, and $i+1$ to the edge x_iy where $1 \leq i \leq \ell$ and $y \in Y$. Color

$E[S, Y]$ with color $\ell+2$ and $E(G[Y])$ with color $\ell+3$. Give a 3-rainbow coloring of $G[S]$ using colors from $\{\ell+4, \ell+5, \ell+6\}$. Then color the remaining edges arbitrarily (e.g., all of them with color 1). Next, we prove that this coloring is a 3-rainbow coloring of G.

Let $W = \{u, v, w\}$ be a 3-subset of V.

(i) $\{u, v, w\} \subseteq S \cup X$. Clearly, there is a rainbow tree containing W.

(ii) $\{u, v\} \subseteq S \cup X, w \in Y$. We can easily find a rainbow tree containing an edge in $E[S, Y]$ that connects W.

(iii) $u \in S \cup X, \{v, w\} \subseteq Y$.

$a)$ If $vw \in E(G)$, then there obviously is a rainbow tree containing the edge vw that connects W.

$b)$ If $vw \notin E(G)$, then we have $|N_X(\{v, w\})| \geq |\{v, w\}| = 2$. So there are two vertices x_i and $x_j (i \neq j)$ in X adjacent to v and w, respectively. As $i+1 \neq j+1$, so either $i+1 \neq c(zu)$ or $j+1 \neq c(zu)$. Without loss of generality, we assume that $i+1 \neq c(zu)$ and s is a neighbor of w in S. Then there is a rainbow tree containing the edges zu, uv, sw, sz if $u = x_i$ or the edges $zu, zx_i, x_i v, sw, sz$ if $u \neq x_i$.

(iv) $\{u, v, w\} \subseteq Y$.

$a)$ If $\{uv, vw, uw\} \cap E(G) \neq \emptyset$, for example, $uv \in E(G)$, then we have a rainbow tree containing the edges $zx_i, x_i u, uv, sw, sz$ where x_i is a neighbor of u in X and s is a neighbor of w in S.

$b)$ If $\{uv, vw, uw\} \cap E(G) = \emptyset$, then we have $|N_X\{u, v, w\}| \geq |\{u, v, w\}| = 3$, so we can find three distinct vertices x_i, x_j, x_k in X such that $\{x_i u, x_j v, x_k w\} \subseteq E(G)$. We may assume that $i < j < k$, so $k+1 \notin \{i, j, k, i+1, j+1\}$ and $k \neq i+1$. Then there is a rainbow tree containing the edges $zx_i, x_i u, zx_k, x_k w, sv, sz$ where s is a neighbor of v in S.

Thus the coloring is a 3-rainbow coloring of G using at most $\ell+6 \leq r+5 \leq \mathrm{sdiam}_3(G)+r+3$ colors. The proof is complete.

Combining Proposition 2 and Theorem 7, we can easily get Theorem 6.

Remark When the maximum independent set of $G[V \setminus S]$, X, satisfies $|X| = \ell \geq 4$, we just need $\ell+5$ colors in the proof of Theorem 7: for the edges $x_\ell y$, we can color them with color 1 instead of color $\ell+1$. It only matters when the case $\{u, v, w\} \subseteq Y$ and $\{uv, vw, uw\} \cap E(G) = \emptyset$ happens. Suppose $\{x_i u, x_j v, x_k w\} \subseteq E(G)$ and $i < j < k$. If $i \neq 1$ or $k \neq \ell$, it is the case in the proof above. So we turn to the case when $i = 1$ and $k = l$. If $j = 2$, then $j+1 < 4 \leq \ell$ (that is why we need the condition $\ell \geq 4$). Thus, there is a rainbow tree containing the edges $zx_j, x_j v, zx_k, x_k w, su, sz$ where s is a neighbor of u in S. If $j \neq 2$, then there is a rainbow tree containing the edges $zx_i, x_i u, zx_j, x_j v, sw, sz$.

5 Forbidden triples

Now, we continue to consider more forbidden subgraphs and obtain an analogous result which characterizes all forbidden triples $\mathcal{F}$ for which there is a constant $C_{\mathcal{F}}$ such that G being $\mathcal{F}$-free implies $\mathrm{rx}_3(G) \leq \mathrm{sdiam}_3(G) + C_{\mathcal{F}}$. We exclude the cases which are covered by Theorems 5 and 6. We set:

$\mathfrak{F}_1 = \{\{P_3\}\}$,

$\mathfrak{F}_2 = \{\{K_{1,r}, P_4\}|\ r \geq 3\}$,

$\mathfrak{F}_3 = \{\{K_{1,r}, Y, P_\ell\}|\ r \geq 3, Y \overset{\text{IND}}{\subseteq} K_s^h, s \geq 3, \ell > 4\}$.

Theorem 8 *Let $\mathcal{F}$ be a family of connected graphs with $|\mathcal{F}| = 3$ such that $\mathcal{F} \not\supseteq \mathcal{F}'$ for any $\mathcal{F}' \in \mathfrak{F}_1 \cup \mathfrak{F}_2$. Then there is a constant $C_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph G satisfies $\mathrm{rx}_3(G) \leq \mathrm{sdiam}_3(G) + C_{\mathcal{F}}$, if and only if $\mathcal{F} \in \mathfrak{F}_3$.*

First of all, we prove the necessity of the triples given by Theorem 8.

Proposition 3 *Let $X, Y, Z \neq P_3$ be connected graphs, $\{X, Y, Z\} \not\supseteq \mathcal{F}'$ for any $\mathcal{F}' \in \mathfrak{F}_2$, for which there is a constant C_{XYZ} such that every connected (X, Y)-free graph G satisfies $\mathrm{rx}_3(G) \leq \mathrm{sdiam}_3(G) + C_{XYZ}$. Then, (up to symmetry) $X = K_{1,r}(r \geq 3), Y \overset{IND}{\subseteq} K_s^h(s \geq 3)$, and $Z = P_\ell(\ell > 4)$.*

Proof Let t be an arbitrarily large integer, and let G_1^t, G_2^t, G_3^t be the graphs defined in the proof of Proposition 2.

Firstly, we consider the graph G_1^t. Up to symmetry, we have $X = K_{1,r}, r \geq 3$ (for the case $r = 2$ is excluded by the assumptions). Secondly, we consider the graph G_2^t. The graph G_2^t does not contain X, since it is $K_{1,3}$-free. Thus, up to symmetry, we have G_2^t contains Y, implying $Y \overset{\text{IND}}{\subseteq} K_s^h$ for some $s \geq 3$ (for the case $s \leq 2$ is excluded by the assumptions). Finally, we consider the graphs G_3^t and G_3^{t+1}. Clearly, they are $(K_{1,3}, K_3^h)$-free, so both of them contain neither X nor Y. Hence, we get that $Z = P_\ell$ for some $\ell > 4$ (for the case $\ell \leq 4$ is excluded by the assumptions).

This completes the proof.

It is easy to observe that if $X \overset{\text{IND}}{\subseteq} X'$, then every (X, Y, Z)-free graph is also (X', Y, Z)-free. Thus, when proving the sufficiency of Theorem 8, we will be always interested in *maximal triples* of forbidden subgraphs, i.e., triples X, Y, Z such that, if replacing one of X, Y, Z, say X, with a graph $X' \neq X$ such that $X \overset{\text{IND}}{\subseteq} X'$, then the statement under consideration is not true for (X', Y, Z)-free graphs.

For every vertex $c \in V(G)$ and $i \in \mathbb{N}$, we set $\alpha_i(G, c)$ =max$\{|M|\big|M \subseteq N_G^i[c], M$ is independent$\}$ and $\alpha_i^0(G, c)$ =max$\{|M^0|\big|M^0 \subseteq N_G^i(c), M^0$ is independent$\}$.

Lemma 1 *[2] Let $r, s, i \in \mathbb{N}$. Then there is a constant $\alpha(r, s, i)$ such that, for every connected $(K_{1,r}, K_s^h)$-free graph G and for every $c \in V(G)$, $\alpha_i(G, c) < \alpha(r, s, i)$.*

We use the proof of Lemma 1 to get the following corollary concerning $\alpha_i^0(G, c)$ for each integer $i \geq 1$.

Corollary 1 *Let $r, s, i \in \mathbb{N}$. Then there is a constant $\alpha^0(r,s,i)$ such that, for every connected $(K_{1,r}, K_s^h)$-free graph G and for every $c \in V(G)$, $\alpha_i^0(G,c) < \alpha^0(r,s,i)$.*

Proof For the sake of completeness, here we give a brief proof concentrating on the upper bound of $\alpha_i^0(G,c)$. We prove the corollary by induction on i.

For $i = 1$, we have $\alpha^0(r,s,1) = r$, for otherwise G contains a $K_{1,r}$ as an induced subgraph.

Let, to the contrary, i be the smallest integer for which $\alpha^0(r,s,i)$ does not exist(i.e., $\alpha_i^0(G,c)$ can be arbitrarily large), choose a graph G and a vertex $c \in V(G)$ such that $\alpha_i^0(G,c) \geq (r-2)R(s(2r-3), \alpha^0(r,s,i-1))$, and let $M^0 = \{x_1^0, \ldots, x_k^0\} \subseteq N_G^i(c)$ be an independent set in G of size $\alpha_i^0(G,c)$. Obviously, $k \geq (r-2)R(s(2r-3), \alpha^0(r,s,i-1))$. Let Q_j be a shortest (x_j^0, c)-path in G, $j = 1, \ldots, k$. We denote $M^1 \subseteq N_G^{i-1}(c)$ the set of all successors of the vertices from M^0 on Q_j, $j = 1, \ldots, k$, and x_j^1 the successor of x_j^0 on Q_j (note that some distinct vertices in M^0 can have a common successor in M^1). Every vertex in M^1 has at most $r-2$ neighbors in M^0 since G is $K_{1,r}$-free. Thus, $|M^1| \geq \frac{k}{r-2} \geq R(s(2r-3), \alpha^0(r,s,i-1))$. By the induction assumption and the definition of Ramsey number, $G[M^1]$ contains a complete subgraph $K_{s(2r-3)}$. Choose the notation such that $V(K_{s(2r-3)}) = \{x_1^1, \ldots, x_{s(2r-3)}^1\}$, and set $\widetilde{M^0} = N_{M^0}(K_{s(2r-3)})$. Using a matching between $K_{s(2r-3)}$ and $\widetilde{M^0}$, we can find in G an induced K_s^h with vertices of degree 1 in $\widetilde{M^0}$, a contradiction. For more details about finding the K_s^h, we refer the reader to [2].

Armed with Corollary 1, we can get the following important theorem.

Theorem 9 *Let $r \geq 3, s \geq 3$, and $\ell > 4$ be fixed integers. Then there is a constant $C(r,s,\ell)$ such that every connected $(K_{1,r}, K_s^h, P_\ell)$-free graph G satisfies $\mathrm{rx}_3(G) \leq \mathrm{sdiam}_3(G) + C(r,s,\ell)$.*

Proof. We have $\mathrm{diam}(G) \leq \ell - 2$ since G is P_ℓ-free. Let c be a central vertex of G, i.e., $\mathrm{ecc}(c) = \mathrm{rad}(G) \leq \mathrm{diam}(G) \leq \ell - 2$. We set $S_i = \cup_{j=1}^{i} N_G^j[c]$ for an integer $i \geq 1$.

Claim: $\mathrm{rx}_3(G[S_i \cup N_G^{i+1}(c)]) \leq \mathrm{rx}_3(G[S_i]) + \alpha_{i+1}^0(G,c) + 3$

Proof Let $X = \{x_1, x_2, \ldots, x_{\alpha_{i+1}^0(G,c)}\}$ be a maximum independent set of $N_G^{i+1}(c)$ and $Y = N_G^{i+1}(c) \setminus X$. Then for any vertex $y \in Y$, y is adjacent to some $x \in X$ and $s \in S$. Furthermore, for any independent set W of the graph $G[Y]$, we have $|N_X(W)| \geq |W|$ since X is maximum.

Now we demonstrate a 3-rainbow coloring of $G[S_i \cup N_G^{i+1}(c)]$ using at most $k + \alpha_{i+1}^0(G,c) + 3$ colors, where $k = \mathrm{rx}_3(G[S_i])$. We color the edges of $G[S_i]$ using colors $1, 2, \ldots, k$. Color $E[S_i, Y]$ with color $k+1$ and $E(G[Y])$ with color $k+2$. Then assign color $j+k+2$ to the edges $E[\{x_j\}, S_i]$, and $j+k+3$ to the edges $E[\{x_j\}, Y]$ where $1 \leq j \leq \alpha_{i+1}^0(G,c)$. With the same argument as the proof of Theorem 7, we can prove that this coloring is a 3-rainbow coloring of $G[S_i \cup N_G^{i+1}(c)]$.

From the proof of Corollary 1, it follows that $\alpha_1^0(G,c) \leq r-1$ and $\alpha_i^0(G,c) \leq (r-2)R(s(2r-3), \alpha^0(r,s,i-1)) - 1$ for each integer $i \geq 2$. Let $\mathcal{R}(r,s) = \Sigma_{i=2}^{\mathrm{ecc}(c)} R(s(2r-3), \alpha^0(r,s,i-1))$. Recall that $\mathrm{ecc}(c) \leq \ell - 2$. Repeated application of Claim gives the following:

$$\begin{aligned} \mathrm{rx}_3(G) &\leq \mathrm{rx}_3(G[N_G^{\mathrm{ecc}(c)-1}[c]]) + \alpha^0_{\mathrm{ecc}(c)}(G,c) + 3 \\ &\leq \dots \\ &\leq \mathrm{rx}_3(c) + \alpha_1^0(G,c) + \cdots + \alpha^0_{\mathrm{ecc}(c)}(G,c) + 3\mathrm{ecc}(c) \\ &\leq 0 + r + (r-2)\mathcal{R}(r,s) + 2(\ell - 2) \\ &\leq \mathrm{sdiam}_3(G) + (r-2)(\mathcal{R}(r,s)+1) + 2(\ell-1). \end{aligned}$$

Thus, we complete our proof. ■

Remark The same as the remark in Section 4: for $i \geq 1$, every time $\alpha^0_{i+1}(G,c) \geq 4$ happens, we can save one color in the Claim of Theorem 9.

6 Forbidden k-tuples for any $k \in \mathbb{N}$

Let $\mathcal{F} = \{X_1, X_2, X_3, \dots, X_k\}$ be a finite family of connected graphs with $k \geq 4$ for which there is a constant $k_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph satisfies $\mathrm{rx}_3(G) \leq \mathrm{sdiam}_3(G) + C_{\mathcal{F}}$. Let t be an arbitrarily large integer, and let G_1^t, G_2^t and G_3^t be defined in Proposition 2. For the graph G_1^t, up to symmetry, we suppose that $X_1 = K_r, r \geq 3$ (for the case $r = 2$ has been discussed in Section 3). Then, we consider the graphs G_2^t and G_3^t. Notice that G_2^t and G_3^t are both $K_{1,3}$-free, so neither of them contains X_1, implying that G_2^t or G_3^t contains X_i, where $i \neq 1$. We may assume that X_2 is an induced subgraph of G_2^t. If G_3^t contains X_2, then $X_2 = P_4$, which is just the case in Section 4. So we turn to the case that G_3^t contains X_i for some $i > 2$. Now consider the graphs $G_3^t, G_3^{t+1}, G_3^{t+2}, \dots, G_3^{t+k}$, each of which contains at least one of $X_3, X_4, \dots, X_k$ as an induced subgraph due to the analysis above. So it is forced that at least one of these $X_i (i \geq 3)$ is isomorphic to P_l for some $l \geq 5$, which goes back to the case in Section 5. Thus, the conclusion comes out.

Theorem 10 *Let $\mathcal{F}$ be a finite family of connected graphs. Then there is a constant $C_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph satisfies* $\mathrm{rx}_3(G) \leq \mathrm{sdiam}_3(G) + C_{\mathcal{F}}$*, if and only if $\mathcal{F}$ contains a subfamily $\mathcal{F}' \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$.*

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