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# THE TWO-ARC-TRANSITIVE GRAPHS OF SQUARE-FREE ORDER ADMITTING ALTERNATING OR SYMMETRIC GROUPS

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**Abstract**

Let $G$ be a finite group with $\mathsf{soc}(G) = \mathrm{A}_c$ for $c \geq 5$. A characterization of the subgroups with square-free index in $G$ is given. Also, it is shown that a $(G, 2)$-arc-transitive graph of square-free order is isomorphic to a complete graph, a complete bipartite graph with a matching deleted or one of 11 other graphs.

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## 1. Introduction

Let $\Gamma$ be a graph with vertex set $V\Gamma$ and edge set $E\Gamma$. We use $\mathsf{Aut}\Gamma$ to denote the automorphism group of $\Gamma$. For a positive integer $s$, an *s-arc* of $\Gamma$ is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that $\{v_{i-1}, v_i\} \in E\Gamma$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$.

Let $G \leq \mathsf{Aut}\Gamma$. The graph $\Gamma$ is said to be $(G, s)$-*arc-transitive* if it has at least one $s$-arc and $G$ is transitive on both the vertices and the $s$-arcs of $\Gamma$, and $\Gamma$ is $(G, s)$-*transitive* if it is $(G, s)$-arc-transitive but not $(G, s+1)$-arc-transitive. For the case when $G = \mathsf{Aut}\Gamma$, a $(G, s)$-arc-transitive graph or a $(G, s)$-transitive graph is simply called *s-arc-transitive* or *s-transitive*, respectively.

Praeger [24] gave a reduction for finite nonbipartite two-arc-transitive graphs into four types, say, HA, AS, PA and TW. For the bipartite case, Praeger [25] gave a reduction into five types. Praeger's reductions indicate that a two-arc-transitive graph involved in the nine types either has a complete bipartite quotient graph or admits a group acting faithfully and quasiprimitively (of type HA, AS, PA or TW) on the vertex set or on each of its two orbits. Since then, characterizing or classifying finite two-arc-transitive graphs have been an active topic in algebraic graphtheory, which

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is highly attractive from the group-theoretic and combinatorial viewpoint and has received considerable attention (see [1, 7, 8, 11, 13, 15, 16, 25] for more references).

Another main motivation stems from the recently increasing interest in the study of permutation groups of square-free degree and their application to graphs. The class of graphs of square-free order has been studied in some special cases. In 1967, Turner [31] gave the classification of symmetric graphs with order a prime number $p$. The classification of symmetric graphs with order $2p$ was not completed until 1987 by Cheng and Oxley [3]. The classification of symmetric graphs with order $3p$ in [32] and some other graphs with order a product of two distinct primes were classified in [22, 27, 28]. The graphs of order a product of three distinct primes are determined by a series of articles [10, 12, 23, 30]. Further, see [17, 19–21] for the case of order four or more distinct primes.

In particular, the cases of two-arc-transitive graphs admitting a Suzuki simple group and a Ree simple group are classified in [7, 8], and the case of two-arc-transitive graphs admitting a two-dimensional projective linear group is studied in [11].

The object of this paper is to describe the subgroups of square-free index in $G$ and classify the $(G, 2)$-arc-transitive graphs of square-free order, where $G$ is an almost simple group with the alternating socle.

Theorem 1.1. *Let $G$ be a finite group with $\mathsf{soc}(G) = \mathrm{A}_c$ for $c \geq 5$. If $H$ is a square-free index subgroup of $G$, then $H$ is described in Lemmas 3.6 and 3.7. If $\Gamma$ is a connected $(G, 2)$-arc-transitive graph of square-free order, then $\Gamma$ is isomorphic to one of the graphs given in Section 2.2: $\mathsf{K}_c$ with square-free $c$; $\mathsf{K}_{c,c} - c\mathsf{K}_2$ with odd square-free $c$; $\mathsf{K}_6$ with $c = 5$; $\mathsf{K}_{10}$ with $c = 6$; Tutte's 8-cage with $c = 6$; a symmetric coset graph with $c = 7$; the point-hyperplane incidence graph of $\mathrm{PG}(3, 2)$ and its complement graph in $\mathsf{K}_{15,15}$ with $c = 7, 8$; and $\mathrm{O}_k$ with $c = 2k - 1$ for $k \in \{3, 4, 6, 9, 10, 12, 36\}$.*

In the following sections, bold-face $\mathbf{c}$ always means the set $\{1, 2, \ldots, c\}$; for $\Delta \subseteq \mathbf{c}$, we denote by $\mathrm{Alt}(\Delta)$ or $\mathrm{Sym}(\Delta)$, or sometimes just $\mathrm{A}_{|\Delta|}$ or $\mathrm{S}_{|\Delta|}$, the alternating group or symmetric group on $\Delta$, respectively.

## 2. Coset graphs, examples and stabilizers

**2.1. Coset graphs and examples.** We sometimes represent a graph as a coset graph introduced by Sabidussi [29]. Let $G$ be a finite group and let $H$ be a core-free subgroup of $G$, that is, $\bigcap_{x\in G} H^x = 1$. Let $g \in G\backslash H$ be of order a power of two with $g^2 \in H$. Then the *symmetric coset graph* $\mathsf{Cos}(G, H, HgH)$ is defined to be the graph with vertex set $[G : H] = \{Hx \mid x \in G\}$ such that $Hx$ and $Hy$ are adjacent while $yx^{-1} \in HgH$. Then $\mathsf{Cos}(G, H, HgH)$ is a well-defined $G$-arc-transitive graph, where $G$ is viewed as a subgroup of $\mathsf{Aut}\Gamma$ acting on $[G : H]$ by right multiplication. The follow lemma is formulated from several well-known facts on coset graphs (see [18] for example).

Lemma 2.1. *Let $\Gamma$ be a connected graph and $G \leq \mathsf{Aut}\Gamma$. Let $\{\alpha, \beta\} \in E\Gamma$, $H = G_\alpha$ and $K = G_{\alpha\beta}$. Assume that $G$ acts transitively on both the vertices and the arcs of $\Gamma$. Then $\Gamma \cong \mathsf{Cos}(G, H, HxH)$ for some $x \in \mathrm{N}_G(K)\backslash H$ of two-power order such that $x^2 \in K$ and $G = \langle x, H\rangle$.*

**2.2. Examples.** We collect several examples of two-arc-transitive graphs with square-free order and admitting the alternating group $A_c$.

Example 2.2. $\mathsf{K}_n$, the complete graph of order $n$ for a square-free $n \geq 5$. Assume that $G \leq \mathsf{Aut}\mathsf{K}_n$ acts transitively on the two-arcs of $\mathsf{K}_n$. Then $G$ is a three-transitive subgroup of $S_n$. Thus $\mathsf{soc}(G) = A_c$ implies $(c, n) = (c, c)$, $(5, 6)$ or $(6, 10)$.

Example 2.3. $\mathsf{K}_{c,c} - c\mathsf{K}_2$, the complete bipartite graph with a matching deleted. $\mathsf{Cos}(S_c, A_c, A_c(1\,2)A_c) \cong \mathsf{K}_{c,c} - c\mathsf{K}_2$ with square-free order if $c$ is odd square-free.

Example 2.4. Point-hyperplane incidence graph of the projective geometry $\mathrm{PG}(3, 2)$. This graph and its complement graph in $\mathsf{K}_{15,15}$ admit $S_8 \cong \mathrm{GL}(4, 2) \cdot 2$ acting transitively on both their two-arcs.

Example 2.5. Tutte's 8-cage. Let $U$ consist of the two-subsets of $\mathbf{6}$ and let $V$ consist of the partitions of $\mathbf{6}$ into three parts with size 2. Then Tutte's 8-cage may be defined as the bipartite graph with vertex set $U \cup V$ such that $\alpha \in U$ and $\beta \in V$ are joined by an edge if $\alpha$ is a part of $\beta$. This graph is a cubic five-transitive graph with automorphism group $\mathsf{Aut}(A_6) = \mathrm{P\Gamma L}(2, 9)$.

Example 2.6. $O_k$, odd graph for $k \in \{3, 4, 6, 9, 10, 12, 36\}$. Let $c = 2k - 1$ for $k \geq 3$ and let $V$ consist of $(k - 1)$-subsets of $\mathbf{c}$. Then $O_k$ is defined with vertex set $V$ such that $\alpha, \beta \in V$ are adjacent if and only if $\alpha \cap \beta = \emptyset$ (see [2, 8f], for example). Further, $\mathsf{Aut}O_k = S_c$ and $O_k$ is two-arc-transitive, and further, by Corollary 3.2, $|V| = c!/[k!(k - 1)!]$ is square-free if and only if $k \in \{3, 4, 6, 9, 10, 12, 36\}$.

Example 2.7. $\mathsf{Cos}(A_7, \mathrm{PSL}(2, 5), \mathrm{PSL}(2, 5)(1\,4\,5\,2)(6\,7)\mathrm{PSL}(2, 5))$, a two-arc-transitive graph of valency six and order 42. We identify $H = \mathrm{PSL}(2, 5)$ with a transitive subgroup of $A_6$ containing $K = \langle\sigma, \tau\rangle$, where $\sigma = (1\,2\,3\,4\,5)$ and $\tau = (1\,5)(2\,4)$. Then $N_{A_7}(K) = \langle\sigma, \pi\rangle$, $\langle\pi, H\rangle = A_7$ and $\pi^2 \in K$, where $\pi = (1\,4\,5\,2)(6\,7)$. Thus $\mathsf{Cos}(A_7, H, H\pi H)$ is a connected two-arc-transitive graph.

**2.3. Stabilizers.** Let $\Gamma$ be a graph, $G \leq \mathsf{Aut}\Gamma$ and $\{\alpha, \beta\} \in E\Gamma$. Then the stabilizer $G_\alpha$ induces an action on the neighborhood $\Gamma(\alpha)$ of $\alpha$ in $\Gamma$. Let $G_\alpha^{\Gamma(\alpha)}$ denote the permutation group on $\Gamma(\alpha)$ induced by $G_\alpha$, let $G_\alpha^{[1]}$ be the kernel of this action and set $G_{\alpha\beta}^{[1]} = G_\alpha^{[1]} \cap G_\beta^{[1]}$. Then

$$(G_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}^{\Gamma(\alpha)}, \quad G_\alpha = G_\alpha^{[1]} \cdot G_\alpha^{\Gamma(\alpha)} = (G_{\alpha\beta}^{[1]} \cdot (G_\alpha^{[1]})^{\Gamma(\beta)}) \cdot G_\alpha^{\Gamma(\alpha)}, \tag{2.1}$$

where $X \cdot Y$ means a group extension of $X$ by $Y$.

Lemma 2.8. *If $G$ is transitive on $V\Gamma$, then $\Gamma$ is $(G, 2)$-arc-transitive if and only if $G_\alpha^{\Gamma(\alpha)}$ is a two-transitive permutation group.*

Lemma 2.9 [9, 34]. *Let $\Gamma$ be a $(G, s)$-transitive graph for $s = 2$ or 3. Then, for an edge $\{\alpha, \beta\}$ of $\Gamma$, either $G_{\alpha\beta}^{[1]} = 1$ or $G_{\alpha\beta}^{[1]}$ is a nontrivial $p$-group for some prime $p$, $\mathrm{PSL}(n, q) \leq G_\alpha^{\Gamma(\alpha)} \leq \mathrm{P\Gamma L}(n, q)$ and $|\Gamma(\alpha)| = q^n - 1/q - 1$ for some $n \geq 2$ and a power $q$ of $p$.*

TABLE 1. Stabilizers of $s$-transitive graph of valency $k$.

| $k$ | $s$ | $G_\alpha$ | $G_{\alpha\beta}$ |
|---|---|---|---|
| $q+1$ | 4 | $[q^2] \rtimes Z_{(q-1)/(3,q-1)} \cdot \mathrm{PGL}(2,q) \cdot Z_e$ | $[q^3] \rtimes (Z_{q-1} \times Z_{(q-1)/(3,q-1)}) \cdot Z_e$ |
| $2^f+1$ | 5 | $[q^3] \times \mathrm{GL}(2,q) \cdot Z_e$ | $[q^4] \times Z_{q-1}^2 \cdot Z_e$ |
| $3^f+1$ | 7 | $[q^5] \rtimes \mathrm{GL}(2,q) \cdot Z_e$ | $[q^6] \rtimes Z_{q-1}^2 \cdot Z_e$ |

All finite two-transitive permutation groups are precisely known; the reader is referred to [14] for a complete list. Then, by Equation (2.1) and Lemmas 2.8 and 2.9, we have shown the following result.

COROLLARY 2.10. *If $\Gamma$ is a $(G,2)$-arc-transitive graph, then the stabilizer $G_\alpha$ has at most two insoluble composition factors. Further, if there are two insoluble factors, then either they are not isomorphic when $G_\alpha^{\Gamma(\alpha)}$ is almost simple or they are isomorphic when $G_\alpha^{\Gamma(\alpha)}$ is an affine group.*

PROOF. By Lemma 2.9, $G_{\alpha\beta}^{[1]}$ is a $p$-group. Then, by (2.1), all possible insolvable composition factors are involved in $(G_\alpha^{[1]})^{\Gamma(\beta)}$ and $G_\alpha^{\Gamma(\alpha)}$. Note that $(G_\alpha^{[1]})^{\Gamma(\beta)} \lhd G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}^{\Gamma(\alpha)} \cong (G_\alpha^{\Gamma(\alpha)})_\beta$. Then the two-transitive permutation group $G_\alpha^{\Gamma(\alpha)}$ and its a stabilizer acting on $\Gamma(\alpha)$ give all possible insolvable composition factors of $G_\alpha$. Thus our result follows from checking the two-transitive permutation groups one by one. □

LEMMA 2.11 [33, 35]. *Suppose that $\Gamma$ is a connected $(G,s)$-transitive graph of valency $k$ with $s \geq 4$. Then $k = q+1$, $s = 4, 5$ or $7$, and, for an edge $\{\alpha,\beta\}$, the vertex stabilizer $G_\alpha$ and arc stabilizer $G_{\alpha\beta}$ are listed in Table 1, where $q = p^f$ is a power of some prime $p$ and $e$ is a divisor of $f$.*

The structure of stabilizers for cubic $s$-transitive graphs is explicitly known due to Tutte's result (see [2, 18f], for example). For the four-valent case, we have the following result, which is a consequence from Lemmas 2.9 and 2.11.

LEMMA 2.12. *Let $\Gamma$ be a four-valent $(G,s)$-transitive graph with $s = 2$ or $3$. Let $\alpha \in V\Gamma$. Then either $s = 2$ and $\mathrm{A}_4 \leq G_\alpha \leq \mathrm{S}_4$ or $s = 3$ and $\mathrm{A}_4 \times Z_3 \leq G_\alpha \leq \mathrm{S}_4 \times \mathrm{S}_3$.*

## 3. Subgroups with square-free index in $\mathrm{S}_c$ or $\mathrm{A}_c$

The purpose of this section is to describe the subgroups of square-free index in $G$, where $\mathsf{soc}(G) = \mathrm{A}_c$ for $c \geq 5$. Several results on elementary number theory are necessary. The first lemma is formulated from [21].

LEMMA 3.1. *Let $a \geq 2$ and $b \geq 2$ be two integers. Then $(ab)!/[(a!)^b b!]$ is not square-free except that either $a = 2$ and $b \in \{3,4\}$ or $b = 2$ and $a \in \{2,3,4,6,9,10,12,36\}$.*

COROLLARY 3.2. *If $a \geq 2$, then $(2a-1)!/[a!(a-1)!]$ is not square-free except for $a \in \{2,3,4,6,9,10,12,36\}$.*

Lemma 3.3. *Let* $\mathrm{p}(d,t)=\prod_{i=1}^{t}(d+i)$ *be the product of* $t$ *consecutive positive integers. Then the following statements hold.*

(1) *If* $\mathrm{p}(d,4)/8$ *is square-free, then* $d\equiv 0,1,3,4 \bmod 9$.
(2) *If* $\mathrm{p}(d,5)/20$ *is square-free, then* $d=6m$ *with* $m\equiv 0,3,12,15 \bmod 8$.
(3) *If* $\mathrm{p}(d,6)/48$ *is square-free, then* $d=4m$ *with* $m\equiv 0,14,25 \bmod 9$*, or* $d=4n+1$ *with* $n\equiv 0,16,20 \bmod 9$.
(4) *If* $d\geq 2$ *and* $\mathrm{p}(d,6)/24$ *is square-free, then* $d=8m$ *with* $m\equiv 7,17,27 \bmod 9$*, or* $d=8n+1$ *with* $n\equiv 8,10,27 \bmod 9$.
(5) *If* $\mathrm{p}(d,6)/120$ *is square-free, then* $d=8m$ *with* $m\equiv 0,7,8 \bmod 9$*, or* $d=8n+1$ *with* $n\equiv 0,1,8 \bmod 9$.
(6) *If* $\mathrm{p}(d,6)/72$ *is square-free, then* $d\equiv 0,1 \bmod 8$.
(7) *If* $\mathrm{p}(d,7)/168$ *is square-free, then* $d=72m$ *with* $m\geq 3$ *and* $m\equiv 0,3,6 \bmod 5$*, or* $d=72n+64$ *with* $n\geq 1$ *and* $n\equiv 1,3,4 \bmod 5$.
(8) *If* $\mathrm{p}(d,7)/120$ *is square-free, then* $d=8m$ *with* $m\equiv 0,8 \bmod 9$.
(9) *If* $\mathrm{p}(d,7)/72$ *is square-free, then* $d=8m$ *with* $m\equiv 0,2,9 \bmod 5$.
(10) *If* $\mathrm{p}(d,7)/48$ *is square-free, then* $d=4m$ *with* $m\equiv 0,25 \bmod 9$.
(11) *If* $\mathrm{p}(d,8)/(2^6\cdot 3\cdot 7)$ *is square-free, then* $d=45m$ *or* $d=45n+36$ *for* $m,n\geq 0$.
(12) *If* $\mathrm{p}(d,8)/(2^6\cdot 3^2)$ *is square-free, then* $15n+6$ *with* $n\equiv 2,3,4,5,15,17,22 \bmod 16$*, or* $d=15m$ *with* $m\equiv 0,9,10,12,14,15,27,29 \bmod 16$.
(13) *If* $\mathrm{p}(d,8)/(2^7\cdot 3)$ *is square-free, then* $d=15m$ *with* $m=0$ *or* $m\geq 9$*, or* $15n+6$ *with* $n\geq 2,5,17$.
(14) *If* $\mathrm{p}(d,12)/[(6!)^2\cdot 2]$ *is square-free, then* $d=7m$ *with* $m=0$ *or* $m\geq 21$*, or* $d=7n+1$ *with* $n=0$ *or* $n\geq 23$.
(15) *If* $\mathrm{p}(d,24)/[(12!)^2\cdot 2]$ *is square-free, then* $d=0$ *or* $d>99$.
(16) *If* $\mathrm{p}(d,2a)/[(a!)^2\cdot 2]$ *is square-free, then* $d=0,1$ *or* $d>99$*, where* $a\in\{9,10,36\}$.

Proof. As examples, we prove (7) and (12) only; the others can be proved by similar arguments and (or) checking by GAP.

Assume that $\mathrm{p}(d,7)/168$ is square-free. If 8 divides some $d+i$, then $2^5$ divides $\mathrm{p}(d,7)$ by noting that at least three of seven consecutive integers are even, and so 4 divides $\mathrm{p}(d,7)/168$, which contradicts the hypothesis. It follows that $d=8k$ for some $k$. If 9 divides some $d+i$, then $3^3$ divides $\mathrm{p}(d,7)$, so $3^2$ divides $\mathrm{p}(d,7)/168$, which contradicts the hypothesis. Then $d=9l$ or $9l+1$ for some $l$. It yields $d=72m$ or $d=72n+64$ with $m,n\geq 0$. If $0\neq m\leq 2$ or $n=0$ then $5^2$ divides $\mathrm{p}(d,7)$, which contradicts the hypothesis. Thus (7) follows by noting that 5 does not divide both $d+1$ and $d+2$.

Assume that $\mathrm{p}(d,8)/(2^6\cdot 3^2)$ is square-free. Then none of $d+1$, $d+2$ and $d+3$ is divisible by 5, and hence $d=5l$ or $5l+1$. If 3 divides one of $d+1$ and $d+2$, then three of these eight consecutive integers are divisible by 3. This yields that $3^4$ divides $\mathrm{p}(d,8)$, which contradicts the hypothesis. Thus $d=3k$. Then $d=15m$ or $15n+6$. If $2^4$ divides some $d+i$, then $2^8$ divides $\mathrm{p}(d,8)$, which contradicts the hypothesis.

It yields $m \equiv 0, 9, 10, 11, 12, 13, 14, 15 \bmod 16$ and $n \equiv 1, 2, 3, 4, 5, 6, 15 \bmod 16$. Noting that both $5^2$ and $7^2$ do not divide $\mathrm{p}(d, 8)$, (12) follows. □

Let $c$ be a positive integer and $P$ a partition of $c$ into positive parts. We define $\mathrm{f}(c; P) = (\sum_{d\in P} d)!/\prod_{d\in P} d!$. Then the following result holds.

Lemma 3.4. *Let $k \geq 2$ and $c \geq 5$ be integers. Let $c = \sum_{i=1}^{k} c_i$ and $c_i = \sum_{j=1}^{t_i} d_{ij}$ for $1 \leq i \leq k$ and positive integers $d_{ij}$. Then $\mathrm{f}(c; d_{11}, \ldots, d_{kt_k}) = \mathrm{f}(c; c_1, \ldots, c_k) \prod_{i=1}^{k} \mathrm{f}(c_i; d_{i1}, \ldots, d_{it_i})$. Assume, further, that $\mathrm{f}(c; d_{11}, \ldots, d_{kt_k})$ is square-free. Then the following statements hold.*

(1) *$\mathrm{f}(c; c_1, \ldots, c_k)$ and $\mathrm{f}(c_i; d_{i1}, \ldots, d_{it_i})$, $1 \leq i \leq k$, are pairwise coprime square-free numbers; so at most one of them is even.*
(2) *If $d_{i_1 j_1} = d_{i_2 j_2}$ for $(i_1, j_1) \neq (i_2, j_2)$, then $d_{i_1 j_1} = d_{i_2 j_2} = 4$, $2$ or $1$.*
(3) *If $l_r$ if the number of $d_{ij}$ with value r, then $l_4 \leq 2$, $l_3 \leq 1$, $l_2 \leq 2$, $l_1 \leq 3$, $\sum_{r=1}^{4} l_r \leq 4$ and $\sum_{r=1}^{4} r l_r \leq 8$.*

Proof. Note that $\mathrm{S}_c \geq \mathrm{S}_{c_1} \times \cdots \times \mathrm{S}_{c_k}$ and $\mathrm{S}_{c_j} \geq \mathrm{S}_{d_{i1}} \times \cdots \times \mathrm{S}_{d_{it_i}}$. Then the first part of this lemma holds by checking that $|\mathrm{S}_c : (\mathrm{S}_{d_{11}} \times \cdots \times \mathrm{S}_{d_{kt_k}})|$. And then (1) follows. Assume that $d_{i_1 j_1} = d_{i_2 j_2} := a$ for some $(i_1, j_1) \neq (i_2, j_2)$. Then $\mathrm{f}(2a; a, a)$ is square-free by (1). Of course, $\mathrm{f}(2a; a, a)/2$ is odd square-free. By Lemma 3.1, $a$ is known. It yields $a = 4$ or 2 if $a \neq 1$, and (2) follows. Let $c'$ be one of $\sum_{d_{i,j}=r} d_{ij}$ and $\sum_{d_{i,j}\leq 4} d_{ij}$. Then (3) follows from (1). □

The following facts about primitive permutation groups (see [6, Theorem 3.3.A, Example 3.3.1]) are known.

Lemma 3.5. *Let $G$ be a primitive subgroup of $\mathrm{S}_c$. If $G$ contains one of $(ij)$, $(ijk)$ and $(ij)(kl)$, then either $G \geq \mathrm{A}_c$ or $c \leq 8$.*

Lemma 3.6. *Let $c \geq 5$ be an integer. Let $G$ be almost simple with $\mathsf{soc}(G) = \mathrm{A}_c$ and let $H < G$ with $|G : H|$ being square-free. If either $G \not\leq \mathrm{S}_c$ or $H$ is transitive on $\mathbf{c}$, then one of the following holds.*

(1) *$G = \mathrm{PGL}(2, 9)$, $\mathrm{M}_{10}$ or $\mathrm{P\Gamma L}(2, 9)$ and $H = Z_3^2 \rtimes Z_8$, $Z_3^2 \rtimes \mathrm{Q}_8$ or $Z_3^2 \rtimes [2^4]$, respectively, where $[2^4]$ is a 2-group of order $2^4$.*
(2) *Either $\mathsf{soc}(G) = \mathsf{soc}(H) = \mathrm{A}_6$ or $(G, H)$ is one of $(\mathrm{PGL}(2, 9), \mathrm{S}_4)$, $(\mathrm{M}_{10}, \mathrm{S}_4)$, and $(\mathrm{P\Gamma L}(2, 9), \mathrm{S}_4 \times Z_2)$.*
(3) *$(G, H)$ is one of $(\mathrm{S}_c, \mathrm{A}_c)$, $(\mathrm{A}_5, \mathrm{D}_{10})$, $(\mathrm{S}_5, Z_5 \rtimes Z_4)$, $(\mathrm{A}_6, \mathrm{PSL}(2, 5))$, $(\mathrm{S}_6, \mathrm{PGL}(2, 5))$, $(\mathrm{S}_7, \mathrm{PSL}(3, 2))$ $(\mathrm{A}_7, \mathrm{PSL}(3, 2))$, $(\mathrm{S}_8, Z_2^3 \rtimes \mathrm{PSL}(3, 2))$ and $(\mathrm{A}_8, Z_2^3 \rtimes \mathrm{PSL}(3, 2))$.*
(4) *$H$ is not primitive on $\mathbf{c}$, and either $c \leq 8$ and $H$ is a $\{2, 3\}$-group or $c = 2a$ and $H = (\mathrm{S}_a \wr \mathrm{S}_2) \cap G$ for $a \in \{6, 9, 10, 12, 36\}$.*

Proof. If $G \not\leq \mathrm{S}_c$, then $c = 6$, and so (1) and (2) follow from checking the subgroups of $G$ of square-free indices in [5]. Thus, in the following, assume that $\mathrm{A}_c \leq G \leq \mathrm{S}_c$ and $H$ is transitive on $\mathbf{c}$.

Assume that $H$ is primitive on $\mathbf{c}$. Since $|G : H|$ is square-free, $H$ contains a maximal subgroup of a Sylow two-subgroup of $\mathrm{A}_c$. Then $H$ contains a permutation with the form of $(ij)(kl)$ and (3) follows from Lemma 3.5 and checking the primitive groups of degree no more than eight.

Assume that $H$ is not primitive on $\mathbf{c}$. Then $\mathrm{A}_c \leq G \leq \mathrm{S}_c$. Let $\mathcal{B}$ be a nontrivial $H$-invariant partition on $\mathbf{c}$ with minimal block size, say, $a$. Then $H \leq (\mathrm{S}_a \wr \mathrm{S}_b) \cap G := M \leq G$, where $b = c/a$. Since $|G : H|$ is square-free, $|G : M|$ and $|M : H|$ are also square-free. It is easy to see that $|\mathrm{S}_c : (\mathrm{S}_a \wr \mathrm{S}_b)| = |G : M|$. Then $|\mathrm{S}_c : (\mathrm{S}_a \wr \mathrm{S}_b)|$ is square-free and $(a, b)$ is given in Lemma 3.1. Clearly, if both $a$ and $b$ are no more than four, then $H$ is a $\{2, 3\}$-group. Thus assume that $b = 2$ and $a \in \{6, 9, 10, 12, 36\}$. In particular, it is easy to know that $|\mathrm{S}_c : (\mathrm{S}_a \wr \mathrm{S}_b)| = |G : M|$ is even square-free.

Set $\mathcal{B} = \{\Delta_1, \Delta_2\}$. Without loss of generality, assume that $\Delta_1 = \mathbf{a}$ and let $\mathrm{S}_a \wr \mathrm{S}_2 = (\mathrm{Sym}(\Delta_1) \times \mathrm{Sym}(\Delta_2)) \rtimes \langle \pi \rangle$, where $\pi = \Pi_{i=1}^{a}(ia + i)$. In particular, $\pi \in \mathrm{A}_c$ if $a$ is even. Let $N = \mathrm{Alt}(\Delta_1) \times \mathrm{Alt}(\Delta_2)$. Then $N \trianglelefteq M$, and so $HN$ is a subgroup of $M$. Thus $|N : (H \cap N)| = |HN : H|$ is a divisor of $|M : H|$. Then $|N : (H \cap N)|$ is square-free. It is easily shown that $H \cap N$ contains a maximal subgroup $Q$ of a Sylow two-subgroup $P$ of $N$. Then $Q \trianglelefteq P$ and $|P : Q| = 2$. Without loss of generality, assume that $P$ contains $(1\,2\,3\,4)(5\,6)$ and $(a + 1 a + 2 a + 3 a + 4)(a + 5 a + 6)$. It follows that $(1\,2)(3\,4)$, $(a + 1 a + 2)(a + 3 a + 4) \in Q$. Thus $(1\,2)(3\,4) \in H_{\Delta_1}^{\Delta_1}$ and $(a + 1 a + 2)(a + 3 a + 4) \in H_{\Delta_2}^{\Delta_2}$. By the choice of $\mathcal{B}$, $H_{\Delta_i}^{\Delta_i}$ is a primitive subgroup of $\mathrm{Sym}(\Delta_i)$. Then, similarly as in (2), either $H_{\Delta_i}^{\Delta_i} \geq \mathrm{Alt}(\Delta_i)$ or $\mathrm{PSL}(2, 5) \leq H_{\Delta_i}^{\Delta_i} \leq \mathrm{PGL}(2, 5)$. But the latter case yields four dividing $|G : H|$. Thus $H_{\Delta_i}^{\Delta_i} \geq \mathrm{Alt}(\Delta_i)$. Noting that $1 \neq (H \cap N)^{\Delta_i} \trianglelefteq H_{\Delta_i}^{\Delta_i}$, $(H \cap N)^{\Delta_i} = \mathrm{Alt}(\Delta_i)$. It follows from [6, Lemma 4.3A] that $H \cap N = \mathrm{Alt}(\Delta_1) \times \mathrm{Alt}(\Delta_2) = N$. It is easy to check that a Sylow two-subgroup of $N$ has index $2^2$ in some Sylow two-subgroup of $\mathrm{A}_c$. Then $N$ is properly contained in $H$. Noting that $|M : H|$ divides $|M : N| = 2^2$ or $2^3$ and $|G : M|$ is even square-free, it follows that $|M : H| = 1$. Then (4) holds. □

Lemma 3.7. *Let $c \geq 5$ be an integer. Let $\mathrm{A}_c \leq G \leq \mathrm{S}_c$ and let $H < G$ with $|G : H|$ being square-free. Assume that $H$ has $t$ orbits $\Delta_1, \ldots, \Delta_t$ on $\mathbf{c}$, where $t \geq 2$. Let $d_j = |\Delta_j|$ for $1 \leq j \leq t$. Let $r$ be such that $b_{r+1} = \cdots = b_t = 1$ and $b_j > 1$ for $j \leq r$. Set $c_1 = \sum_{i=1}^{r} d_j$.*

(1) *If $r \geq 2$ and $c_1 \geq 5$, then, reordering $d_j$ if necessary, either $H$ is one of $(\mathrm{S}_{d_1} \times \cdots \times \mathrm{S}_{d_{r-1}} \times \mathrm{A}_{d_r}) \cap G$ and $(\mathrm{S}_{d_1} \times \cdots \times \mathrm{S}_{d_r}) \cap G$ or, for each $d_j > 1$, the pair $(d_j, H^{\Delta_j})$ is as described in Tables 2, 3, 4 and 5 for $r = t$ and as in Tables 8, 9, 10 and 11 for $r < t$.*

(2) *If $r = 1$ or $c_1 \leq 5$, then $(d_1, H^{\Delta_1})$ is as described in Tables 6 and 7.*

Proof. Set $M_1 := (H^{\Delta_1} \times \cdots \times H^{\Delta_t}) \cap G$ and $M_2 := (\mathrm{S}_{d_1} \times \cdots \times \mathrm{S}_{d_t}) \cap G$. Then $H \leq M_1$ and $H \leq M_2$. Since $|G : H|$ is square-free, $|M_i : H|$, $|M_2 : M_1|$ and $|G : M_i|$ are all square-free, where $i = 1, 2$.

*Case 1*. Assume that $H$ is fixed-point-free on $\mathbf{c}$, that is, $d_j \geq 2$ for all $j \leq t$.

Assume that $H^{\Delta_j} \leq \mathrm{Alt}(\Delta_j)$ for all $1 \leq j \leq t$. Then $H \leq \mathrm{A}_c$ and $M_1 = H^{\Delta_1} \times \cdots \times H^{\Delta_t}$ as $\mathrm{A}_c \leq G$. If $G = \mathrm{S}_c$, then $|G : H|$ is divisible by $2^t$, which contradicts the hypothesis. Thus $G = \mathrm{A}_c$. Then $M_2 = (\mathrm{A}_{d_1} \times \cdots \times \mathrm{A}_{d_t}) \rtimes Z_2^{t-1}$, and hence $t = 2$ and $|\mathrm{A}_{d_j} : H^{\Delta_j}|$ is

TABLE 2. Pairs of orbit length and subgroup transitive restriction Case 1.

| $c$ | $d_1$ | $d_2$ | $H^{\Delta_1}$ | $H^{\Delta_2}$ | Remark |
|---|---|---|---|---|---|
| $d_1+d_2$ | $\geq 5$ | $\geq 3$ | $A_{d_1}$ | $A_{d_2}$ | $d_1-d_2\geq 2$ |
| | | | | | $p(d_1,d_2)/d_2!$ odd square-free |
| $d_1+8$ | $\geq 36$ | 8 | $A_{d_1}$ | $Z_2^3 \rtimes PSL(3,2)$ | $p(d_1,8)/(2^6\cdot 3\cdot 7)$ square-free |
| $d_1+8$ | $\geq 36$ | 8 | $A_{d_1}$ | $Z_2^3 \rtimes S_4$ | $p(d_1,8)/(2^6\cdot 3)$ square-free |
| $d_1+8$ | $\geq 36$ | 8 | $A_{d_1}$ | $(S_4 \wr S_2)\cap A_8$ | $p(d_1,8)/(2^6\cdot 3^2)$ square-free |
| $d_1+7$ | $\geq 136$ | 7 | $A_{d_1}$ | $PSL(3,2)$ | $p(d_1,7)/168$ square-free |
| $d_1+6$ | $\geq 56$ | 6 | $A_{d_1}$ | $S_4$ | $p(d_1,6)/24$ square-free |
| $d_1+4$ | $\geq 9$ | 4 | $A_{d_1}$ | $Z_2^4$ | $p(d_1,4)/4$ square-free |
| 7 | 4 | 3 | $A_4$ | $A_3$ | |
| 7 | 4 | 3 | $Z_2^2$ | $A_3$ | |

TABLE 3. Pairs of orbit length and subgroup transitive restriction Case 2.

| $d_j$ | $d_t$ | $H^{\Delta_j}$ | $H^{\Delta_t}$ | Remark |
|---|---|---|---|---|
| $>99$ | $2a$ | $S_{d_j}$ | $S_a \wr S_2$, $a=6,9,10,12,36$ | $p(d_j,2a)/[2\cdot(a!)^2]$ square-free |
| | | | $S_4 \wr S_2$ | $p(d_j,8)/[2\cdot(4!)^2]$ square-free |
| | | | $(S_4\wr S_2)\cap A_8$ | $p(d_j,8)/[(4!)^2]$ square-free |
| $\geq 36$ | 8 | $S_{d_j}$ | $Z_2^3\rtimes S_4$, $Z_2^4\rtimes[2^2\cdot 3]$, $Z_2^4\rtimes A_4$ | $p(d_j,8)/(3\cdot 2^6)$ square-free |
| | | | $Z_2^4 \rtimes S_4$ | $p(d_j,8)/(3\cdot 2^7)$ square-free |
| | | | $Z_2^3\rtimes PSL(3,2)$ | $p(d_j,8)/(3\cdot 7\cdot 2^6)$ square-free |
| $\geq 136$ | 7 | $S_{d_j}$ | $PSL(3,2)$ | $p(d_j,7)/(3\cdot 7\cdot 2^3)$ square-free |
| $\geq 36$ | 6 | $S_{d_j}$ | $S_4\times Z_2$ | $p(d_j,6)/48$ square-free |
| $\geq 56$ | | | $S_4$ | $p(d_j,6)/24$ square-free |
| $\geq 9$ | | | $PGL(2,5)$ | $p(d_j,6)/120$ square-free |
| $\geq 8$ | | | $Z_3^2\rtimes D_8$ | $p(d_j,6)/72$ square-free |
| $\geq 18$ | 5 | $S_{d_j}$ | $Z_5\rtimes Z_4$ | $p(d_j,5)/20$ square-free |
| $\geq 9$ | 4 | $S_{d_j}$ | $D_8$ | $p(d_j,4)/8$ square-free |
| | | | $[2^2]$ | $p(d_j,4)/4$ square-free |

odd square-free for $i=1$ and 2. Thus either $H^{\Delta_j}=A_{d_i}$ or $H^{\Delta_j}$ is known as in (2) or (3) as it is transitive on $\Delta_i$. Calculating $|A_{d_j}:H^{\Delta_j}|$ shows that $H^{\Delta_j}$ is one of $A_{d_j}$, $PSL(3,2)$ for $d_j=7$, $Z_2^3\rtimes PSL(3,2)$ for $d_j=8$, $(S_{d_i/2}\wr S_2)\cap A_{d_i}$ for $d_i\in\{12,18,20,24,72\}$, $Z_2^3\rtimes S_4$ for $d_j=8$, $(S_4\wr S_2)\cap A_8$ for $d_j=8$, $S_4$ for $d_j=6$, or $Z_2^2$ for $d_j=4$.

Since $|A_c:M_1|$ and $|M_2:M_1|=2|A_{d_1}:H^{\Delta_1}||A_{d_2}:H^{\Delta_2}|$ are square-free, with the help of Lemma 3.1, Corollary 3.2 and Lemma 3.3, $(c,d_1,d_2;H^{\Delta_1},H^{\Delta_2})$ are listed in Table 2. Assume that $H^{\Delta_i}\nleq \mathrm{Alt}(\Delta_i)$ for some $1\leq i\leq t$. Then $M_1$ has index two or one in $L_1:=H^{\Delta_1}\times\cdots\times H^{\Delta_t}$ depending on $G=A_c$ or not, respectively; and the same thing occurs for $M_2$ and $L_2:=S_{d_1}\times\cdots\times S_{d_t}$. Thus $|L_2:L_1|$, $|S_c:L_2|$, $|S_c:L_1|$ and $|S_{d_j}:H^{\Delta_j}|$ are all square-free. Then $(d_j,H^{\Delta_j})$ is one of the following pairs: $(d_j,S_{d_j})$, $(d_j,A_{d_j})$, $(S_7,PSL(3,2))$, $(S_8,Z_2^3\rtimes PSL(3,2))$, $(5,Z_5\rtimes Z_4)$, $(6,PGL(2,5))$,

TABLE 4. Pairs of orbit length and subgroup transitive restriction Case 3.

| $d_j$ | $d_{t-1}$ | $d_t$ | $H^{\Delta_j}$ | $H^{\Delta_{t-1}}$ | $H^{\Delta_t}$ | Remark |
|---|---|---|---|---|---|---|
| ≥36 | ≥36 | 8 | $S_{d_j}$ | $A_{d_{t-1}}$ | $S_4 \wr S_2$ | $p(d_j, 8)/[2 \cdot (4!)^2]$ odd square-free |
| | | | | | $Z_2^4 \rtimes S_4$ | $p(d_j, 8)/(2^7 \cdot 3)$ odd square-free |
| ≥36 | ≥36 | 6 | $S_{d_j}$ | $A_{d_{t-1}}$ | $S_4 \times Z_2$ | $p(d_j, 6)/48$ odd square-free |
| ≥9 | ≥9 | 4 | $S_{d_j}$ | $A_{d_{t-1}}$ | $D_8$ | $p(d_j, 4)/8$ odd square-free |
| ≥36 | 4 | 4 | $S_{d_j}$ | $S_4$ | $D_8$ | $p(d_j, 8)/(3 \cdot 2^6)$ square-free |
| ≥36 | 4 | 4 | $S_{d_j}$ | $A_4$ | $D_8$ | $p(d_j, 8)/(3 \cdot 2^5)$ square-free |
| ≥36 | 4 | 3 | $S_{d_j}$ | $D_8$ | $S_3$ | $p(d_j, 7)/48$ square-free |
| ≥136 | 4 | 3 | $S_{d_j}$ | $D_8$ | $A_3$ | $p(d_j, 7)/24$ square-free |
| | | | | $[2^2]$ | $S_3$ | |

TABLE 5. Pairs of orbit length and subgroup transitive restriction Case 4.

| c | $d_1$ | $d_2$ | $d_3$ | $H^{\Delta_1}$ | $H^{\Delta_2}$ | $H^{\Delta_3}$ | Remark |
|---|---|---|---|---|---|---|---|
| $d_1 + 7$ | $d_1 \geq 36$ | 4 | 3 | $A_{d_1}$ | $D_8$ | $S_3$ | $p(d_1, 7)/48$ odd square-free |
| $d_1 + 8$ | $d_1 \geq 36$ | 4 | 4 | $A_{d_1}$ | $D_8$ | $S_4$ | $p(d_1, 8)/(2^6 \cdot 3)$ odd square-free |
| $d_1 + 8$ | $d_1 \geq 36$ | 8 | | $A_{d_1}$ | $S_4 \wr S_2$ | | $p(d_1, 8)/[(4!)^2 \cdot 2]$ odd square-free |
| $d_1 + 8$ | $d_1 \geq 36$ | 8 | | $A_{d_1}$ | $Z_2^4 \rtimes S_4$ | | $p(d_1, 8)/(2^7 \cdot 3)$ odd square-free |
| $d_1 + 6$ | $d_1 \geq 36$ | 6 | | $A_{d_1}$ | $S_4 \times Z_2$ | | $p(d_1, 6)/48$ odd square-free |
| $d_1 + 4$ | $d_1 \geq 9$ | 4 | | $A_{d_1}$ | $D_8$ | | $p(d_1, 4)/8$ odd square-free |
| 8 | 4 | 4 | | $S_4$ | $D_8$ | | |
| 7 | 4 | 3 | | $D_8$ | $S_3$, $A_3$ | | |
| 7 | 4 | 3 | | $[2^2]$ | $S_3$ | | |

TABLE 6. Pairs of orbit length and subgroup transitive restriction Case 5.

| $c$ | $d_1$ | $c - c_1$ | $G$ | $H$ | Remark |
|---|---|---|---|---|---|
| $c$ | $d_1$ | ≤3 | $S_c$ | $S_{c_1}$ | $p(c_1, c - c_1)$ square-free |
| $c$ | $c - 1$ | 1 | $S_c$ | $A_{c_1}$ | $c$ odd square-free |
| | $d_1$ | ≤3 | $A_c$ | $A_{c_1}$ | $p(c_1, c - c_1)$ square-free |
| $2a + 1$ | $2a$ | 1 | $S_{2a+1}$ | $S_a \wr S_2$ | $a \in \{6, 9, 10, 36\}$ |
| | | | $A_{2a+1}$ | $(S_a \wr S_2) \cap A_{2a}$ | |
| 7 | 6 | 1 | $S_7$ | $PGL(2, 5)$ | |
| | | | | $Z_3^2 \rtimes D_8$ | |
| | | | | $S_4 \times Z_2$, $S_4$ | |
| | | | $A_7$ | $Z_3^2 \rtimes Z_4$, $A_4$, $S_4$ | |
| | | | | $PSL(2, 5)$ | |

$(2a, S_a \wr S_2)$ for $a \in \{6, 9, 10, 12, 36\}$, $(8, S_4 \wr S_2)$, $(8, (S_4 \wr S_2) \cap A_8)$, $(8, Z_2^4 \rtimes [2^2 \cdot 3])$, $(8, Z_2^4 \rtimes S_4)$, $(8, Z_2^4 \rtimes A_4)$, $(8, Z_2^3 \rtimes S_4)$, $(6, S_4)$, $(6, Z_3^2 \rtimes D_8)$, $(6, S_4 \times Z_2)$, $(4, S_4)$, $(4, D_8)$ or $(4, [2^2])$. Noting that $|L_2 : L_1| = \prod_{i=1}^{t} |S_{d_j} : H^{\Delta_j}|$, all $|S_{d_j} : H^{\Delta_j}|$ are pairwise coprime,

TABLE 7. Pairs of orbit length and subgroup transitive restriction Case 6.

| $c$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $H^{\Delta_1}$ | $H^{\Delta_2}$ | $H^{\Delta_3}$ | $H^{\Delta_4}$ | $G$ | $H$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 3 | 1 | 1 | | $S_3$ | 1 | 1 | | $A_5$ | $S_3$ |
| | 2 | 2 | 1 | | $Z_2$ | $Z_2$ | 1 | | $S_5$ | $Z_2^2$ |
| | | | | | $Z_2$ | $Z_2$ | 1 | | $A_5$ | $Z_2$ |
| 5 | 4 | 1 | | | $S_4$ | 1 | | | $S_5$ | $S_4$ |
| | | | | | $A_4$ | 1 | | | $S_5$ | $A_4$ |
| | | | | | $D_8$ | 1 | | | $S_5$ | $D_8$ |
| | | | | | $[2^2]$ | 1 | | | $S_5$ | $[2^2]$ |
| | | | | | $A_4$ | 1 | | | $A_5$ | $A_4$ |
| | | | | | $Z_2^2$ | 1 | | | $S_5$ | $Z_2^2$ |
| 6 | 4 | 1 | 1 | | $S_4$ | 1 | 1 | | $S_6$ | $S_4$ |
| | | | | | $A_4$ | 1 | 1 | | $A_6$ | $A_4$ |
| 7 | 4 | 2 | 1 | | $S_4$ | $Z_2$ | 1 | | $S_7$ | $S_4 \times S_2$, $S_4$ |
| | | | | | $A_4$ | $Z_2$ | 1 | | $S_7$ | $A_4 \times S_2$ |
| | | | | | $S_4$ | $Z_2$ | 1 | | $A_7$ | $S_4$, $A_4$ |
| 7 | 4 | 1 | 1 | 1 | $S_4$ | 1 | 1 | 1 | $S_7$ | $S_4$ |
| | | | | | $A_4$ | 1 | 1 | 1 | $A_7$ | $A_4$ |

TABLE 8. Pairs of orbit length and subgroup transitive restriction Case 7.

| $c$ | $t-r$ | $d_1$ | $d_2$ | $H^{\Delta_1}$ | $H^{\Delta_2}$ | Remark |
|---|---|---|---|---|---|---|
| $d_1+d_2+1$ | 1 | $\geq 5$ | $\geq 3$ | $A_{d_1}$ | $A_{d_2}$ | $d_1-d_2\geq 2$ |
| | | | | | | $p(d_1, d_2+1)/d_2!$ odd square-free |
| $d_1+7$ | 1 | $\geq 136$ | 6 | $A_{d_1}$ | $S_4$ | $p(d_1, 7)/24$ square-free |
| $d_1+5$ | 1 | $\geq 18$ | 4 | $A_{d_1}$ | $Z_2^4$ | $p(d_1, 5)/4$ square-free |

TABLE 9. Pairs of orbit length and subgroup transitive restriction Case 8.

| $d_j$ | $d_r$ | $H^{\Delta_j}$ | $H^{\Delta_r}$ | Remark |
|---|---|---|---|---|
| $>99$ | $2a$ | $S_{d_j}$ | $S_a \wr S_2$, $a=6,9,10,36$ | $p(d_j, 2a+1)/[2(a!)^2]$ square-free |
| $\geq 36$ | 6 | $S_{d_j}$ | $S_4\times Z_2$ | $p(d_j, 7)/48$ square-free |
| $\geq 136$ | | | $S_4$ | $p(d_j, 7)/24$ square-free |
| $\geq 64$ | 6 | $S_{d_j}$ | $PGL(2,5)$ | $p(d_j, 7)/120$ square-free |
| $\geq 16$ | | | $Z_3^2 \rtimes D_8$ | $p(d_j, 7)/72$ square-free |
| $\geq 9$ | 4 | $S_{d_j}$ | $D_8$ | $p(d_j, 5)/8$ square-free |
| $\geq 18$ | | | $[2^2]$ | $p(d_j, 5)/4$ square-free |

and so at most one of them is even square-free. If $H^{\Delta_j}\geq A_{d_j}$ for all $j$, then $H = (S_{d_1}\times\cdots\times S_{d_t})\cap G$ or, reordering $d_j$ if necessary, $H = (S_{d_1}\times\cdots\times S_{d_{t-1}}\times A_{d_t})\cap G$. For the other cases, with the help of Lemma 3.1, Corollary 3.2 and Lemmas 3.3 and 3.4, $(d_j, H^{\Delta_j})$ is as described in Tables 3, 4 and 5.

TABLE 10. Pairs of orbit length and subgroup transitive restriction Case 9.

| $d_j$ | $d_{r-1}$ | $d_r$ | $H^{\Delta_j}$ | $H^{\Delta_{r-1}}$ | $H^{\Delta_r}$ | Remark |
|---|---|---|---|---|---|---|
| ≥136 | ≥136 | 6 | $S_{d_j}$ | $A_{d_{t-1}}$ | $S_4 \times Z_2$ | $p(d_j,7)/48$ odd square-free |
| ≥18 | ≥18 | 4 | $S_{d_j}$ | $A_{d_{t-1}}$ | $D_8$ | $p(d_j,5)/8$ odd square-free |

TABLE 11. Pairs of orbit length and subgroup transitive restriction Case 10.

| c | $d_1$ | $d_2$ | $d_3$ | $H^{\Delta_1}$ | $H^{\Delta_2}$ | $H^{\Delta_3}$ | Remark |
|---|---|---|---|---|---|---|---|
| $d_1+7$ | $d_1 \geq 136$ | 6 | 1 | $A_{d_1}$ | $S_4 \times Z_2$ | 1 | $p(d_1,7)/48$ odd square-free |
| $d_1+5$ | $d_1 \geq 18$ | 4 | 1 | $A_{d_1}$ | $D_8$ | 1 | $p(d_1,5)/8$ odd square-free |

*Case 2*. Assume that $H$ fixes at least one point in $\mathbf{c}$. Assume that $d_{r+1} = \cdots = d_t = 1$ and $d_j > 1$ for $1 \leq j \leq r$. Then, as $c \geq 5$, $r \geq 1$ and $t - r \leq 3$ by Lemma 3.4. If $\sum_{i=1}^{r} d_i \leq 4$, then $t \leq 4$ and $\sum_{i=1}^{t} d_i \leq 8$, and then $(c; d_1, \ldots, d_t; G, H)$ is as listed in Table 7. Assume that $c_1 := \sum_{i=1}^{r} d_i \geq 5$. Then $H \leq G_1 := S_{c_1} \cap G$ and $|G : H| = c(c-1)\cdots(c-t+r+1)|G_1 : H| = p(c_1, t-r-1)|G_1 : H|$ is square-free.

Assume that $r = 1$, that is, $c_1 = d_1$ and $t - r = c - d_1$. Then, by Lemma 3.6, either $5 \leq c_1 = d_1 \leq 8$ and $H$ is a transitive $\{2,3\}$-subgroup of square-free index in $G_1$ or $(G_1, H)$ is one of $(S_{c_1}, S_{c_1})$, $(S_{c_1}, A_{c_1})$, $(A_{c_1}, A_{c_1})$, $(A_5, D_{10})$, $(S_5, Z_5 \rtimes Z_4)$, $(A_6, \mathrm{PSL}(2,5))$, $(S_6, \mathrm{PGL}(2,5))$, $(S_7, \mathrm{PSL}(3,2))$ $(A_7, \mathrm{PSL}(3,2))$, $(S_8, Z_2^3 \rtimes \mathrm{PSL}(3,2))$, $(A_8, Z_2^3 \rtimes \mathrm{PSL}(3,2))$, $(S_{2a}, S_a \wr S_2)$ or $(A_{2a}, (S_a \wr S_2) \cap A_{2a})$, where $a \in \{6, 9, 10, 12, 36\}$. Noting that $c|G : H|$ is square-free, then $(c; c_1, c - c_1; G, H)$ is as listed in Table 6.

Assume that $r \geq 2$. Consider the restrictions of $H$ on $\Delta_j$ for $1 \leq j \leq r$. Then, by Case 1, consider all possible pairs $(d_j, H^{\Delta_j})$. If a pair $(d_j, H^{\Delta_j})$ appears in Tables 2 to 5, then $\mathbf{p}(d_1, d_j)/|H^{\Delta_j}| \cdot p(d_1 + d_j, t - r - 1) = \mathbf{p}(d_1, c - d_1)/|H^{\Delta_j}|$ should be square-free, and then we get Tables 8–11. If $H^{\Delta_j} \geq A_{d_j}$ for all $j \leq r$ and $H^{\Delta_i} = S_{d_i}$ for some $i \leq r$, then $H = (S_{d_1} \times \cdots \times S_{d_r}) \cap G$ or, reordering $d_j$ if necessary, $H = (S_{d_1} \times \cdots \times S_{d_{r-1}} \times A_{d_r}) \cap G$. This concludes the proof. □

## 4. Proof of Theorem 1.1

Let $G$ be a finite group with $\mathrm{soc}(G) = A_c$ for $c \geq 5$. The first part of Theorem 1.1 follows form Lemmas 3.6 and 3.7. In the following, assume that $\Gamma$ is a connected $(G, 2)$-arc-transitive graph on square-free number vertices and sometimes, setting $H = G_\alpha$ for some $\alpha \in V\Gamma$, write $\Gamma = \mathrm{Cos}(G, H, HxH)$ for some $x \in G$ satisfying Lemma 2.1. Then the second part of Theorem 1.1 follows from Lemmas 4.1, 4.2, 4.3, 4.4 and 4.5.

LEMMA 4.1. *Assume that $G$ is one of* $\mathrm{PGL}(2,9)$, $M_{10}$ *and* $\mathrm{P\Gamma L}(2,9)$. *Then $\Gamma$ is isomorphic to $\mathsf{K}_{10}$ or the Tutte's 8-cage.*

PROOF. If $G$ is primitive on $V\Gamma$, then, by [26, Main-Theorem (1)], we know that $G$ is three-transitive on $V\Gamma$ and $\Gamma \cong \mathsf{K}_{10}$.

Thus we assume that $H$ is not maximal in $G$. Then$(G, H)$ is one of $(\mathrm{PGL}(2,9), \mathrm{S}_4)$, $(\mathrm{M}_{10}, \mathrm{S}_4)$ and $(\mathrm{P\Gamma L}(2,9), \mathrm{S}_4 \times Z_2)$. Further, for these three cases, $G$ has a subgroup of index two which contains $H$, say, $X = \mathrm{S}_6$ for $G = \mathrm{P\Gamma L}(2,9)$ and $X = \mathrm{A}_6$ for the other two cases. Thus $\Gamma$ is a bipartite graph with two parts, say, $U$ and $V$, each having size 15. It is easy to see that $X$ acts primitively on both $U$ and $V$. In particular, $X$ acts transitively on the edges of $\Gamma$. We claim that the actions of $X$ on $U$ and $V$ are not permutation equivalent; otherwise, $X$ will have a primitive permutation representation of degree 15 with a two-transitive subconstituent, which contradicts the main theorem of [26]. Thus assume that $U$ consists of two-subsets of $\mathbf{6}$ while $V$ is the set of partitions of $\mathbf{6}$ into three parts with the same size. Let $\{\alpha, \beta\}$ be an edge of $\Gamma$ with $\alpha \in U$ and $\beta \in V$. Then two possible cases arise. If $\alpha$ is not a part of $\beta$, then it is easily shown that $\Gamma(\alpha) = \beta^H = \{\beta^h \mid h \in H\}$ contains 12 partitions of $\mathbf{6}$, but $H$ cannot act two-transitively on $\Gamma(\alpha)$, which contradicts the hypothesis. Thus $\alpha$ must be a part of $\beta$ and, in this case, $\Gamma$ is isomorphic to Tutte's 8-cage. □

Lemma 4.2. *If $H$ is a transitive subgroup of $\mathrm{S}_c$, then $c = 5, 6$ and $\Gamma \cong \mathsf{K}_6$; or $c = 6$ and $\Gamma \cong \mathsf{K}_{10}$; or $c = 7, 8$ and $\Gamma$ or its complement graph in $\mathsf{K}_{15,15}$ is isomorphic to the point-hyperplane incidence graph of* $\mathrm{PG}(3,2)$.

Proof. Assume that $H$ is transitive on $\mathbf{c}$ with respect to the natural action of $\mathrm{S}_c$. Since $\Gamma$ is $(G, 2)$-arc-transitive, $|H| = |G_\alpha|$ has at least one odd prime divisor. It follows from Lemma 3.6 and checking the imprimitive groups of degrees six and eight that one of the following three cases occurs: (i) $H$ is maximal in $G$ and $H$ is one of $(\mathrm{S}_a \wr \mathrm{S}_2) \cap G$ for $c = 2a$ and $a \in \{6, 9, 10, 12, 36\}$, $(Z_5 \rtimes Z_4) \cap G$ for $c = 5$, $\mathrm{PGL}(2,5) \cap G$ for $c = 6$, $(Z_3^2 \rtimes \mathrm{D}_8) \cap G$ for $c = 6$, $(\mathrm{S}_4 \times Z_2) \cap G$ for $c = 6$, $\mathrm{PSL}(3,2)$ for $c = 7$ and $G = \mathrm{A}_7$, $(\mathrm{S}_4 \wr \mathrm{S}_2) \cap G$ for $c = 8$, and $Z_2^4 \rtimes \mathrm{S}_4$ for $c = 8$, $Z_2^3 \rtimes \mathrm{PSL}(3,2)$ for $c = 8$ and $G = \mathrm{A}_8$; (ii) $H$ is not maximal in $G$ and $(G, H)$ is one of $(\mathrm{S}_7, \mathrm{PSL}(3,2))$ and $(\mathrm{S}_8, Z_2^3 \rtimes \mathrm{PSL}(3,2))$; and (iii) $H$ is not maximal in $G$ and $(G, H)$ is one of $(\mathrm{A}_6, \mathrm{A}_4)$, $(\mathrm{S}_6, \mathrm{S}_4)$, $(\mathrm{S}_6, \mathrm{A}_4 \times Z_2)$, $(\mathrm{A}_8, Z_2^3 \rtimes \mathrm{S}_4)$, $(\mathrm{A}_8, Z_2^3 \rtimes \mathrm{A}_4)$, $(\mathrm{S}_8, Z_2^3 \rtimes \mathrm{S}_4)$ and $(\mathrm{S}_8, Z_2^4 \rtimes \mathrm{A}_4)$.

*Case 1*. Assume, first, that $H$ is maximal in $G$. Then $G$ is primitive on $V\Gamma$. Noting that $H$ is transitive on $\mathbf{c}$, it follows from [26] that $c = 5$ and $\Gamma \cong \mathsf{K}_6$, or $c = 6$, $G = \mathrm{P\Sigma L}(2,9) = \mathrm{S}_6$ and $\Gamma \cong \mathsf{K}_{10}$ (noting that this case was missed in [26]), or $H$ is almost simple and primitive on $\mathbf{c}$, so $H$ is one of $\mathrm{PGL}(2,5) \cap G$ and $\mathrm{PSL}(3,2)$. If $H = \mathrm{PGL}(2,5) \cap G$, then $\Gamma \cong \mathsf{K}_6$. Suppose that $G = \mathrm{A}_7$ and $H = \mathrm{PSL}(3,2)$. Then $|V\Gamma| = |G : H| = 15$ is odd and $\Gamma$ is of even valency. It yields $|\Gamma(\alpha)| = 8$, and hence $H_\beta = G_{\alpha\beta} \cong Z_7 \rtimes Z_3$ for some $\beta \in \Gamma(\alpha)$. It is easily shown that $\mathrm{N}_G(G_{\alpha\beta}) = G_{\alpha\beta}$. Then there is no $x \in \mathrm{N}_G(G_{\alpha\beta})$ with $\langle H, x\rangle = G$, which contradicts the hypothesis.

*Case 2*. Assume that $G = \mathrm{S}_7$ or $\mathrm{S}_8$ and $H = \mathrm{PSL}(3,2)$ or $Z_2^3 \rtimes \mathrm{PSL}(3,2)$, respectively. Then $H \le \mathsf{soc}(G) = \mathrm{A}_c$, $c = 7$ or 8. Then $\Gamma$ is a bipartite graph with two parts, say, $U$ and $V$, each having size 15. Further, $\mathrm{A}_c$ is primitive on both $U$ and $V$ and transitive on $E\Gamma$.

Assume that the actions of $\mathrm{A}_c$ on $U$ and on $V$ are permutation equivalent. Then $\mathrm{A}_c$ is a primitive permutation group with degree 15 and a suborbit of size $|\Gamma(\alpha)|$.

It is known that such a primitive permutation group is two-transitive. Thus $|\Gamma(\alpha)| = 14$ and $\Gamma \cong \mathsf{K}_{15,15} - 15\mathsf{K}_2$, but such a graph cannot admit $\mathrm{S}_c$ acting transitively on its two-arcs, which contradicts the hypothesis.

Therefore, assume that $U$ is the point set while $V$ is the hyperplane set of the projective geometry $\mathrm{PG}(3, 2)$, respectively. (Note that $\mathrm{A}_7$ is viewed as a transitive subgroup of $\mathrm{PSL}(4, 2) \cong \mathrm{A}_8$ on projective points or on hyperplanes.) Then $\Gamma$ or its complement graph in $\mathsf{K}_{15,15}$ is isomorphic to the point-hyperplane incidence graph of $\mathrm{PG}(3, 2)$.

*Case 3*. Assume that $c = 6$ or $8$ and $H$ is soluble. Then $H^{\Gamma(\alpha)}$ is a two-transitive affine group. Further, by checking one by one the possible $H = G_\alpha$ here, $\Gamma$ is of valency three or four.

Suppose that $\Gamma$ is of valency three. Note that the stabilizers for cubic two-arc-transitive graphs are explicitly known (see [2, 18f], for example). Then the only possible case is $(G, H) = (\mathrm{S}_6, \mathrm{S}_4)$, and so $\Gamma$ is $(\mathrm{S}_6), 4)$-arc-transitive. By [4], all cubic two-arc-transitive graphs of order 30 are isomorphic and five-transitive. Thus $\Gamma$ is isomorphic to the graph given in Example 2.5, but such a graph cannot admit $\mathrm{S}_6$ acting transitively on vertices, which contradicts the hypothesis.

Now let $\Gamma$ be of valency four. If $\Gamma$ is $(G, s)$-transitive for $s \geq 4$, then $H$ should contain a subgroup with quotient $\mathrm{GL}(2, 3)$ by checking the stabilizers listed in Table 1, which is impossible. Thus $\Gamma$ is $(G, 2)$-transitive or $(G, 3)$-transitive. Then, by Lemma 2.12, $(G, H) = (\mathrm{A}_6, \mathrm{A}_4)$ or $(\mathrm{S}_6, \mathrm{S}_4)$.

Suppose that $G = \mathrm{S}_6$ and $H = \mathrm{S}_4 \leq \mathsf{soc}(G) = \mathrm{A}_6$. Then $\Gamma$ is a bipartite graph with $\mathrm{A}_6$ acting primitively on both two parts, say, $U$ and $V$. If the actions of $\mathrm{A}_6$ on $U$ and $V$ are not permutation equivalent, then a similar argument as in Lemma 4.1 yields that $\Gamma$ is of valency three, which contradicts the hypothesis. Thus the actions of $\mathrm{A}_6$ on $U$ and $V$ are permutation equivalent. So $\mathrm{A}_c$ is a primitive group with degree 15 and a suborbit of size four, which is impossible.

The above argument implies that $\Gamma$ is $(\mathrm{A}_6, 2)$-arc-transitive, and it is easily shown that $(\mathrm{A}_6)_\alpha = H \cap \mathrm{A}_6 \cong \mathrm{A}_4$ is transitive on $\mathbf{6}$. Then, replacing $G$ by $\mathrm{A}_6$ if necessary, assume that $H = \langle \sigma, \tau \rangle$ and $G_{\alpha\beta} = \langle \sigma \rangle$, where $\sigma = (1\,2\,3)(4\,5\,6)$ and $\tau = (1\,4)(2\,5)$. Calculation indicates that there is no $x \in \mathrm{N}_G(G_{\alpha\beta}) = \langle (1\,2\,3), (4\,5\,6) \rangle \rtimes \langle (2\,3)(4\,5) \rangle$ with $\langle x, H \rangle = G$, which contradicts the hypothesis. □

By Lemmas 4.1 and 4.2, assume that $G \leq \mathrm{S}_c$ and $H$ is intransitive on $\mathbf{c}$ in the following three lemmas. Let $\Delta_1, \ldots, \Delta_t$ be $H$-orbits on $\mathbf{c}$, where $t \geq 2$. Let $d_j = |\Delta_j|$ for $1 \leq j \leq t$. Then Lemma 3.7 is available for our further argument. By Lemma 2.10, $H = G_\alpha$ has at most two insoluble composition factors. It follows that at most two of $H^{\Delta_j}$ are insoluble.

Lemma 4.3. *If $H$ is soluble, then $\Gamma$ is isomorphic to one of $\mathsf{K}_5$, $\mathrm{O}_3$ and $\mathsf{K}_{5,5} - 5\mathsf{K}_2$ for $c = 5$, or to $\mathrm{O}_4$ for $c = 7$.*

Proof. Assume that $G \leq \mathrm{S}_c$ and $H$ is a soluble intransitive subgroup of $\mathrm{S}_c$.

*Case 1*. $H$ is fixed-point-free on $\mathbf{c}$. In this case, it is shown that $d_j \le 4$ for $1 \le j \le t$ by checking all possible $H^{\Delta_j}$ in Lemma 3.7. Thus $t \le 4$ and $c = \sum_{j=1}^{t} \le 8$ by Lemma 3.4. Further, $\Gamma$ is of valency three or four by considering the possible two-transitive affine group $H^{\Gamma(\alpha)}$, and the fact that $\Gamma$ is not $(G, s)$-transitive for $s \ge 4$, by Lemma 2.11, if $\Gamma$ is of valency four.

Assume that $\Gamma$ is valency three. Then $(c, G, H)$ is one of $(5, \mathrm{S}_5, \mathrm{S}_3 \times \mathrm{S}_2)$, $(5, \mathrm{A}_5, (\mathrm{S}_3 \times \mathrm{S}_2) \cap \mathrm{A}_5)$, $(6, \mathrm{A}_6, (\mathrm{S}_4 \times \mathrm{S}_2) \cap \mathrm{A}_6)$, $(6, \mathrm{S}_6, \mathrm{S}_4 \times \mathrm{S}_2)$ and $(7, \mathrm{A}_7, ([2^2] \times \mathrm{S}_3) \cap \mathrm{A}_7)$. If $c = 7$, then $|V\Gamma| = |G : H| = 210$, but there is no cubic arc-transitive graph with order 210 by [4], which contradicts the hypothesis. Each of the first four triples imply that $G$ is primitive on $V\Gamma$, so then, by [26], the only possible case is that $c = 5$ and $\Gamma \cong \mathrm{O}_3$.

Assume that $\Gamma$ is valency four. Then $(c, G, H)$ is one of $(6, \mathrm{A}_6, (\mathrm{S}_4 \times \mathrm{S}_2) \cap \mathrm{A}_6)$, $(7, \mathrm{S}_7, \mathrm{S}_4 \times \mathrm{S}_3)$, $(7, \mathrm{A}_7, (\mathrm{S}_4 \times \mathrm{S}_3) \cap \mathrm{A}_7)$, $(7, \mathrm{A}_7, \mathrm{A}_4 \times \mathrm{A}_3)$, $(7, \mathrm{S}_7, \mathrm{A}_4 \times \mathrm{S}_3)$, $(7, \mathrm{S}_7, \mathrm{S}_4 \times \mathrm{A}_3)$ and $(7, \mathrm{A}_7, \mathrm{A}_4 \times \mathrm{A}_3)$. Each of the first three triples imply that $G$ is primitive on $V\Gamma$, so then, by [26], $c = 7$ and $\Gamma \cong \mathrm{O}_4$. Each of the last four triples imply that $\Gamma$ is $(\mathrm{A}_7, 3)$-transitive. Thus suppose that $G = \mathrm{A}_7$ and $H = \mathrm{A}_4 \times \mathrm{A}_3$. Then, for $\beta \in \Gamma(\alpha)$, calculation shows that $G_{\alpha\beta} = Z_3^2$, $\mathrm{N}_G(G_{\alpha\beta}) = Z_3^4 \rtimes Z_4$ and there is no $x \in \mathrm{N}_G(G_{\alpha\beta})$ with $x^2 \in G_{\alpha\beta}$ and $\langle x, H\rangle = G$, which contradicts the hypothesis.

*Case 2*. $H$ fixes exactly one point in $\mathbf{c}$ and $(c, G, H)$ is one of $(5, \mathrm{S}_5, \mathrm{S}_4)$, $(5, \mathrm{A}_5, \mathrm{A}_4)$, $(5, \mathrm{S}_5, \mathrm{A}_4)$, $(7, \mathrm{S}_7, Z_3^2 \rtimes \mathrm{D}_8)$, $(7, \mathrm{A}_7, Z_3^2 \rtimes Z_4)$, $(7, \mathrm{S}_7, \mathrm{S}_4 \times \mathrm{S}_2)$, $(7, \mathrm{S}_7, \mathrm{A}_4 \times \mathrm{S}_2)$, $(7, \mathrm{S}_7, \mathrm{S}_4)$, $(7, \mathrm{A}_7, \mathrm{S}_4)$, $(7, \mathrm{A}_7, \mathrm{A}_4)$. The first two triples yield $\mathrm{G} = \mathsf{K}_5$. The third triple yields $\Gamma \cong \mathsf{K}_{5,5} - 5\mathsf{K}_2$.

Thus assume that $c = 7$. The first two triples for $c = 7$ imply that $\Gamma$ is of valency nine, while the others yield that $\Gamma$ is of valency three or four and $H \ne \mathrm{A}_4 \times \mathrm{S}_2$. Assume that $H$ fixes the point 7 in $\mathbf{7}$.

Suppose that $\Gamma$ is of valency nine. Then, for $\beta \in \Gamma(\alpha)$, $H_\beta = G_{\alpha\beta} = \mathrm{D}_8$ or $Z_4$ and $\mathrm{N}_G(G_{\alpha\beta})$, contained in $\mathrm{S}_6$, is a Sylow two-subgroup of $\mathrm{S}_7$. Thus $\langle x, H\rangle \le \mathrm{S}_6$ and so $\langle x, H\rangle \ne G$ for each $x \in \mathrm{N}_G(G_{\alpha\beta})$, which contradicts the hypothesis.

Suppose that $\Gamma$ is of valency three. Then $|V\Gamma|$ is even. By inspecting the stabilizers of cubic arc-transitive graphs, the only possible case is that $G = \mathrm{S}_7$ and $H = \mathrm{S}_4$, which leads to a similar contradiction to that above by considering the normalizer of an arc stabilizer in $G$.

Suppose that $\Gamma$ is of valency four. Then there are three triples, say, $(7, \mathrm{S}_7, \mathrm{S}_4)$, $(7, \mathrm{A}_7, \mathrm{S}_4)$, $(7, \mathrm{A}_7, \mathrm{A}_4)$. Since $H$ fixes 7 and is transitive on $\mathbf{6}$, so $G_{\alpha\beta}$ fixes 7 and has two orbits on $\mathbf{6}$ with size three. Then each $x \in \mathrm{N}_G(G_{\alpha\beta})$ also fixes 7, yielding $\langle x, H\rangle \ne G$, which contradicts the hypothesis.

*Case 3*. $H$ fixes at least two points in $\mathbf{c}$ and $(c, G, H)$ is one of $(7, \mathrm{S}_7, \mathrm{S}_4)$, $(7, \mathrm{A}_7, \mathrm{A}_4)$, $(6, \mathrm{S}_6, \mathrm{S}_4)$, $(6, \mathrm{A}_6, \mathrm{A}_4)$. Let $\beta \in \Gamma(\alpha)$. Each of these four cases yields that $H \le \mathrm{S}_4$ and $\mathrm{N}_G(G_{\alpha\beta}) \le \mathrm{S}_4 \times \mathrm{S}_{c-4}$. Thus there is no $x \in \mathrm{N}_G(G_{\alpha\beta})$ with $\langle x, H\rangle = G$, which contradicts the hypothesis. □

Lemma 4.4. *If $H$ is intransitive on* $\mathbf{c}$ *and $H$ has only one insoluble composition factor, then $\Gamma \cong \mathsf{K}_c$, $\mathsf{K}_{c,c} - c\mathsf{K}_2$ or the graph in Example 2.7.*

Proof. Assume that $G \leq S_c$, $H$ is intransitive on $\mathbf{c}$ and $H$ has only one insoluble composition factor. Assume that $H^{\Delta_1}$ is insoluble and each $H^{\Delta_j}$ is soluble for $j \geq 2$. Then, by Lemmas 3.4 and 3.7, $c_2 := \sum_{j=2}^{t} \leq 8$.

*Case 1*. Assume that $d_1 > 9$, or $d_1 = 9$ and $c_2 \leq d_1 - 2$. In this case, since $A_{d_1}$ is not a simple group of Lie type, $H^{\Gamma(\alpha)} = G_\alpha^{\Gamma(\alpha)} \cong S_{d_1}$ or $A_{d_1}$, by checking possible $H^{\Delta_1}$ in Lemma 3.7. In particular, $\Gamma$ is of valency $d_1$. Further, by Lemma 2.11, $\Gamma$ is not $(G, s)$-transitive for $s \geq 4$. Let $\beta \in \Gamma(\alpha)$. Then $G_{\alpha\beta}^{[1]} = 1$ by Lemma 2.9. Recalling that $(G_\alpha^{[1]})^{\Gamma(\beta)} \trianglelefteq G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}^{\Gamma(\alpha)}$ and $G_\alpha = G_{\alpha\beta}^{[1]} \cdot (G_\alpha^{[1]})^{\Gamma(\beta)} \cdot G_\alpha^{\Gamma(\alpha)} = G_\alpha^{\Gamma(\alpha)}$, $G_{\alpha\beta} \cong S_{d_1-1}$ or $A_{d_1-1}$.

Suppose that some $d_j \neq 1$. Assume that $d_2 \geq \cdots \geq d_r > d_{r+1} = \cdots = d_t = 1$ for a suitable $r \geq 2$. Then $H = \Gamma$ fixes set-wise a subset $\Delta = \Delta_2 \cup \cdots \cup \Delta_r$ of $\mathbf{c}$. Noting that $|\Delta| \leq 8 < d_1 - 1$, $L := (H^{\Delta_2} \times \cdots \times H^{\Delta_r}) \cap H \leq G_\alpha^{[1]} \leq G_{\alpha\beta} \leq H$ and $L$ has no fixed point on $\Delta$, this implies that each $x \in N_G(G_{\alpha\beta})$ also fixes $\Delta$ set-wise, and hence $\langle x, H\rangle \neq G$, which contradicts the hypothesis.

Assume that $d_j = 1$ for $t \geq j \geq 2$. Then $H = G_{\Delta_1}$ and $G_{\alpha\beta}$ fixes a $\delta$ in $\Delta_1$. Let $\Delta_1 = \mathbf{d_1}$ and $\delta = d_1$. Then $N_G(G_{\alpha\beta}) \leq S_{d_1-1} \times \mathrm{Sym}(\{d_1, \delta_1 + 1, \ldots, c\})$. Thus $\langle x, H\rangle \neq G$ for $x \in N_G(G_{\alpha\beta})$ with $x^2 \in G_{\alpha\beta}$ unless $c - d_1 = 1$. It follows that $c = d_1 + 1$ and either $\Gamma \cong \mathsf{K}_{c,c}$ if $H = A_{c-1}$ and $G = S_c$ or $\Gamma = \mathsf{K}_c$ otherwise.

*Case 2*. Assume that $5 \leq d_1 \leq 8$, or $d_1 = 9$ and $c_2 = 8$. By Lemma 3.7, noting that $|G : H|$ is square-free, $d_1 \leq 8$ and three cases arise.

(1) $H$ is maximal in $G$ and $H$ is one of $S_{c-1} \cap G$ for $c = 6$ and 7, $(S_5 \times S_2) \cap G$ for $c = 7$, $(S_6 \times S_4) \cap G$ for $c = 10$, $(S_7 \times S_4) \cap G$ or $S_8 \times S_3$ for $c = 11$. Then $\Gamma = \mathsf{K}_c$ for $c = 6, 7$ follows from [26].

(2) $t = 2$ or 3, $d_2 > 1$ and $H$ is one of $(S_8 \times Z_3^2 \rtimes D_8) \cap G$ for $c = 14$, $(A_8 \times S_3) \cap G$ or $(S_8 \times A_3) \cap G$ for $c = 11$, $(S_6 \times S_4) \cap G$ for $c = 11$, and $A_5 \times S_2$ for $c = 7$. Then $G^{\Gamma(\alpha)} \cong A_{d_1} = \mathrm{PSL}(m, q)$ for suitable $m$ and $q$, and $\Gamma$ is of valency $d_1$ or $q^m - 1/(q-1)$. It is easily shown that $N_G(G_{\alpha\beta}) \leq \mathrm{Sym}(\mathbf{c}\backslash\Delta_2) \times \mathrm{Sym}(\Delta_2)$. Thus there is no $x \in N_G(G_{\alpha\beta})$ with $\langle x, H\rangle = G$, which contradicts the hypothesis.

(3) $t = 2$ or 3, $d_j = 1$ for $j > 1$, $c = c$ and either $(G, H) = (S_7, A_6)$ or $H$ is one of $\mathrm{PGL}(2, 5) \cap G$ for $t = 2$, and $S_5 \cap G$ for $t = 3$. The first case, that is, $(G, H) = (S_7, A_6)$, yields $\Gamma \cong \mathsf{K}_{7,7} - 7\mathsf{K}_2$.

Suppose that $t = 3$. Then either $N_G(G_{\alpha\beta}) \leq \mathrm{Sym}(\Delta_1) \times \mathrm{Sym}(\mathbf{7}\backslash\Delta_1)$ when $\Gamma$ is of valency six or, for some $\delta \in \Delta_1$, $N_G(G_{\alpha\beta}) \leq \mathrm{Sym}(\Delta_1\backslash\{\delta\}) \times \mathrm{Sym}((\mathbf{7}\backslash\Delta_1)) \cup \{\delta\})$ when $\Gamma$ is of valency five. It is easily shown that there is no $x \in N_G(G_{\alpha\beta})$ with $x^2 \in N_G(G_{\alpha\beta})$ and $\langle x, H\rangle = G$, which contradicts the hypothesis.

Assume that $t = 2$ and $H = \mathrm{PGL}(2, 5) \cap G$. Then $H \leq \mathrm{Sym}(\Delta_1)$. If $\Gamma$ is of valency five, then $G_{\alpha\beta} \cong S_4$ or $A_4$ is transitive on $\Delta_1$, and so $N_G(G_{\alpha\beta}) \leq \mathrm{Sym}(\Delta_1)$ yields a similar contradiction to that above. Thus $\Gamma$ is of valency six. It is easy to see that $\Gamma$ is $(A_7, 2)$-arc-transitive. Then, replacing $G$ by $\mathsf{soc}(G)$ if necessary, $G_{\alpha\beta} \cong Z_5 \rtimes Z_2$, and $G_{\alpha\beta}$ fixes a point $\delta \in \Delta_1$. Set $\Delta_1 = \mathbf{6}$, $\delta = 6$ and $G_{\alpha\beta} = \langle\sigma, \tau\rangle$, where $\sigma = (1\,2\,3\,4\,5)$ and $\tau = (1\,5)(2\,4)$. Then $N_G(G_{\alpha\beta}) = \langle\sigma, \pi\rangle \cong Z_5 \rtimes Z_4$, where $\pi = (1\,4\,5\,2)(6\,7)$. It is easy to show $\langle x, H\rangle = A_7$ and $x^2 \in G_{\alpha\beta}$ for $x \in N_G(G_{\alpha\beta})\backslash H$, and $x = h\pi$ for some $h \in G_{\alpha\beta}$. Then $\Gamma \cong \mathsf{Cos}(A_7; A_5, A_5\pi A_5)$, as in Example 2.7. □

LEMMA 4.5. *If $H$ is an intransitive subgroup of* $S_c$ *and $H$ has at least two insoluble composition factors, then* $\Gamma \cong O_k$, $k \in \{6, 9, 10, 12, 36\}$.

PROOF. Assume that $H$ is intransitive on **c** and $H$ has at least two insoluble composition factors. By Corollary 2.10, $H$ has exactly two insoluble composition factors. Consider the restrictions of $H$ on its orbits $\Delta_j$ on **c**. Then one or two of those restrictions are insoluble, and the others are soluble.

Suppose that $H$ has two isomorphic insoluble composition factors. Then $H^{\Gamma(\alpha)} = G_\alpha^{\Gamma(\alpha)}$ is an affine two-transitive group. By Lemmas 3.4 and 3.7, $t = 2$, $d_1 = 2a$, $d_2 = 1$, $H = (S_a \wr S_2) \cap G$ and $G = S_{2a+1}$ or $A_{2a+1}$, where $a \in \{6, 9, 10, 36\}$. But such an $H$ can not have an insoluble affine quotient, which contradicts the hypothesis.

Therefore, $H$ has two nonisomorphic insoluble composition factors. Then $H^{\Gamma(\alpha)} = G_\alpha^{\Gamma(\alpha)}$ is an almost simple two-transitive group. Further, by Lemma 3.7, assume that $H^{\Delta_1}$ and $H^{\Delta_2}$ is insoluble and any other $H^{\Delta_j}$ is soluble. Assume, further, that $d_1 = |\Delta_1| \geq d_2 = |\Delta_2|$. Noting that $H \leq S_{d_1} \times \cdots \times S_{d_t} \cap G$ and $|G : H|$ is square-free, $\mathrm{f}(c; d_1, \ldots, d_t)$ is square-free. Then $d_1 > d_2$ and $H^{\Delta_1} = A_{d_1}$ or $S_{d_1}$ by Lemma 3.4. So $G_\alpha^{\Gamma(\alpha)} \cong A_{d_1}$ or $S_{d_1}$.

Assume that $d_1 \leq 8$. Then either $A_{d_1} \times \cdots \times A_{d_r} \leq H \leq S_{d_1} \times \cdots \times S_{d_r}$ for some $2 \leq r \leq t$ such that $d_1, \ldots, d_r \geq 2$ and $d_j = 1$ for $j > r$ or the pair $(H^{\Delta_1}, H^{\Delta_2})$ appears in Table 2 for $c = d_1 + d_2$ and in Table 8 for $c = d_1 + d_2 + 1$. By calculation, these two cases yield $t = 2 = r$, $H = (S_6 \times S_5) \cap G$ for $c = 11$ and $A_8 \times A_6 \leq H \leq S_8 \times S_6$ for $c = 14$. If $c = 14$, then $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)}) \cong A_8$ and the other insoluble composition factor of $H$ should be $A_7$ or $\mathrm{PSL}(3, 2)$, which contradicts the hypothesis. Thus $c = 11$, and $H = (S_6 \times S_5) \cap G$ is maximal in $G$. Then $\Gamma \cong O_6$ follows from [26].

Assume that $d_1 \geq 9$. Then $\Gamma$ is of valency $d_1$, and $\Gamma$ is not $(G, s)$-transitive for $s \geq 4$ by Lemma 2.11, so $G_{\alpha\beta}^{[1]} = 1$ by Lemma 2.9. Then, by (2.1), we conclude that $H = G_\alpha = G_{\alpha\beta}^{[1]} \cdot (G_\alpha^{[1]})^{\Gamma(\beta)} \cdot G_\alpha^{\Gamma(\alpha)} \cong (A_{d_1} \times A_{d_1-1}) \rtimes Z_2^l$ for some $l \leq 2$. In particular, $d_2 = d_1 - 1$. By Lemma 3.4, $\mathrm{f}(d_1 + d_2; d_1, d_2) = (2d_1 - 1)!/(d_1!(d_1 - 1)!)$ is square-free. Then $d_1 \in \{9, 10, 12, 36\}$ by Corollary 3.2. It is easy to see that $|G : H| = c!/(d_1!(d_1 - 1)! \cdot 2^{l-i})$ for $i = 1$ or $2$. Since $|G : H|$ is square-free, calculation indicates that $1 \leq i \leq l$ and $c = 2d_1 - 1$. It implies that $H = (S_{d_1} \times S_{d_1-1}) \cap G$ is maximal in $G$. Then $\Gamma \cong O_{d_1}$ follows from [26]. □

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