Nordhaus-Gaddum-type theorem for total proper connection number of graphs*

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Abstract

A graph is said to be *total-colored* if all the edges and the vertices of the graph are colored. A path P in a total-colored graph G is called a *total-proper path* if (i) any two adjacent edges of P are assigned distinct colors; (ii) any two adjacent internal vertices of P are assigned distinct colors; (iii) any internal vertex of P is assigned a distinct color from its incident edges of P. The total-colored graph G is *total-proper connected* if any two distinct vertices of G are connected by a total-proper path. The *total-proper connection number* of a connected graph G, denoted by $tpc(G)$, is the minimum number of colors that are required to make G total-proper connected. In this paper, we first characterize the graphs G on n vertices with $tpc(G) = n-1$. Based on this, we obtain a Nordhaus-Gaddum-type result for total-proper connection number. We prove that if G and $\overline{G}$ are connected complementary graphs on n vertices, then $6 \leq tpc(G) + tpc(\overline{G}) \leq n + 2$. Examples are given to show that the lower bound is sharp for $n \geq 4$. The upper bound is reached for $n \geq 4$ if and only if G or $\overline{G}$ is the tree with maximum degree $n-2$.

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1 Introduction

All graphs considered in this paper are simple, finite, and undirected. We follow the terminology and notation of Bondy and Murty in [1] for those not defined here. For a connected graph G, let $V(G), E(G)$ and $\Delta(G)$ denote the vertex set, the edge set and the maximum degree of G, respectively. If G is a graph and $A \subseteq V(G)$, then $G[A]$ denotes the subgraph of G induced by the vertex set A, and $G - A$ denotes the graph $G[V(G) \setminus A]$. If $A = \{v\}$, then we write $G - v$ for short. An edge xy is called a *pendent edge* if one of its end vertices, say x, has degree one, and x is called a *pendent vertex*. For a vertex $v \in V(G)$, we use $N_G(v)$ to denote the neighborhood of v in G and use $d_G(v)$ to denote the degree of v in G, sometimes we simply write $N(v)$ and $d(v)$ if G is clear. For graphs X and G, we write $X \cong G$ if X is isomorphic to G. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers.

Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \to \{1, 2, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored with the same color. If adjacent edges of G receive different colors by c, then c is a *proper coloring*. The minimum number of colors needed in a proper coloring of G is referred as the *chromatic index* of G and denoted by $\chi'(G)$. Meanwhile, a path in G is called *a rainbow path* if no two edges of the path are colored with the same color. The graph G is called *rainbow connected* if for any two distinct vertices of G, there is a rainbow path connecting them. For a connected graph G, the *rainbow connection number* of G, denoted by $rc(G)$, is defined as the minimum number of colors that are needed to make G rainbow connected. These concepts were first introduced by Chartrand et al. in [3] and have been well-studied since then. For further details, we refer the reader to a book [9].

Motivated by rainbow connection coloring and proper coloring in graphs, Borozan et al. [2] introduced the concept of proper-path coloring. Let G be a nontrivial connected graph with an edge-coloring. A path in G is called a *proper path* if no two adjacent edges of the path are colored with the same color. The *proper connection number* of a connected graph G, denoted by $pc(G)$, is defined as the minimum number of colors that are needed in an edge-coloring of G such that any two distinct vertices of G are connected by a proper path. For more details, we refer to a dynamic survey [8].

Jiang et al. [7] introduced the analogous concept of total-proper connection of graphs. Let G be a nontrivial connected graph with a *total-coloring* $c : E(G) \cup V(G) \rightarrow \{1, 2, \ldots, t\}$, $t \in \mathbb{N}$. We use $c(u), c(uv)$ to denote the colors assigned to the vertex $u \in V(G)$ and the edge $uv \in E(G)$, respectively. A path P is called a *total-proper path* if (i) any two adjacent edges of P are assigned distinct colors; (ii) any two adjacent internal vertices of P are assigned distinct colors; (iii) any internal vertex of P is assigned a distinct color from its incident edges of P. A total-coloring c is a *total-proper coloring* of G if every pair of distinct vertices u, v of G is connected by a total-proper path in G. A graph with a total-proper coloring is said to be *total-proper connected*. If k colors are used, then c is referred as a *total-proper k-coloring*. The *total-proper connection number* of a connected graph G, denoted by $tpc(G)$, is the minimum number of colors that are required to make G total-proper connected. For the total-proper connection number of graphs, the following observations are immediate.

Proposition 1 *Let G be a nontrivial connected graph with n vertices. Then*

(i) $tpc(G) = 1$ *if and only if* $G = K_n$;

(ii) $tpc(G) \geq 3$ *if and only if G is noncomplete.*

A Nordhaus-Gaddum-type result is a (tight) lower or upper bound on the sum or product of the values of a parameter for a graph and its complement. The name "Nordhaus-Gaddum-type" is given because Nordhaus and Gaddum [10] first established the following type of inequalities for chromatic number of graphs in 1956. They proved that if G and $\overline{G}$ are complementary graphs on n vertices whose chromatic number are $\chi(G)$ and $\chi(\overline{G})$, respectively, then $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$. Since then, many analogous inequalities of other graph parameters have been considered, such as diameter [5], domination number [4], proper connection number [6], and so on. In this paper, we consider analogous inequalities concerning total-proper connection number of graphs. We prove that if both G and $\overline{G}$ are connected graphs on $n \geq 4$ vertices, then

$$6 \leq tpc(G) + tpc(\overline{G}) \leq n + 2.$$

The rest of this paper is organized as follows: In Section 2, we list some useful known results on total-proper connection number. In Section 3, we first characterize the graphs G on n vertices with $tpc(G) = n - 1$. Based on this result, we give the upper bound and show that this bound is reached for $n \geq 4$ if and only if G or $\overline{G}$ is the tree with maximum degree $n-2$. Then we give the lower bound and show that it is sharp for $n \geq 4$.

2 Preliminaries

In this section, we list some preliminary results and definitions on the total-proper coloring which can be found in [7].

Proposition 2 *[7] If G is a nontrivial connected graph and H is a connected spanning subgraph of G, then $tpc(G) \leq tpc(H)$. In particular, $tpc(G) \leq tpc(T)$ for every spanning tree T of G.*

Proposition 3 *[7] Let G be a connected graph of order $n \geq 3$ that contains a bridge. If b is the maximum number of bridges incident with a single vertex in G, then $tpc(G) \geq b+1$.*

In [7], the authors determined the total-proper connection numbers of trees and complete bipartite graphs.

Theorem 1 *[7] If T is a tree of order $n \geq 3$, then $tpc(T) = \Delta(T)+1$.*

A *Hamiltonian path* in a graph G is a path containing every vertex of G and a graph having a Hamiltonian path is a *traceable graph*.

Corollary 1 *[7] If G is a traceable graph that is not complete, then $tpc(G) = 3$.*

Theorem 2 *[7] Let $G = K_{s,t}$ denote a complete bipartite graph with $s \geq t \geq 2$. Then $tpc(G) = 3$.*

Given a total-coloring c of a path $P = v_1v_2\ldots v_{s-1}v_s$ between any two vertices v_1 and v_s, we denote by $start_e(P)$ the color of the first edge in the path, i.e., $c(v_1v_2)$, and by $end_e(P)$ the last color, i.e., $c(v_{s-1}v_s)$. Moreover, let $start_v(P)$ be the color of the first internal vertex in the path, i.e., $c(v_2)$, and $end_v(P)$ the last color, i.e., $c(v_{s-1})$. If P is just the edge v_1v_s, then $start_e(P) = end_e(P) = c(v_1v_s)$, $start_v(P) = c(v_s)$ and $end_v(P) = c(v_1)$.

Definition 1 *Let c be a total-coloring of a graph G that makes G total-proper connected. We say that G has the* strong property *if for any pair of vertices $u, v \in V(G)$, there exist two total-proper paths P_1, P_2 between them (not necessarily disjoint) such that (1) $c(u) \neq start_v(P_i)$ and $c(v) \neq end_v(P_i)$ for $i = 1, 2$, and (2) both $\{c(u), start_e(P_1), start_e(P_2)\}$ and $\{c(v), end_e(P_1), end_e(P_2)\}$ are 3-sets.*

The authors in [7] studied the total-proper connection number of 2-connected graphs and gave an upper bound.

Theorem 3 *[7] Let G be a 2-connected graph. Then $tpc(G) \leq 4$ and there exists a total-coloring of G with 4 colors such that G has the strong property.*

From Definition 1 and Theorem 3, we get the following.

Corollary 2 *Let G and H be connected graphs such that $G = H - v$. If there is a total-proper k-coloring c of G such that G has the strong property, then $tpc(H) \leq k$.*

In particular, we study the total-proper connection number of H when G is a complete bipartite graph, and get the exact value of $tpc(H)$.

Lemma 1 *Let H be a connected graph such that $H - v = K_{s,t}$, where $s \geq t \geq 2$. Then $tpc(H) = 3$. Moreover, $tpc(H') = 3$, where H' is the graph shown in Fig. 1.*

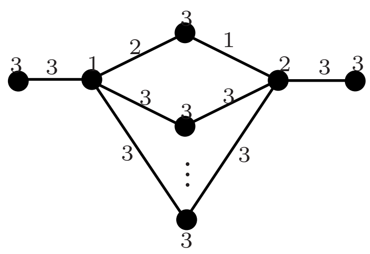

Figure 1: The graph H'

Proof. Let U and W be the two partite sets of $K_{s,t}$, where $U = \{u_1, \ldots, u_s\}$ and $W = \{w_1, \ldots, w_t\}$. Since H and H' are both noncomplete, we only need to prove $tpc(H) \leq 3$ and $tpc(H') \leq 3$, i.e., demonstrating a total-proper 3-coloring of H or H'. We divide our discussion according to the value of t.

Case 1. $t = 2$

If v is adjacent to W, say $vw_1 \in E(H)$, then set $c(w_1) = c(u_1w_2) = 1$, and $c(w_2) = c(u_1w_1) = 2$. Assign all the remaining vertices and edges with color 3. Thus, there is a total-proper path $u_iw_1u_1w_2u_j$ connecting u_i and u_j, where $2 \leq i, j \leq s$. As for the rest of vertex pairs, we can always find a path contained in the path $vw_1u_1w_2u_i$ for some $2 \leq i \leq s$. If there is another vertex v' adjacent to w_2, based on the above coloring, set $c(v') = c(v'w_2) = 3$, then we obtain a total-proper 3-coloring of H', see Fig.1.

If v is adjacent to U, say $vu_1 \in E(H)$, then set $c(w_1) = c(u_2) = c(u_1w_2) = 1$, and $c(w_2) = c(u_1w_1) = c(u_2w_1) = c(vu_1) = 2$. Assign all the remaining vertices and edges with color 3. Thus, there is a total-proper path, contained in the path $vu_1w_2u_2w_1$ or $vu_1w_2u_i$ for some $3 \leq i \leq s$, connecting v or u_1 and any other vertex in H. For the remaining vertex pairs in $U \cup W$, there is a total-proper path contained in the path $u_iw_2u_1w_1u_j$ for some $2 \leq i < j \leq s$.

Case 2. $t \geq 3$

If $s = t = 3$, then H is traceable so that $tpc(H) = 3$. If $s \geq 4$, we consider two subcases.

1) Assume there is a 6-cycle C_6 in $K_{s,t}$ such that $H - C_6$ is still connected. Without loss of generality, we suppose $C_6 = u_1w_1u_2w_2u_3w_3$. We color C_6 with the colors $1, 2, 3$ by the sequence of vertices and edges on the cycle. That is, set $c(u_1) = c(w_2) = c(w_1u_2) = c(u_3w_3) = 1$, $c(u_2) = c(w_3) = c(u_1w_1) = c(w_2u_3) = 2$, and $c(w_1) = c(u_3) = c(u_2w_2) = c(w_3u_1) = 3$. Let $i, j \geq 4$ be two integers. Assign u_i and u_3w_j (if any) with color 1, and assign w_j and w_1u_i with color 2. The remaining vertices and edges are all colored 3. Then we claim that this total-coloring makes H total-proper connected. Any pair $(u_i, w_j) \in U \times W$ is connected by the edge u_iw_j. The total-proper path for the pairs from $U \times U$ is contained in the path $P = u_iw_1u_2w_2u_3w_3u_j$ for some $1 \leq i, j \leq s$. The total-proper path for the pairs from $W \times W$ is contained in the path $P = w_iu_1w_1u_2w_2u_3w_j$ for some $1 \leq i, j \leq t$. Now consider the pairs of $\{v\} \times (U \cup W)$. By the assumption, we know that $vu_\ell \in E(H)$ or $vw_\ell \in E(H)$ for $\ell \geq 4$. Without loss of generality, suppose $\ell = 4$. If $vu_4 \in E(H)$, then for pairs (v, u_i) $(1 \leq i \leq s)$ there is a total-proper path contained in the path $P = vu_4w_1u_2w_2u_3w_3u_j$ for some $1 \leq j \leq s$, and for pairs (v, w_i) $(1 \leq i \leq t)$ there is a total-proper path contained in the path $P = vu_4w_1u_2w_2u_3w_j$ for some $1 \leq j \leq t$.

The case when $vw_4 \in E(H)$ is similar.

2) Assume there is no such a 6-cycle in subcase 1). As $s \geq 4$ we can deduce that $t = 3$ and v is only adjacent to W, say $vw_2 \in E(H)$. We color H as above. Then it is sufficient to check the pairs in $\{v\} \times (U \cup W)$. For pairs in $\{v\} \times U$, there is a total-proper path $P = vw_2u_3w_3u_i$ for some $1 \leq i \leq s$, and for pairs in $\{v\} \times W$, we can find a total-proper path contained in the path $P = vw_2u_3w_3u_1w_1$.

The proof is complete. $\square$

3 Bounds on $tpc(G) + tpc(\overline{G})$

To begin this section, we give total-proper connection numbers of four unicyclic graphs, which are useful to characterize the graphs on n vertices that have total-proper connection number $n - 1$.

Lemma 2 *Let H_1, H_2, H_3 and H_4 be the graphs on $n \geq 5$ vertices shown in the Fig. 2, respectively. Then $tpc(H_1) = n - 2$; $tpc(H_2) = n - 2$ if $n = 5$, $tpc(H_2) = n - 3$ if $n \geq 6$; and for $i = 3, 4$, $tpc(H_i) = n - 2$ if $n = 5$ or 6, $tpc(H_i) = n - 3$ if $n \geq 7$.*

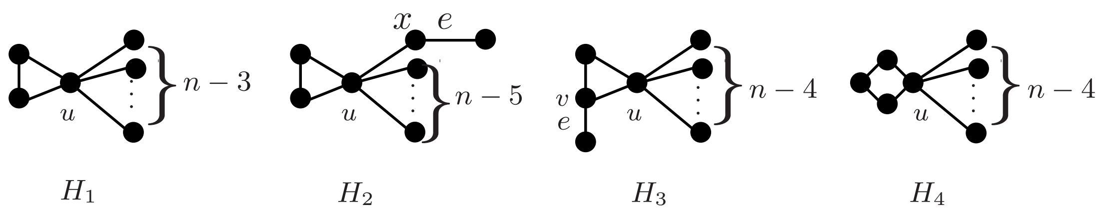

Figure 2: The graphs H_1, H_2, H_3 and H_4.

Proof. By Proposition 3, we get $tpc(H_1) \geq n-2$ and $tpc(H_i) \geq n-3$ for $i \in \{2, 3, 4\}$.

For $i = 1, 2, 3$, let uvw be the triangle in H_i and let e_1, e_2,..., and e_{n-3} denote the bridges in H_i. Assume that $e = e_{n-3}$ in the graphs H_2 and H_3, and the edge e is incident with the vertex x and adjacent to the bridge e_1 in H_2, and e is incident with the vertex v in H_3. We first consider the graph H_1 and demonstrate a total-coloring of it with $n - 2$ colors. Let $c(u) = c(vw) = 1$, $c(e_j) = j + 1$ for $1 \leq j \leq n - 3$, $c(uv) = c(w) = 2$ and $c(v) = c(wu) = 3$. The remaining vertices are all colored 1. It is easy to check this total-coloring makes H_1 total-proper connected. Hence, we have $tpc(H_1) = n - 2$ when $n \geq 5$.

We should point out that for $i = 2,3,4$, the graph H_i is traceable when $n = 5$, hence $tpc(H_i) = 3$ by Corollary 1. So we assume $n \geq 6$. Consider the graph H_2. Color H_2 as H_1 only with the exception that $c(e_{n-3}) = 1$ and $c(x) = 3$. It is easy to check that under this total-coloring, H_2 is total-proper connected. Hence, we have $tpc(H_2) = n-2$ when $n = 5$ and $tpc(H_2) = n-3$ when $n \geq 6$.

Consider the graph H_3. When $n = 6$, we claim that $tpc(H_3) = 4$. From Proposition 2, we get that $tpc(H_3) \leq 4$. If we use 3 colors to total-color H_3, no matter how we color it, there always exist two pendent vertices not being connected by a total-proper path. When $n \geq 7$, it can be easily checked that the total-coloring of H_2, only with the exception that $c(e) = 4$, makes H_3 total-proper connected. Hence, we have $tpc(H_3) = n-2$ when $n = 5,6$ and $tpc(H_3) = n-3$ when $n \geq 7$.

Now we consider the graph H_4. We use e_1, e_2,..., and e_{n-4} to denote the bridges incident with u, respectively, and use $uvwx$ to denote the quadrangle in H_4. First, we consider the case $n \geq 7$. We demonstrate a total-coloring of H_4 with $n-3$ colors. Let $c(e_j) = j$ for $1 \leq j \leq n-4$, $c(u) = n-3$, $c(v) = c(x) = 2$, $c(vw) = c(xu) = 3$ and $c(w) = 4$. The remaining edges and vertices are all colored 1. It is easy to check that under this total-coloring, H_4 is total-proper connected. When $n = 6$, we claim that $tpc(H_4) = 4$. From Proposition 2, we get that $tpc(H_4) \leq 4$. If we use 3 colors to total-color H_4, no matter how we color it, there always exists a vertex pair not being connected by a total-proper path. Hence, we have $tpc(H_4) = n-2$ when $n = 5,6$ and $tpc(H_4) = n-3$ when $n \geq 7$. $\square$

We use C_n and S_n to denote the cycle and the star on n vertices, respectively, and use $T(a,b)$ to denote the double star that is obtained by adding an edge between the center vertices of S_a and S_b. Given a cycle $C_r = v_1v_2\ldots v_r$, let $C_r(T_1,T_2,\ldots,T_r)$ be the graph obtained from C_r and rooted trees T_i by identifying the root, say r_i, of T_i with v_i on C_r, $i = 1,2,\ldots,r$. We assume that $|T_i| = n_i, n_i \geq 1, i = 1,2,\ldots,r$. Then $|C_r(T_1,T_2,\ldots,T_r)| = \sum_{i=1}^{r}|T_i|$. In particular, if $|T_i| = 1$ for each $i \in \{1,2,\ldots,r\}$, the graph $C_r(T_1,T_2,\ldots,T_r)$ is just the cycle C_r. For a nontrivial graph G such that $G + uv \cong G + xy$ for every two pairs (u,v), (x,y) of nonadjacent vertices of G, we use $G + e$ to denote the graph obtained from G by joining two nonadjacent vertices of G.

Theorem 4 *Let G be a connected graph of order $n \geq 4$. Then $tpc(G) = n-1$ if and only if $G \in \{T(2,n-2), C_4, C_4+e, S_4+e\}$.*

*Proof.*By Theorem 1 and Corollary 1, we can easily check that $tpc(G) = n-1$ if G is one of the above four graphs. So we concentrate on the verification of the converse of the theorem. Suppose that $tpc(G) = n-1$. Then G cannot be complete, so $tpc(G) \geq 3$. If G is a tree, then by Theorem 1, we have $\Delta(G) = n-2$, thus $G \cong T(2,n-2)$. Now, we consider the case that G contains cycles. Pick a longest cycle $C_k = v_1v_2...v_k$ of G, where $k \geq 3$. If $k = n$, then $3 = tpc(C_k) = tpc(G) = n-1$. So $n = 4$. Thus $G \cong C_4$ or C_4+e. If $k < n$, consider a unicyclic spanning subgraph H of G containing the cycle C_k. Then H can be written as $C_k(T_1, T_2, ..., T_k)$. Set $r = max\{\Delta(T_i) : 1 \leq i \leq k\}$ and let T_ℓ be a tree with $\Delta(T_\ell) = r$. Notice that $\Delta(T_\ell) \leq |T_\ell| - 1 \leq n-k$, so $r \leq n-k$. Then delete an edge e of H, which is incident with v_ℓ in C_k, and denote the obtained graph as H', so H' is a spanning tree of G and $\Delta(H') \leq n-k+1$, and the equality holds if and only if there is only one non-trivial subtree $T_\ell = S_{n-k+1}$ in H whose center is v_ℓ or there are exactly two pendent edges attached to C_k. Thus $n-1 = tpc(G) \leq tpc(H') = \Delta(H')+1 \leq n-k+2$, therefore we have $k \leq 3$. So $k = 3$ and all the equalities must hold. Hence, there is only one non-trivial subtree in H and $\Delta(H) = n-1$ or H is traceable on 5 vertices, the latter contradicting the condition $tpc(G) = n-1$. So we can identify H as S_n+e, and when $n \geq 5$, the graph H is just the graph H_1 in Fig. 2. By Lemma 3 and Proposition 2, we have $tpc(G) \leq tpc(H_1) = n-2$, a contradiction. So $n = 4$ and $G \cong S_4+e$ since C_3 is a longest cycle of G. □

We know that if G and $\overline{G}$ are connected complementary graphs on n vertices, then n is at least 4, and $\Delta(G) \leq n-2$. Therefore, we get that $tpc(G) \leq n-1$. Similarly, we have $tpc(\overline{G}) \leq n-1$. Hence, we obtain that $tpc(G)+tpc(\overline{G}) \leq 2(n-1)$. For $n = 4$, it is obvious that $tpc(G)+tpc(\overline{G}) = 6$ if both G and $\overline{G}$ are connected. In the rest of this section, we always assume that all graphs have at least 5 vertices, and both G and $\overline{G}$ are connected.

Lemma 3 *Let G be a graph on 5 vertices. If both G and $\overline{G}$ are connected, then we have*

$$tpc(G)+tpc(\overline{G}) = \begin{cases} 7 & \text{if } G \cong T(2,3) \text{ or } \overline{G} \cong T(2,3); \\ 6 & \text{otherwise.} \end{cases}$$

Proof. If $G \cong T(2,3)$ or $\overline{G} \cong T(2,3)$, then from Theorem 4, we can easily get that $tpc(G)+tpc(\overline{G}) = 7$. Otherwise, we have $tpc(G) \leq n-2 = 3$ and $tpc(\overline{G}) \leq n-2 = 3$. Combining with Proposition 1, we get $tpc(G) + tpc(\overline{G}) = 3 + 3 = 6$ if $G \ncong T(2,3)$ and $\overline{G} \ncong T(2,3)$. □

Now we are ready to give the upper bound on $tpc(G) + tpc(\overline{G})$.

Theorem 5 *Let G be a graph of order $n \geq 5$. If both G and $\overline{G}$ are connected, then we have $tpc(G)+tpc(\overline{G}) \leq n+2$, and the equality holds if and only if $G \cong T(2, n-2)$ or $\overline{G} \cong T(2, n-2)$.*

Proof. It follows from Lemma 3 that the result holds for $n = 5$. So we assume that $n \geq 6$. If $G \cong T(2, n-2)$, then $\overline{G}$ contains a spanning subgraph H that is obtained by attaching a pendent edge to the complete bipartite graph $K_{2,n-3}$. So we have $tpc(G) = 3$ by Lemma 1. Combining with Theorem 4, the result is clear. Similarly, we get that $tpc(G) + tpc(\overline{G}) = n + 2$ if $\overline{G} \cong T(2, n-2)$. In the following, we prove that $tpc(G) + tpc(\overline{G}) < n+2$ when $G \ncong T(2, n-2)$ and $\overline{G} \ncong T(2, n-2)$. Under this assumption, we have $3 \leq tpc(G) \leq n - 2$ and $3 \leq tpc(\overline{G}) \leq n - 2$ by Proposition 1 and Theorem 4.

We first consider the case that both G and $\overline{G}$ are 2-connected. When $n = 6$, we claim that $tpc(G) = 3$. Suppose that the circumference of G is k. If $k = 6$, then $tpc(G) \leq tpc(C_6) = 3$. If $k = 4$, then G contains a spanning $K_{2,4}$, contradicting the fact that $\overline{G}$ is connected. Next, we assume that G contains a 5-cycle $C = v_1v_2v_3v_4v_5$. Then G is traceable, so $tpc(G) = 3$ by Corollary 1. Thus, we have $tpc(G) + tpc(\overline{G}) \leq 3 + n - 2 < n + 2$. For $n \geq 7$, we have $tpc(G) \leq 4$ and $tpc(\overline{G}) \leq 4$ by Theorem 3. Hence, we get $tpc(G) + tpc(\overline{G}) \leq 4 + 4 < n + 2$.

Now, we consider the case that at least one of G and $\overline{G}$ has cut vertices. Without loss of generality, we suppose that G has cut vertices. Let u be a cut vertex of G, let $G_1, G_2, \ldots, G_k$ be the components of $G - u$, and let n_i be the number of vertices in G_i for $1 \leq i \leq k$ with $n_1 \leq \cdots \leq n_k$. We consider the following two cases.

Case 1. There exists a cut vertex u of G such that $n-1-n_k \geq 2$. Since $\Delta(G) \leq n-2$, we have $n_k \geq 2$. We know that $\overline{G} - u$ contains a spanning complete bipartite graph K_{n-1-n_k,n_k}. Hence, it follows from Lemma 1 that $tpc(\overline{G}) = 3$. Combining with the fact that $tpc(G) \leq n - 2$, we get that $tpc(G) + tpc(\overline{G}) < n + 2$.

Case 2. Every cut vertex u of G satisfies that $n-1-n_k=1$.

First, we suppose that G has at least two cut vertices, say u_1 and u_2. Let u_1v_1 and u_2v_2 be two pendent edges of G. Obviously, the edges u_1v_1 and u_2v_2 are disjoint. So $u_1v_2, u_2v_1 \in E(\overline{G})$, and $\overline{G}-\{u_1,u_2\}$ contains a spanning complete bipartite graph $K_{2,n-4}$ with partition classes $U=\{v_1,v_2\}$ and $W=V(G)\backslash\{u_1,v_1,u_2,v_2\}$. By Lemma 1, we have that $tpc(\overline{G})=3$. Together with the fact that $tpc(G)\leq n-2$, we get that $tpc(G)+tpc(\overline{G})<n+2$.

Now, we consider the subcase that G has only one cut vertex u and let uv be the pendent edge of G. Then $G-v$ is 2-connected. By Theorem 3 and Corollary 2, we have $tpc(G)\leq 4$, thus $tpc(G)+tpc(\overline{G})\leq n+2$. Now, we prove that the equality cannot hold. Note that $d_{\overline{G}}(v)=n-2$. Let $N_{\overline{G}}(v)=\{w_1,w_2,\ldots,w_{n-2}\}$. Since $\Delta(G)\leq n-2$, there exists a vertex w_i $(1\leq i\leq n-2)$ not adjacent to u in G, say $uw_1\notin E(G)$. Then $uw_1\in E(\overline{G})$. If there is a vertex w_j $(2\leq j\leq n-2)$ adjacent to w_1 in $\overline{G}$, then $\overline{G}$ contains an H_3 in Fig. 2 as its spanning subgraph, so $tpc(\overline{G})\leq n-3$. If there is a vertex w_j $(2\leq j\leq n-2)$ adjacent to u in $\overline{G}$, then $\overline{G}$ contains an H_4 in Fig. 2 as its spanning subgraph, so $tpc(\overline{G})\leq max\{4,n-3\}$. If there are two vertices $w_j,w_k(2\leq j\neq k\leq n-2)$ are adjacent in $\overline{G}$, then $\overline{G}$ contains an H_2 in Fig. 2 as its spanning subgraph, so $tpc(\overline{G})\leq n-3$. We conclude that $tpc(\overline{G})\leq max\{4,n-3\}$ if $G-v$ is 2-connected. For $n\geq 7$, we get the result $tpc(G)+tpc(\overline{G})\leq n+1<n+2$. For $n=6$, since $G-v$ is a 2-connected graph on 5 vertices, $G-v$ contains a spanning 5-cycle or a spanning $K_{2,3}$, implying that $tpc(G)=3$ by Corollary 1 and Lemma 1. Thus, we have $tpc(G)+tpc(\overline{G})\leq 3+4=7<8$. □

For the lower bound on $tpc(G)+tpc(\overline{G})$, we note that $tpc(G)=1$ if and only if G is a complete graph, in which case the graph $\overline{G}$ is not connected. So, if G and $\overline{G}$ are both connected, then $tpc(G)\geq 3$. Similarly, we have $tpc(\overline{G})\geq 3$. Hence, we obtain that $tpc(G)+tpc(\overline{G})\geq 6$.

Theorem 6 *Let G be a graph of order $n\geq 5$. If both G and $\overline{G}$ are connected, then we have $tpc(G)+tpc(\overline{G})\geq 6$, and the lower bound is sharp.*

Proof. We only need to prove that there are graphs G and $\overline{G}$ on $n\geq 5$ vertices such that $tpc(G)=tpc(\overline{G})=3$.

Let G be the graph with vertex set $\{v\} \cup U \cup W$, where $U = \{u_1, \dots, u_{\lfloor \frac{n-1}{2} \rfloor}\}$ and $W = \{w_1, \dots, w_{\lceil \frac{n-1}{2} \rceil}\}$, such that $N(v) = U$ and U is an independent set and $G[W]$ is a clique, and for each vertex u_i, u_i is adjacent to $w_i, w_{i+1}, \dots, w_{i+\lfloor \frac{n-3}{4} \rfloor}$ where the subscripts are taken modulo $\lceil \frac{n-1}{2} \rceil$. Obviously, the graphs G and $\overline{G}$ are both traceable. It follows from Corollary 1 that $tpc(G) = tpc(\overline{G}) = 3$. □

Remark: Clearly, both Theorems 5 and 6 are valid for $n = 4$. So if both G and $\overline{G}$ are connected graphs on $n \geq 4$ vertices, then $6 \leq tpc(G) + tpc(\overline{G}) \leq n + 2$; moreover, both bounds are sharp.

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References

[1] J.A. Bondy, U.S.R. Murty, *Graph Theory with Applications*, The Macmillan Press, London and Basingstoker, 1976.

[2] V. Borozan, S. Fujita, A. Gerek, C. Magnant, Y. Manoussakis, L. Montero, Zs. Tuza, *Proper connection of graphs*, Discrete Math. 312(2012), 2550-2560.

[3] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, *Rainbow connection in graphs*, Math. Bohem. 133(1)(2008), 85-98.

[4] F. Harary, T.W. Haynes, *Nordhaus-Gaddum inequalities for domination in graphs*, Discrete Math. 155(1996), 99-105.

[5] F. Harary, R.W. Robinson, *The diameter of a graph and its complement*, Amer. Math. Monthly 92(1985), 211-212.

[6] F. Huang, X. Li, S. Wang, *Proper connection numbers of complementary graphs*, Bull. Malays. Math. Sci. Soc., DOI 10.1007/s40840-016-0381-8, in press.

[7] H. Jiang, X. Li, Y. Zhang, *Total proper connection of graphs*, arXiv preprint arXiv:1512.00726 [math.CO].

[8] X. Li, C. Magnant, *Properly colored notions of connectivity - a dynamic survey*, Theory and Appl. Graphs 0(1)(2015), Art. 2.

[9] X. Li, Y. Sun, *Rainbow Connections of Graphs*, New York, SpringerBriefs in Math., Springer, 2012.

[10] E.A. Nordhaus, J.W. Gaddum *On complementary graphs*, Amer. Math. Monthly 63(1956), 175-177.