

# On alternatively connected edge-transitive graphs of square-free order

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Received: date / Accepted: date

**Abstract** An edge-transitive graph  $\Gamma$  is called alternatively connected if a subgroup  $G$  of the automorphism group of  $\Gamma$  has two orbits on the arc set of  $\Gamma$ , and there exists an alternative walk (with respect to a given  $G$ -orbit on arcs) between every pair of vertices of  $\Gamma$ . Employing the standard double covers of digraphs, we give some basic properties of alternatively connected edge-transitive graphs. The main result of this paper is a reduction result on alternatively connected edge-transitive graphs of square-free order. As an application of this result, we give a characterization for alternatively connected edge-transitive graphs of square-free order and valency 6. It is proved that such a graph is either a circulant or constructed from  $\text{PSL}(2, p)$ .

**Keywords** edge-transitive graph · half-transitive graph · locally-primitive graph · alternatively connected graph · (almost) simple group.

## 1 Introduction

In this paper we consider only finite and simple graphs.

Let  $\Gamma$  be a graph. We use  $V\Gamma$ ,  $E\Gamma$  and  $\text{Aut}\Gamma$  to denote its vertex set, edge set and automorphism group, respectively. Recall that an *arc* in a graph is an ordered pair of adjacent vertices. We denote by  $A\Gamma$  the arc set of  $\Gamma$ . For a vertex  $u \in V\Gamma$ , set  $\Gamma(u) = \{v \mid (u, v) \in A\Gamma\}$ . Then  $\Gamma(u)$  is called the *neighborhood* of  $u$  in  $\Gamma$ , and the size  $|\Gamma(u)|$  is called the *valency* of  $u$ . The

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Supported by National Natural Science Foundation of China (11371204, 11731002).

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graph  $\Gamma$  is said to be *regular* of valency  $k$  if all vertices have the same valency  $k$ .

Let  $\Gamma$  be a graph, and let  $G$  be a subgroup of  $\text{Aut}\Gamma$ . Then  $G$  acts on both  $E\Gamma$  and  $A\Gamma$  naturally by

$$\{u, v\}^g = \{u^g, v^g\} \quad \text{and} \quad (u, v)^g = (u^g, v^g), \quad g \in G, \{u, v\} \in E\Gamma,$$

respectively. The graph  $\Gamma$  is said to be  *$G$ -vertex-transitive*,  *$G$ -edge-transitive* or  *$G$ -arc-transitive* if  $G$  acting transitively on  $V\Gamma$ ,  $E\Gamma$  or  $A\Gamma$ , respectively. (Note that, for graphs without isolated vertices, the arc-transitivity yields the vertex-transitivity.) If  $\Gamma$  is  $G$ -edge-transitive but not  $G$ -arc-transitive, then  $G$  has two orbits on  $A\Gamma$ ; in this case,  $\Gamma$  is said to be  *$G$ -semisymmetric* when  $\Gamma$  is regular and  $G$  is intransitive on  $V\Gamma$ , and  $\Gamma$  is said to be  *$G$ -half-transitive* when  $G$  is transitive on  $V\Gamma$ . For  $u \in V\Gamma$ , set

$$G_u = \{g \in G \mid u^g = u\},$$

called the *vertex-stabilizer* of  $u$  in  $G$ . Then  $G_u$  fixes  $\Gamma(u)$  setwise. The graph  $\Gamma$  is said to be  *$G$ -locally-primitive* if  $G_u$  acts primitively on  $\Gamma(u)$  for every  $u \in V\Gamma$ .

This paper is devoted to characterizing edge-transitive graphs of square-free order. In the literature, vertex- or edge-transitive graphs of square-free order has been studied extensively, and many interesting results have appeared. See for example [1, 3, 4, 18, 19, 21] for those graphs of order a prime or a product of two primes. Recently, several classification results were given about edge-transitive graphs of square-free order. For arc-transitive graphs of square-free order, Feng and Li [7] gave a classification of one-regular graphs and prime valency, and Li et al. [11–13, 15] gave a classification of locally-primitive graphs of valency no more than 7. For half-transitive graphs of square-free order, one may deduce a classification of tetravalent graphs from [11, 13]. For semisymmetric graphs of square-free order, Liu and Lu [16] gave a explicit list of such graphs of valency 3. Some of the mentioned results were in fact motivated by the observation in [14].

Let  $\Gamma$  be a connected  $G$ -edge-transitive regular graph of square-free order. Then  $\Gamma$  is  $G$ -arc-transitive,  $G$ -semisymmetric or  $G$ -half-transitive. In [14], the first two cases were considered under the ‘locally-primitive’ assumption, see [14, Theorems 4 and 30]. In the present paper, we shall deal with the half-transitive case under some restrictions. We first introduce several concepts.

Let  $\Gamma$  be a  $G$ -half-transitive graph. Then  $G$  has two orbits on  $A\Gamma$ , say  $\Delta$  and  $\Delta^* := \{(u, v) \mid (v, u) \in \Delta\}$ . For  $u, v \in V\Gamma$ , a  $\Delta$ -*alternative walk* between  $u$  and  $v$  means a sequence  $u = v_0 v_1 \dots v_{2l} = v$  of odd number of vertices such that  $(v_{2i}, v_{2i+1}), (v_{2i+2}, v_{2i+1}) \in \Delta$  for  $0 \leq i \leq l-1$ . The graph  $\Gamma$  is called ( $\Delta$ )-*alternatively connected* if there exists a  $\Delta$ -alternative walk between each pair of distinct vertices. For  $u \in V\Gamma$ , set

$$\Delta(u) = \{v \mid (u, v) \in \Delta\} \quad \text{and} \quad \Delta^*(u) = \{v \mid (u, v) \in \Delta^*\}.$$

Then  $\Gamma(u) = \Delta(u) \cup \Delta^*(u)$ , and  $G_u$  fixes both  $\Delta(u)$  and  $\Delta^*(u)$  setwise. The graph  $\Gamma$  is called *G-locally-biprimitive* if for every  $u \in V\Gamma$ , the stabilizer  $G_u$  acts primitively on both  $\Delta(u)$  and  $\Delta^*(u)$ .

We now outline the main results of this paper. The following reduction result is proved in Section 4.

**Theorem 1** *Let  $\Gamma$  be a G-half-transitive graph of square-free order and valency  $2d$ , where  $d > 2$ . Let  $M$  be a maximal intransitive normal subgroup of  $G$ . Assume that  $\Gamma$  is alternatively connected and G-locally-biprimitive. Then  $M$  is semiregular on  $V\Gamma$  and one of the following holds.*

- (1)  *$G$  has a regular normal cyclic subgroup  $\langle a \rangle$ ,  $|G| = d|V\Gamma|$ ,  $d$  is a prime and  $q - 1$  is divisible by  $d$  for each prime divisor  $q$  of  $|V\Gamma|$ . Setting  $G_u = \langle b \rangle$  for  $u \in V\Gamma$ , we have  $b^{-1}ab = a^r$  for some  $r$  with  $r^d \equiv 1 \pmod{|V\Gamma|}$  and  $(r - 1, |V\Gamma|) = 1$ .*
- (2)  *$G = M:X$  for some subgroup  $X$  of  $G$ , where  $X$  is almost simple with socle  $T$  isomorphic to one of the following simple groups:*
  - $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1; A_n$  with  $n < 3d$ ;
  - $\text{PSL}(2, p)$  for prime  $p \geq 5$ ;  $\text{PSL}(2, p^2)$  with  $p > 3$  and  $d$  divisible by  $p + 1$ ;  $\text{PSL}(2, p^f)$  with  $f \geq 3$ ,  $p^f > 9$  and  $d$  divisible by  $p^{f-1}$ ;
  - simple classical groups of dimension  $n$  over  $\text{GF}(p^f)$  with  $p \leq d$ ,  $n \geq 3$  and  $\lfloor \frac{n}{2} \rfloor f < d$ ;
  - simple exceptional groups  $G_2(p^f), {}^3D_4(p^f), F_4(p^f), {}^2E_6(p^f)$  and  $E_7(p^f)$  with  $2f < d$  and  $p \leq d$ .

Moreover,  $MT = M \times T$  and  $\Gamma$  is  $T$ -half-transitive.

As an application of Theorem 1, we give in Section 5 a characterization of alternatively connected edge-transitive graphs of square-free order and valency 6.

**Theorem 2** *Let  $\Gamma$  be a G-half-transitive graph of square-free order and valency 6, let  $u \in V\Gamma$ . Assume that  $G$  is insoluble and  $\Gamma$  is alternatively connected. Then one of the following holds.*

- (1)  *$G \cong \text{PSL}(2, p)$  and  $G_u \cong D_{12}$ , where  $p$  is a prime with  $p^2 - 1$  not divisible by 32.*
- (2) *Either  $G \cong \text{PSL}(2, p)$  and  $G_u \cong S_3$ , or  $G \cong \text{PGL}(2, p)$  and  $G_u \cong D_{12}$ , where  $p$  is a prime with  $p \equiv \pm 3 \pmod{8}$  and  $p \pm 1$  not a power of 2.*
- (3)  *$G \cong \mathbb{Z}_2 \times \text{PSL}(2, p)$ , where  $p$  is a prime with  $p \equiv \pm 11 \pmod{24}$  and  $p \pm 1$  not a power of 2.*

## 2 Alternative walks and standard double covers

In this section, we always assume that  $\Delta$  is a simple digraph with vertex set  $V\Delta$ . By  $(u, v) \in \Delta$  we mean that  $(u, v)$  is an arc (directed edge) of  $\Delta$ . By  $\Delta^*$  we denote the digraph on  $V\Delta$  with arc set  $\{(v, u) \mid (u, v) \in \Delta\}$ .

For vertices  $u, v \in V\Delta$ , an *alternative walk* between  $u$  and  $v$  means a sequence  $u = v_0 v_1 \dots v_{2l} = v$  of odd number of vertices such that

$(v_{2i}, v_{2i+1}), (v_{2i+2}, v_{2i+1}) \in \Delta$  for  $0 \leq i \leq l-1$ . Define a relation ' $\sim$ ' on  $V\Delta$  as follows:

$$u \sim v \Leftrightarrow u = v, \text{ or there is an alternative walk between } u \text{ and } v.$$

It is easily shown that this gives an equivalent relation among the elements of  $V\Delta$ . Every equivalent class is called an *alternative component* of  $\Delta$ , and the number of alternative components is called the *alternative index* of  $\Delta$ , denoted by  $\text{alt}(\Delta)$ . The digraph  $\Delta$  is called *alternative connected* if  $\text{alt}(\Delta) = 1$ , that is, there exists an alternative walk between each pair of distinct vertices of  $\Delta$ .

The *standard double cover* of  $\Delta$ , denoted by  $\Delta^{(2)}$ , is the (undirected) graph defined on  $V\Delta \times \{1, 2\}$  such that  $\{(u, 1), (v, 2)\}$  is an edge if and only if  $(u, v) \in \Delta$ . For  $u \in V\Delta$ , set  $\Delta(u) = \{v \mid (u, v) \in \Delta\}$  and  $\Delta^*(u) = \{v \mid (v, u) \in \Delta\}$ . Then we have the following simple observation.

**Lemma 1** *If  $|\Delta^*(u)| \geq 1$  holds for every  $u \in V\Delta$ , then  $\text{alt}(\Delta)$  is equal to the number of connected components of  $\Delta^{(2)}$ .*

Let  $\text{Aut}\Delta$  be the automorphism group of  $\Delta$ , and  $G \leq \text{Aut}\Delta$ . Then  $\Delta$  is called  *$G$ -vertex-transitive* or  *$G$ -arc-transitive* if  $G$  acts transitively on the vertices or the arcs of  $\Delta$ , respectively. (If  $\Delta$  has no isolated vertex, then the arc-transitivity yields the vertex-transitivity.) For each  $g \in G$ , we obtain an automorphism  $\hat{g}$  of  $\Delta^{(2)}$  by  $(u, i)^{\hat{g}} = (u^g, i)$ . Set  $\widehat{G} = \{\hat{g} \mid g \in G\}$ . Then  $\widehat{G}$  is a subgroup of  $\text{Aut}\Delta^{(2)}$  and isomorphic to  $G$ , and

$$\widehat{G}_u = \widehat{G}_{(u,i)}$$

for  $u \in V\Delta$  and  $i = 1, 2$ . Moreover, the following lemma is easily shown.

**Lemma 2** *Let  $\Delta$  be a  $G$ -arc-transitive digraph without isolated vertices. Then*

- (1)  $\Delta^{(2)}$  is  $\widehat{G}$ -semisymmetric;
- (2)  $\text{alt}(\Delta) = |G : \langle G_u, G_v \rangle|$  for  $(u, v) \in \Delta$ ;
- (3)  $\Delta$  is alternatively connected if and only if  $G = \langle G_u, G_v \rangle$  for  $(u, v) \in \Delta$ ; in particular,  $\Delta$  is alternatively connected if  $G$  is primitive on  $V\Delta$  unless  $\Delta$  is a directed cycle of prime length.

*Proof* (1) is trivial.

Since  $\Delta^{(2)}$  is  $\widehat{G}$ -edge-transitive,  $\widehat{G}$  acts transitively on the set of connected components of  $\Delta^{(2)}$ . Let  $(u, v) \in \Delta$ , and let  $\Sigma$  be the connected component which contains the vertices  $(u, 1)$  and  $(v, 2)$ . Let  $H$  be the subgroup of  $G$  such that  $\widehat{H}$  preserves  $\Sigma$  invariantly. Then  $\Sigma$  is  $\widehat{H}$ -edge-transitive, and  $|\widehat{G} : \widehat{H}|$  is the number of connected components of  $\Delta^{(2)}$ . Thus  $\text{alt}(\Delta) = |\widehat{G} : \widehat{H}|$  by Lemma 1. Since  $\Sigma$  is a connected bipartite graph, by [22],  $\langle \widehat{H}_{(u,1)}, \widehat{H}_{(v,2)} \rangle$  acts transitively on  $E\Sigma$ . This implies that  $\widehat{H} = \langle \widehat{H}_{(u,1)}, \widehat{H}_{(v,2)} \rangle$ . Note that  $\widehat{H}_u = \widehat{H}_{(u,1)} = \widehat{G}_{(u,1)} = \widehat{G}_u$  and  $\widehat{H}_v = \widehat{H}_{(v,2)} = \widehat{G}_{(v,2)} = \widehat{G}_v$ . Then  $\text{alt}(\Delta) = |\widehat{G} : \widehat{H}| = |G : H| = |G : \langle G_u, G_v \rangle|$ . Thus (2) holds, and then the first part of (3) follows.

Assume that  $G$  is primitive on  $V\Delta$ . Then both  $G_u$  and  $G_v$  are maximal subgroups of  $G$ , and so either  $G = \langle G_u, G_v \rangle$  or  $G_u = G_v$ . The former case says that  $\Gamma$  is alternatively connected. The latter case yields that  $G_u = 1$  and  $G$  is a cyclic group of prime order, and then  $\Delta$  is a directed cycle of length  $|G|$ . Thus (3) is proved.

**Lemma 3** *Let  $\Delta$  be a  $G$ -arc-transitive digraph with  $|\Delta(u)| = 2$  for all  $u \in V\Delta$ . If  $\Delta$  is alternatively connected, then  $|V\Delta|$  is odd and  $(u, v) \in \Delta$  yields  $(v, u) \in \Delta$ .*

*Proof* Assume that  $\Delta$  is alternatively connected. Then  $\Delta^{(2)}$  is a cycle of length  $2n$ , where  $n = |V\Delta|$ . Thus  $\text{Aut}\Delta^{(2)} \cong D_{4n}$ , the dihedral group of order  $4n$ . Noting that  $\Delta^{(2)}$  is  $\widehat{G}$ -semisymmetric, it implies that  $\widehat{G} \cong D_{2n}$ , and so  $G \cong D_{2n}$ . Set  $G = \langle a, b \rangle$ , where  $a$  has order  $n$  and  $b$  is an involution with  $bab = a^{-1}$ . Then  $\langle a \rangle$  is a regular subgroup of  $G$ , and there is  $u \in V\Delta$  with  $G_u = \langle b \rangle$ . Take  $u^{a^i} \in \Delta(u)$ . Then  $G_{u^{a^i}} = G_u^{a^i} = \langle ba^{2i} \rangle$ . By Lemma 2,  $G = \langle b, ba^{2i} \rangle = \langle b, a^{2i} \rangle$ . Then  $\langle a \rangle = \langle a^{2i} \rangle$ , it implies that  $n$  is odd. Note that  $(u, u^{a^{i^b}}) \in \Delta$ . We have  $(u^{a^{i^b}}, u) = (u, u^{a^{i^b}})^{a^{i^b}} \in \Delta$ . Then our result follows from the arc-transitivity of  $\Delta$ .

For a subgroup  $G$  of  $\text{Aut}\Delta$ , the digraph  $\Delta$  is called  $G$ -locally-primitive if for every vertex  $u$ , neither  $\Delta(u) = \emptyset$  nor  $\Delta^*(u) = \emptyset$ , and the stabilizer  $G_u$  acts primitively on both  $\Delta(u)$  and  $\Delta^*(u)$ . It is easily shown that  $\Delta$  is  $G$ -locally-primitive if and only if  $\Delta^{(2)}$  is  $\widehat{G}$ -locally-primitive.

Assume that  $\Delta$  is  $G$ -locally-primitive and alternatively connected. Then  $\Delta^{(2)}$  is a connected  $\widehat{G}$ -locally-primitive graph. (Note that a general analyzing about the class of locally-primitive graphs is given in [8].) It is easy to see that  $\Delta^{(2)}$  is a regular bipartite graph. By edge-transitivity of  $\Delta^{(2)}$ , we know that  $\Delta$  is  $G$ -arc-transitive.

Note that every normal subgroup of  $\widehat{G}$  is transitive on  $V\Delta \times \{1\}$  if and only if it is transitive on  $V\Delta \times \{2\}$ . We have the following result.

**Theorem 3** *Let  $\Delta$  be a  $G$ -locally-primitive digraph, and let  $N$  be an intransitive normal subgroup of  $G$ . Let  $\mathcal{B}$  be the set of  $N$ -orbits, and  $G^{\mathcal{B}}$  the permutation group on  $\mathcal{B}$  induced by  $G$ . Assume that  $\Delta$  is alternatively connected. Then  $\Delta$  is  $G$ -arc-transitive; in particular,  $\Delta$  is  $G$ -vertex-transitive. Assume further that  $|\Delta(u)| > 2$  for some (and so for all)  $u \in V\Delta$ . Then  $G^{\mathcal{B}} \cong G/N$ , every  $N$ -orbit is an independent set of  $\Delta$ , and  $N$  is regular on each of its orbits. Moreover, one the following statements hold.*

- (1) *For  $B_1, B_2 \in \mathcal{B}$ , the subdigraph  $[B_1, B_2]$  of  $\Delta$  induced by  $B_1 \cup B_2$  is either empty or a directed matching. Define a digraph on  $\mathcal{B}$ , denoted by  $\Delta_N$ , such that  $(B_1, B_2) \in \Delta_N$  if and only if  $(u_1, u_2) \in \Delta$  for some  $u_1 \in B_1$  and  $u_2 \in B_2$ . Then  $\Delta_N$  is  $G^{\mathcal{B}}$ -locally-primitive,  $G^{\mathcal{B}}$ -arc-transitive and alternatively connected.*
- (2) *For  $B_1, B_2 \in \mathcal{B}$ , the subdigraph  $[B_1, B_2]$  is either empty or a union of disjoint directed cycles. Define a graph on  $\mathcal{B}$ , denoted by  $\Delta_N$ , such that  $\{B_1, B_2\} \in E\Delta_N$  if and only if  $[B_1, B_2]$  has a directed cycle. Then  $\Delta_N$  is  $G^{\mathcal{B}}$ -locally-primitive,  $G^{\mathcal{B}}$ -arc-transitive, non-bipartite and of valency  $|\Delta(u)|$ .*

*Proof* By the argument above this theorem, we know that  $\Delta$  is  $G$ -arc-transitive. Assume that  $|\Delta(u)| > 2$  for  $u \in V\Delta$ . Then  $\Delta^{(2)}$  is a connected  $\widehat{G}$ -locally-primitive graph of valency no less than 3. Note that for every  $B \in \mathcal{B}$ , both  $B \times \{1\}$  and  $B \times \{2\}$  are  $\widehat{N}$ -orbits on  $V\Delta \times \{1, 2\}$ . By [8, Lemma 5.1],  $\widehat{N}$  is semiregular and has at least  $|\Delta(u)|$  orbits on  $V\Delta \times \{1, 2\}$ , and  $\widehat{N}$  is the kernel of the action of  $\widehat{G}$  on the  $\widehat{N}$ -orbits. Thus  $N$  is semiregular on  $V\Delta$ , and  $G^{\mathcal{B}} \cong G/N$ .

Let  $B \in \mathcal{B}$ . Since  $\Delta$  is  $G$ -arc-transitive,  $G$  is transitive on  $V\Delta$ , and so  $\mathcal{B}$  is a  $G$ -invariant partition of  $V\Delta$ . Then for each  $g \in G$ , either  $B^g = B$  or  $B \cap B^g = \emptyset$ . Noting that  $\Delta$  is connected, there is  $(u, v) \in \Delta$  with  $|\{u, v\} \cap B| = 1$ . If  $\{u^g, v^g\} \subseteq B$  for some  $g \in G$ , then  $B \cap B^{g^{-1}} \neq \emptyset$ , and so  $B = B^{g^{-1}}$ , yielding  $\{u, v\} \subseteq B$ , a contradiction. Thus, since  $\Delta$  is  $G$ -arc-transitive, we know that  $B$  is an independent set of  $\Delta$ . Then the first part of this theorem follows.

Let  $B_1, B_2 \in \mathcal{B}$  such that  $u_i \in B_i$  with  $(u_1, u_2) \in \Delta$ . Note that each  $B_i \times \{j\}$  is an  $\widehat{N}$ -orbit on  $V\Delta \times \{1, 2\}$ . Consider the subgraphs  $\Sigma_1$  and  $\Sigma_2$  of  $\Delta^{(2)}$  induced by  $(B_1 \times \{1\}) \cup (B_2 \times \{2\})$  and  $(B_2 \times \{1\}) \cup (B_1 \times \{2\})$ , respectively. Since  $\Delta^{(2)}$  is a connected  $\widehat{G}$ -locally-primitive graph, by [8, Lemma 5.1],  $\Sigma_1$  is a matching, and  $\Sigma_2$  is either empty or a matching. If  $\Sigma_2$  is empty then  $[B_1, B_2]$  is a directed matching. If  $\Sigma_2$  is a matching then  $[B_1, B_2]$  is a union of disjoint directed cycles.

Since  $\Delta$  is  $G$ -arc-transitive, there are no  $B_1, B_2, B_3, B_4 \in \mathcal{B}$  such that  $[B_1, B_2]$  is a directed matching and  $[B_3, B_4]$  has a directed cycle. Thus  $\Delta_N$  is well-defined in each of cases (1) and (2). Let  $u \in B \in \mathcal{B}$ . Then  $G_B = NG_u$ , and so  $G_B^{\mathcal{B}} \cong G_B/N \cong G_u$ . Define

$$\theta : \Delta(u) \cup \Delta^*(u) \rightarrow \Delta_N(B) \cup (\Delta_N)^*(B), v \mapsto v^N.$$

Then  $\theta$  gives a well-defined bijection between  $\Delta(u)$  and  $\Delta_N(B)$  and a well-defined bijection between  $\Delta^*(u)$  and  $(\Delta_N)^*(B)$ . (Note that  $\Delta_N(B) = (\Delta_N)^*(B)$  for case (2).) Moreover,

$$\theta(v^g) = (v^g)^N = v^{gNg^{-1}} = (v^N)^g = \theta(v)^g$$

for  $g \in G_u$  and  $v \in \Delta(u) \cup \Delta^*(u)$ . It follows that  $G_u^{\Delta(u)}$  and  $(G_B^{\mathcal{B}})^{\Delta_N(B)}$  are permutation isomorphic, and so do for  $G_u^{\Delta^*(u)}$  and  $(G_B^{\mathcal{B}})^{(\Delta_N)^*(B)}$ . Thus  $\Delta_N$  is  $G^{\mathcal{B}}$ -locally-primitive.

Since  $\Delta$  is alternatively connected,  $G = \langle G_{u_1}, G_{u_2} \rangle$  for  $(u_1, u_2) \in \Delta$ . Set  $B_i = u_i^N$  for  $i = 1, 2$ . Then  $G = \langle G_{B_1}, G_{B_2} \rangle$ , yielding  $G^{\mathcal{B}} = \langle G_{B_1}^{\mathcal{B}}, G_{B_2}^{\mathcal{B}} \rangle$ . If  $\Delta_N$  is a digraph, then  $(B_1, B_2)$  is an arc of  $\Delta_N$ , and so  $\Delta_N$  is alternatively connected by Lemma 2. Thus (1) holds.

Assume that  $\Delta_N$  is a graph. Then  $\{B_1, B_2\}$  is an edge of  $\Delta_N$ . Suppose that  $\Delta_N$  is a bipartite graph. Then both  $G_{B_1}^{\mathcal{B}}$  and  $G_{B_2}^{\mathcal{B}}$  fix the bipartition of  $\Delta_N$ . In particular,  $\langle G_{B_1}^{\mathcal{B}}, G_{B_2}^{\mathcal{B}} \rangle$  is intransitive on the vertex set of  $\Delta_N$ . Noting that  $\Delta_N$  is  $G^{\mathcal{B}}$ -vertex-transitive, we have  $G^{\mathcal{B}} \neq \langle G_{B_1}^{\mathcal{B}}, G_{B_2}^{\mathcal{B}} \rangle$ , a contradiction. Thus  $\Delta_N$  is not bipartite, and hence (2) holds.

### 3 Alternatively connected edge-transitive graphs

In this section, we let  $\Gamma$  be a  $G$ -half-transitive graph of valency  $2d$ . Then  $G$  has exactly two orbits on  $A\Gamma$ . Let  $\Delta$  be the digraph on  $V\Gamma$  with arc set being one of the  $G$ -orbits on  $A\Gamma$ . (Then  $\Delta^*$  is the digraph on  $V\Gamma$  with arc set being the other  $G$ -orbit on  $A\Gamma$ .) The *alternative index*  $\text{alt}(\Gamma)$  of  $\Gamma$  is defined as  $\text{alt}(\Delta)$ .

By Lemma 3, we have the following result.

**Corollary 1** *If  $\Gamma$  is a  $G$ -half-transitive graph of valency 4, then  $\text{alt}(\Gamma) > 1$ .*

Take an edge  $\{u, v\} \in E\Gamma$ . Since  $\Gamma$  is  $G$ -vertex-transitive  $v = u^g$  for some  $g \in G$ . Consider the arc-stabilizers of  $(u, u^g)$  and  $(u^{g^{-1}}, u)$ , which are  $G_u \cap G_u^g$  and  $G_u \cap G_u^{g^{-1}}$  respectively. Suppose that  $G_u \cap G_u^g$  and  $G_u \cap G_u^{g^{-1}}$  are conjugate in  $G_u$ . Set  $G_u \cap G_u^g = (G_u \cap G_u^{g^{-1}})^x$  for some  $x \in G_u$ . Then

$$G_u \cap G_u^g = G_u^x \cap G_u^{g^{-1}x} = G_u \cap G_u^{g^{-1}x} = (G_u^{(g^{-1}x)^{-1}} \cap G_u)^{g^{-1}x} = (G_u \cap G_u^g)^{g^{-1}x},$$

that is,  $g^{-1}x$  lies in the normalizer  $\mathbf{N}_G(G_u \cap G_u^g)$ . Noting that  $(u^{g^{-1}x}, u) \in A\Gamma$ , since  $\Gamma$  is not  $G$ -arc-transitive,  $(u^{g^{-1}x}, u)^{g^{-1}x} \neq (u, u^{g^{-1}x})$ . This implies that  $(g^{-1}x)^2 \notin G_u \cap G_u^g$ . In particular,  $\mathbf{N}_G(G_u \cap G_u^g)/(G_u \cap G_u^g)$  is not a 2-group. Then, by Lemma 2, we have the following lemma.

**Lemma 4** *Let  $\Gamma$  be a  $G$ -half-transitive graph of valency  $2d$  with  $\text{alt}(\Gamma) = 1$ , where  $d > 2$ . Let  $\{u, u^g\} \in E\Gamma$ . Suppose that  $G_u \cap G_u^g$  and  $G_u \cap G_u^{g^{-1}}$  are conjugate in  $G_u$ . Then  $\mathbf{N}_G(G_u \cap G_u^g)/(G_u \cap G_u^g)$  is not a 2-group, and  $\mathbf{N}_G(G_u \cap G_u^g)$  contains an element  $h$  such that  $u^h \in \Gamma(u)$  and  $G = \langle G_u, G_u^h \rangle$ .*

Take  $(u, v) \in \Delta$ . Since  $G$  is transitive on  $V\Gamma$ ,  $G_u$  and  $G_v$  are conjugate in  $G$ , and so  $\widehat{G}_u$  and  $\widehat{G}_v$  are conjugate in  $\widehat{G}$ . Note that  $\widehat{G}_{(u,1)} = \widehat{G}_u$  and  $\widehat{G}_{(v,2)} = \widehat{G}_v$ . Then  $\widehat{G}_{(u,1)}$  and  $\widehat{G}_{(v,2)}$  are conjugate in  $\widehat{G}$ . Let  $d = 3$  and  $\text{alt}(\Gamma) = 1$ . Note that  $\Delta^{(2)}$  is a connected  $\widehat{G}$ -semisymmetric cubic graph. By [9],  $G_u \cong \widehat{G}_u \cong \mathbb{Z}_3, \mathbb{S}_3, \mathbb{D}_{12}, \mathbb{S}_4$  or  $\mathbb{Z}_2 \times \mathbb{S}_4$ . In this case, noting that  $|G_u : (G_u \cap G_v)| = 3$ , we know that  $G_u \cap G_v$  is a Sylow 2-subgroup of  $G_u$ . Since all Sylow 2-subgroup of  $G_u$  are conjugate, by Lemma 4, we have the following result.

**Lemma 5** *Let  $\Gamma$  be a  $G$ -half-transitive graph of valency 6 with  $\text{alt}(\Gamma) = 1$ . Let  $\{u, v\} \in E\Gamma$ . Then*

- (1)  $G_u \cong \mathbb{Z}_3, \mathbb{S}_3, \mathbb{D}_{12}, \mathbb{S}_4$  or  $\mathbb{Z}_2 \times \mathbb{S}_4$ ;
- (2)  $\mathbf{N}_G(G_u \cap G_v)$  is not a 2-group, and  $\mathbf{N}_G(G_u \cap G_v)$  contains an element  $h$  such that  $u^h \in \Gamma(u)$  and  $G = \langle G_u, G_u^h \rangle$ .

Recall that  $\Gamma$  is  $G$ -locally-biprimitive if  $\Delta$  is  $G$ -locally-primitive. The following result give an undirected version of Theorem 3.

**Theorem 4** *Let  $\Gamma$  be a  $G$ -half-transitive graph of valency  $2d$  with  $\text{alt}(\Gamma) = 1$ , where  $d > 2$ . Assume that  $\Gamma$  is  $G$ -locally-biprimitive. Let  $N$  be an intransitive normal subgroup of  $G$ , and let  $\mathcal{B}$  be the set of  $N$ -orbits. Then  $G^{\mathcal{B}} \cong G/N$ ,*

every  $N$ -orbit is an independent set of  $\Gamma$ , and  $N$  is regular on each of its orbits. Define a graph on  $\mathcal{B}$ , denoted by  $\Gamma_N$ , such that  $\{B_1, B_2\} \in E\Gamma_N$  if and only if  $\{u_1, u_2\} \in E\Gamma$  for some  $u_1 \in B_1$  and  $u_2 \in B_2$ . Then either

- (1)  $\Gamma_N$  is  $G^{\mathcal{B}}$ -locally-biprimitive, alternatively connected and of valency  $2d$ ; or
- (2)  $\Gamma_N$  is  $G^{\mathcal{B}}$ -vertex-transitive,  $G^{\mathcal{B}}$ -locally-primitive and of valency  $d$ ; moreover, for  $B_1, B_2 \in \mathcal{B}$ , the subgraph  $[B_1, B_2]$  is either empty or a union of disjoint cycles.

The next result is a direct consequence from Theorem 4.

**Corollary 2** *Let  $\Gamma$  be a  $G$ -half-transitive graph of valency  $2d$  with  $\text{alt}(\Gamma) = 1$ , where  $d > 2$ . Assume that  $\Gamma$  is  $G$ -locally-biprimitive. Let  $N$  be a normal subgroup of  $G$ . If  $N$  is not semiregular on  $V\Gamma$ , then  $\Gamma$  is  $N$ -half-transitive.*

*Proof* Let  $N$  be a normal subgroup of  $G$ . Assume that  $N$  is not semiregular on  $V\Gamma$ . By [11, Lemma 3.1], for each  $u \in V\Gamma$ , the stabilizer  $N_u$  acts nontrivially on  $\Gamma(u)$ , and all  $N_u$ -orbits on  $\Gamma(u)$  have the same length. Since  $N_u$  is normal in  $G_u$  and  $\Gamma$  is  $G$ -locally-biprimitive,  $N_u$  has two orbits on  $\Gamma(u)$  which are the orbits of  $G_u$ . Moreover, by Theorem 4,  $N$  is transitive on  $V\Gamma$ . Thus  $\Gamma$  is  $N$ -half-transitive.

#### 4 The graphs of square-free order

Let  $\Gamma$  be a  $G$ -half-transitive graph of square-free order and valency  $2d$ , where  $d > 2$ . Assume that  $\Gamma$  is  $G$ -locally-biprimitive, and  $\text{alt}(\Gamma) = 1$ .

Let  $M$  be a maximal intransitive normal subgroup of  $G$ , and let  $\mathcal{B}$  be the set of  $M$ -orbits. Then every nontrivial normal subgroup of  $G^{\mathcal{B}}$  is transitive on  $\mathcal{B}$ . Then, by [14, Lemma 12], either  $|\mathcal{B}| = p$  and  $G^{\mathcal{B}} \cong \mathbb{Z}_p : \mathbb{Z}_l$  for an odd prime  $p$  and a divisor  $l$  of  $p - 1$ , or  $G^{\mathcal{B}}$  is almost simple. For the former case, by [13, Lemma 2.5], we conclude that  $l = d$  is an odd prime.

**Lemma 6** *Assume that  $G^{\mathcal{B}} \cong \mathbb{Z}_p : \mathbb{Z}_d$ . Then  $G$  has a regular normal cyclic subgroup  $\langle a \rangle$ ,  $|G|$  is square-free and  $q - 1$  is divisible by  $d$  for each prime divisor  $q$  of  $|V\Gamma|$ . Moreover, setting  $G_u = \langle b \rangle$  for  $u \in V\Gamma$ , we have  $b^{-1}ab = a^r$  for some  $r$  with  $r^d \equiv 1 \pmod{|V\Gamma|}$  and  $(r - 1, |V\Gamma|) = 1$ .*

*Proof* By Theorem 4,  $G^{\mathcal{B}} \cong G/M$  and  $M$  is semiregular on  $V\Gamma$ . Since  $|V\Gamma|$  is square-free,  $|M|$  is square-free and  $(|M|, p) = 1$ . Clearly,  $G$  has a normal regular subgroup  $R := M:P$ , where  $P \cong \mathbb{Z}_p$ . Let  $q$  be the smallest prime divisor of  $R$ . Then, since  $R$  has square-free order,  $R$  has a unique  $q'$ -Hall subgroup  $N$ . In particular,  $N$  is a characteristic subgroup of  $R$ , and hence  $N$  is normal in  $G$ . Note  $N$  is also a maximal intransitive normal subgroup of  $G$ . Then  $G$  induces a permutation group on the  $N$ -orbits, which is isomorphic to  $\mathbb{Z}_q : \mathbb{Z}_d$  with  $q - 1$  divisible by  $d$ . Thus  $d$  is the smallest prime divisor of  $|G| = dp|M|$ . Without loss of generality, we let  $q = p$ . Note that  $G$  has square-free order. It is well-known and easily shown that  $G = \langle c \rangle \times (\langle a \rangle : \langle b \rangle)$  for some  $a, b, c \in G$ ,



where  $\langle c \rangle$  is the center of  $G$ . Note that an abelian transitive permutation group must be regular. Then  $G$  is not abelian, and so  $a \neq 1$  and  $b \neq 1$ . Let  $a$  and  $b$  have orders  $m$  and  $n$ , respectively.

Take an edge  $\{u, v\} \in E\Gamma$ . Then  $G = \langle G_u, G_v \rangle$  by Lemma 2. Note that for every prime divisor of  $|\langle c \rangle \times \langle a \rangle|$ , the group  $G$  has a unique subgroup of order  $r$ . It follows that both  $G_u$  and  $G_v$  are conjugate to a subgroup of  $\langle b \rangle$ . Without loss of generality, let  $G_u \leq \langle b \rangle$ . Note that  $G_u$  and  $G_v$  are conjugate in  $G$ , and that every element of  $G$  has the form of  $c^k b^j a^i$ . Then  $G_v = G_u^{a^i}$  for some integer  $i$ . Set  $G_u = \langle b^j \rangle$ . Then  $G = \langle G_u, G_v \rangle = \langle b^j, a^{-i} b^j a^i \rangle \leq \langle a^i, b^j \rangle = \langle a^i \rangle : \langle b^j \rangle$ , yielding  $c = 1$ ,  $(i, m) = 1$  and  $(j, n) = 1$ , and so  $G_u = \langle b \rangle$ ,  $n = d$  and  $R = \langle a \rangle$ . Thus, without loss of generality, we choose  $i = 1$  and  $j = 1$ . Set  $b^{-1} a b = a^r$ . Then  $r^d \equiv 1 \pmod{m}$ , and  $G = \langle b, a^{-1} b a \rangle = \langle b, b a^{1-r} \rangle = \langle a^{r-1}, b \rangle$ , and hence  $(r-1, m) = 1$ .

Let  $q$  be an arbitrary prime divisor of  $m$ . Then the  $q'$ -Hall subgroup of  $\langle a \rangle$  is a maximal intransitive normal subgroup of  $G$ . Thus  $q-1$  is divisible by  $d$  by the argument in the first paragraph. This completes the proof.

*Remark 1* Let  $G$  be the group in Lemma 6. Noting that  $G$  has trivial center, it is easily shown that  $G$  is a Frobenius group with Frobenius kernel  $\langle a \rangle$ .

**Lemma 7** *Assume that  $G$  is almost simple. Then  $\text{soc}(G)$  is (isomorphic to) one of the following simple groups:*

- $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1; A_n$  with  $n < 3d$ ;
- $\text{PSL}(2, p)$  for prime  $p \geq 5$ ;  $\text{PSL}(2, p^2)$  with  $p > 3$  and  $d$  divisible by  $p+1$ ;
- $\text{PSL}(2, p^f)$  with  $f \geq 3$ ,  $p^f > 9$  and  $d$  divisible by  $p^{f-1}$ ;
- simple classical groups of dimension  $n$  over  $\text{GF}(p^f)$  with  $p \leq d$ ,  $n \geq 3$  and  $[\frac{n}{2}]f < d$ ;
- simple exceptional groups  $G_2(p^f)$ ,  ${}^3D_4(p^f)$ ,  $F_4(p^f)$ ,  ${}^2E_6(p^f)$  and  $E_7(p^f)$  with  $2f < d$  and  $p \leq d$ .

*Proof* Let  $T = \text{soc}(G)$ . Since  $\Gamma$  has square-free order, by Corollary 2,  $\Gamma$  is  $T$ -half-transitive. It is easy to see that  $\Gamma$  and  $G$  satisfy the assumptions of [14, Theorem 1]. Then  $T$  is (isomorphic to) one of the following simple groups:

- $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1; A_n$  with  $n < 6d$ ;
- $\text{PSL}(2, p)$  for prime  $p \geq 5$ ;  $\text{PSL}(2, p^2)$  with  $p > 3$  and  $2d$  divisible by  $p+1$ ;
- $\text{PSL}(2, p^f)$  with  $f \geq 3$ ,  $p^f > 9$  and  $2d$  divisible by  $p^{f-1}$ ;
- $\text{Sz}(2^f)$  with odd  $f \geq 3$  and  $2d$  divisible by  $2^{2f-1}$ ;
- simple classical groups of dimension  $n$  over  $\text{GF}(p^f)$  with  $p \leq 2d$ ,  $n \geq 3$  and  $[\frac{n}{2}]f < 2d$ ;
- simple exceptional groups  $G_2(p^f)$ ,  ${}^3D_4(p^f)$ ,  $F_4(p^f)$ ,  ${}^2E_6(p^f)$  and  $E_7(p^f)$  with  $2f < 2d$  and  $p \leq 2d$ .

Checking carefully the argument given in [14], we conclude that the restricted conditions ' $n < 6d$ ', ' $[\frac{n}{2}]f < 2d$ ', ' $2f < 2d$ ' and ' $p \leq 2d$ ' are in fact derived from the facts that  $|T : T_u|$  is square-free and each prime divisor of  $|T_u|$  is no more than  $|T_u : T_{uv}|$ . The restricted conditions for  $\text{PSL}(2, p)$  and  $\text{PSL}(2, p^f)$  that

$2d$  is divisible by some special integers are from the fact that those special integers are divisors of  $|T_u : T_{uv}|$ . Given these facts, either  $T$  is one of the simple groups listed in this lemma, or  $T = \text{Sz}(2^f)$  with odd  $f \geq 3$  and  $d$  divisible by  $2^{2f-1}$ .

Suppose that  $T = \text{Sz}(2^f)$ . Let  $\Delta$  be an orbit of  $G$  on  $A\Gamma$ . Then  $G_u$  induces a primitive permutation group  $G_u^{\Delta(u)}$  on  $\Delta(u)$ . By Corollary 2, we conclude that  $T_u^{\Delta(u)}$  is a transitive normal subgroup of  $G_u^{\Delta(u)}$ . Since  $|T : T_u|$  is square-free, inspecting the subgroups of  $T$  (see [20]), we get  $T_u = Q:\mathbb{Z}_l$ , where  $Q$  has order  $2^{2f}$  or  $2^{2f-1}$ , and  $l$  is a divisor of  $2^f - 1$ . It follows  $\text{soc}(G_u^{\Delta(u)}) \cong \mathbb{Z}_2^t$  for some integer  $t \geq 1$ , and  $T_u^{\Delta(u)} \cong \mathbb{Z}_2^t:\mathbb{Z}_{l'}$ , where  $l'$  is a divisor of  $l$ . Moreover,  $d = 2^t$ , and so  $t = 2f$  or  $2f - 1$ .

Let  $P$  be a Sylow 2-subgroup of  $T$  with  $P \geq Q$ . Then  $P = Q$  or  $Q.\mathbb{Z}_2$ . Assume that  $Q \neq P$ . Then  $|Q| = 2^{2f-1}$ , and hence  $Q \cong \mathbb{Z}_2^{2f-1}$ . By [20], all involutions of  $P$  are contained in the center of  $P$ . It follows that  $P$  is abelian, which is impossible as  $\text{Sz}(2^f)$  has no abelian Sylow 2-subgroup. Thus  $P = Q$ . If  $t = 2f$  then  $P \cong \mathbb{Z}_2^{2f}$ , again a contradiction. Let  $t = 2f - 1$ . Then there is a normal subgroup  $K$  of  $P$  of order 2 such that  $P/K \cong \mathbb{Z}_2^{2f-1}$ . Consider the Frattini subgroup  $\Phi(P)$  of  $P$ . By [10, III.3.14], we have  $\mathbb{Z}_2 \cong K \geq \Phi(P) = \langle x^2 \mid x \in P \rangle$ . In particular,  $\Phi(P) = \langle x^2 \rangle$  for each element  $x \in P$  of order 4. However, by [20],  $P$  contains two elements  $x$  and  $y$  of order 4 with  $x^2 \neq y^2$ , a contradiction. Then this lemma follows.

**Lemma 8** *Assume that  $G^{\mathcal{B}}$  is almost simple. Then  $G = M:X$  for some subgroup  $X$  of  $G$ , where  $X$  is almost simple with socle isomorphic to  $\text{soc}(G^{\mathcal{B}})$  and centralizing  $M$ .*

*Proof* By [14, Lemma 28],  $G = M:X$  for some subgroup  $X$  of  $G$ . Then  $X \cong G/M \cong G^{\mathcal{B}}$ , and so  $X$  is almost simple. Since  $|M|$  is square-free,  $M$  has soluble automorphism group  $\text{Aut}(M)$ . Noting that  $G/\mathbf{C}_G(M) = \mathbf{N}_G(M)/\mathbf{C}_G(M) \lesssim \text{Aut}(M)$ , it follows that  $G/\mathbf{C}_G(M)$  is soluble. Thus  $\text{soc}(X) \leq \mathbf{C}_G(M)$ , and the lemma follows.  $\square$

Finally, Theorem 1 follows from Corollary 2 and Lemmas 6-8.

## 5 The graphs of valency 6

Let  $\Gamma$  be a  $G$ -half-transitive graph of square-free order and valency 6. Assume that  $G$  is insoluble and  $\Gamma$  is alternatively connected. By Theorem 1 and Lemma 8, we set  $G = M:X$ , where  $M$  is a maximal intransitive normal subgroup of  $G$ , and  $X$  is an almost simple group with socle  $T$ . Then  $\Gamma$  is  $T$ -half-transitive, where  $T$  normal in  $G$  and described as in Theorem 1.

### 5.1 The simple group $T$

Considering the restricted conditions for  $T$ , we know that either  $T \cong \text{PSL}(2, p)$  for prime  $p \geq 5$ , or  $T$  is one of a finite number of simple groups. Let  $u \in V\Gamma$ .

Since  $T$  is nonabelian simple,  $|T|$  is divisible by 4, so  $T_u$  has even order as  $|T : T_u|$  is square-free. By Lemma 5,  $T_u$  is isomorphic to one of  $S_3$ ,  $D_{12}$ ,  $S_4$  and  $\mathbb{Z}_2 \times S_4$ . Since  $|T : T_u|$  is square-free,  $|T|$  is not divisible by  $2^6$  or  $3^3$ . Checking the orders of the finite number of candidates other than  $\text{PSL}(2, p)$  for  $T$ , up to isomorphism of groups, we may assume that  $T$  is listed in the following lemma.

**Lemma 9**  $T = A_5, A_6, A_7, M_{11}, J_1$  or  $\text{PSL}(2, p)$ , where  $p > 5$  is a prime.

**Lemma 10**  $T \notin \{A_5, A_6, A_7\}$ .

*Proof* Assume that  $T = A_5$ . Then  $|T_u| = 6$  or  $12$ . Note that  $A_5$  has no subgroup isomorphic to  $D_{12}$ , see the Atlas [5]. Then  $T_u \cong S_3$ , and so  $T_u \cap T_v \cong \mathbb{Z}_2$ , where  $v \in \Gamma(u)$ . In this case,  $\mathbf{N}_T(T_u \cap T_v) \cong \mathbb{Z}_2^2$ , which contradicts Lemma 5. The group  $A_6$  is excluded by a similar argument.

Assume that  $T = A_7$ . Then  $|T_u| = 12$  or  $24$ , and so  $T_u \cong D_{12}$  or  $S_4$ . Suppose that  $T_u \cong S_4$ . Then  $T_u \cap T_v \cong D_8$  for  $v \in \Gamma(u)$ . Checking the subgroups of  $A_7$ , we get  $\mathbf{N}_T(T_u \cap T_v) = (T_u \cap T_v)$ , a contradiction. Thus  $T_u \cong D_{12}$ . Set  $T_u = \langle a, b \rangle$ , where  $a$  has order 6 and  $b$  is an involution. Since  $A_7$  consists of even permutations on  $\{1, 2, 3, 4, 5, 6, 7\}$ , we may choose  $a = (1\ 2)(3\ 4)(5\ 6\ 7)$  and  $b = (3\ 4)(5\ 6)$ , and let  $T_u \cap T_v = \langle a^3, b \rangle$ . Then  $\mathbf{N}_T(T_u \cap T_v) = \langle a^3, b \rangle : \langle (1\ 3\ 6)(2\ 4\ 5), (3\ 6)(4\ 5) \rangle \cong S_4$ . It is easily shown that, for each  $x \in \mathbf{N}_T(T_u \cap T_v)$ , the subgroup  $\langle T_u, T_u^x \rangle$  fixes either  $\{1, 2\}$  or  $\{3, 4\}$  setwise. Thus  $T \neq \langle T_u, T_u^x \rangle$  for any  $x \in \mathbf{N}_T(T_u \cap T_v)$ , which contradicts Lemma 5. This completes the proof.

**Lemma 11**  $T \neq M_{11}$ .

*Proof* Assume that  $T = M_{11}$ . Then  $|T_u| = 24$ , and so  $T_u \cong S_4$  and  $T_u \cap T_v \cong D_8$ , where  $v \in \Gamma(u)$ . Let  $P$  be a Sylow 2-subgroup of  $T$  with  $P > T_u \cap T_v$ . Then  $P \leq \mathbf{N}_T(T_u \cap T_v)$ , and so  $\mathbf{N}_T(T_u \cap T_v)$  has odd index in  $T$ . Let  $H$  be a maximal subgroup of  $T$  with  $H \geq \mathbf{N}_T(T_u \cap T_v)$ . Then  $\mathbf{N}_T(T_u \cap T_v) = \mathbf{N}_H(T_u \cap T_v)$ . Noting that  $H$  has odd index in  $T$ , we have  $H \cong M_{10}$  or  $2.S_4$  by the Atlas [5]. Checking the subgroups of  $H$ , we conclude that  $\mathbf{N}_T(T_u \cap T_v) = \mathbf{N}_H(T_u \cap T_v) = P$ , which contradicts Lemma 5. (This was also confirmed by GAP.)

**Lemma 12**  $T \neq J_1$ .

*Proof* Assume that  $T = J_1$ . Then  $|T_u| = 12$ , and so  $T_u \cong D_{12}$  and  $T_u \cap T_v \cong \mathbb{Z}_2^2$ , where  $v \in \Gamma(u)$ . It follows from the information given in the Atlas [5] that all subgroups of  $T$  isomorphic to  $D_{12}$  are conjugate. Thus we may choose a maximal subgroup  $M$  of  $T$  with  $\mathbb{Z}_2 \times A_5 \cong M > T_u$ . Then  $\mathbf{N}_M(T_u \cap T_v) \cong \mathbb{Z}_2 \times A_4$ .

By a similar argument as in the proof of Lemma 11,  $\mathbf{N}_T(T_u \cap T_v) = \mathbf{N}_H(T_u \cap T_v)$ , where  $H$  is a maximal subgroup of  $T$  of odd index. By the Atlas [5],  $H \cong \mathbb{Z}_2 \times A_5$  or  $\mathbb{Z}_2^3 : (\mathbb{Z}_7 : \mathbb{Z}_3)$ . It follows that  $\mathbf{N}_T(T_u \cap T_v) = \mathbf{N}_H(T_u \cap T_v) \cong \mathbb{Z}_2 \times A_4$ . Then  $\mathbf{N}_T(T_u \cap T_v) = \mathbf{N}_H(T_u \cap T_v) \cong \mathbf{N}_M(T_u \cap T_v)$ , and so  $\mathbf{N}_T(T_u \cap T_v) = \mathbf{N}_M(T_u \cap T_v)$ . Recalling that  $M > T_u$ , we have  $\langle T_u, T_u^x \rangle \leq M \neq T$  for any  $x \in \mathbf{N}_T(T_u \cap T_v)$ , which contradicts Lemma 5.

## 5.2 The proof of Theorem 2

By the foregoing argument, we may let  $T = \text{PSL}(2, p)$ , where  $p$  is a prime and  $p \geq 7$ .

Let  $\{u, v\}$  be an edge of  $\Gamma$ . Checking the subgroups of  $\text{PSL}(2, p)$  (see [10, II.8.27]), we conclude that  $T_u, T_v$  and  $T_u \cap T_v$  are listed (up to isomorphism) as follows:

$$\frac{T_u, T_v \mid \text{S}_3 \mid \text{D}_{12} \mid \text{S}_4}{T_u \cap T_v \mid \mathbb{Z}_2 \mid \mathbb{Z}_2^2 \mid \text{D}_8}$$

If  $T_u \cong \text{S}_4$  then we have  $|\mathbf{N}_T(T_u \cap T_v)| = 8$  or  $16$  by checking the subgroups of  $\text{PSL}(2, p)$ , which contradicts Lemma 5. Thus we assume next that  $T_u \cong \text{S}_3$  or  $\text{D}_{12}$ .

Let  $\mathcal{B}$  be the set of  $M$ -orbits, and let  $u \in B \in \mathcal{B}$ . Since  $T$  is transitive on  $V\Gamma$ , the setwise stabilizer  $T_B$  of  $B$  in  $T$  is transitive on  $B$ . Noting that  $M$  centralizes  $T_B$  and  $M$  is regular on  $B$ , it follows that  $T_B$  induces a regular permutation group on  $B$ , see [6, Theorem 4.2A] for example. Then  $T_u$  is normal in  $T_B$ , and  $|T_B : T_u| = |M|$ .

Consider the graph  $\Gamma_M$ . Noting that  $G^{\mathcal{B}} \cong G/M \cong X$ , we identify  $X$  with a subgroup of  $\text{Aut}\Gamma_M$ . By Theorem 4,  $\Gamma_M$  has valency 6 or 3.

Suppose that  $\Gamma_M$  is a cubic graph. Then  $M \neq 1$  and  $\Gamma_M$  is  $X$ -arc-transitive. It is easily shown that  $\Gamma_M$  is also  $T$ -arc-transitive, and so  $T_B \cong \mathbb{Z}_3, \text{S}_3, \text{D}_{12}, \text{S}_4$  or  $\mathbb{Z}_2 \times \text{S}_4$  (see [2, 18C]). Recalling that  $T_B$  has a normal subgroup  $T_u \cong \text{S}_3$  or  $\text{D}_{12}$ , we conclude that  $T_B \cong \text{D}_{12}$  and  $T_u \cong \text{S}_3$ . Thus  $|M| = |T_B : T_u| = 2$ . Then  $\Gamma_M$  has odd order  $\frac{|V\Gamma|}{2}$ , which is impossible as  $\Gamma_M$  is a cubic graph.

By Theorem 4, we assume next that  $\Gamma_M$  is an  $X$ -half-transitive alternatively connected graph of order  $\frac{|V\Gamma|}{|M|}$  and valency 6. Then the foregoing argument in this subsection works for the pair of  $\Gamma_M$  and  $T$ . It follows that  $T_B \cong \text{S}_3, \text{D}_{12}$  or  $\text{S}_4$ . Recalling  $T_B$  has a normal subgroup of index  $|M|$ , we conclude that either  $M = 1$ , or  $M \cong \mathbb{Z}_2, T_u \cong \text{S}_3$  and  $T_B \cong \text{D}_{12}$ .

Let  $T_u \cong \text{D}_{12}$ . Then  $M = 1$ , and so  $G = \text{PSL}(2, p)$  or  $\text{PGL}(2, p)$ . Noting that  $T_u \leq G_u$  and  $|G : G_u| = |T : T_u| = |V\Gamma|$ , we have  $|G_u : T_u| = |G : T|$ . If  $G = \text{PGL}(2, p)$  then  $G_u$  has order 24, and so  $G_u \cong \text{S}_4$ ; however,  $\text{S}_4$  has no subgroup isomorphic to  $\text{D}_{12}$ . Thus  $G = T = \text{PSL}(2, p)$ . Since  $|G : G_u|$  is square-free,  $p^2 - 1$  is not divisible by 32. Then part (1) of Theorem 2 follows.

Let  $T_u \cong \text{S}_3$ . Then  $T_u \cap T_v \cong \mathbb{Z}_2$ , and  $\mathbf{N}_T(T_u \cap T_v) \cong \text{D}_{p+\epsilon}$ , where  $\epsilon = \pm 1$  such that  $p + \epsilon$  is divisible by 4. By Lemma 5,  $p + \epsilon$  is not a power of 2; in particular  $p \neq 7$ . Since  $T$  has order divisible by 4,  $|V\Gamma| = |T : T_u|$  is even and square-free. This implies that  $p + \epsilon$  is not divisible by 8. Then  $p \equiv \pm 3 \pmod{8}$ .

Recall that  $|M| = 1$  or  $2$ . If  $M = 1$  then we get part (2) of Theorem 2. Assume that  $|M| = 2$ . Then  $T_B \cong \text{D}_{12}$  for an  $M$ -orbit  $B$  with  $u \in B$ . Consider the pair  $\Gamma_N$  and  $X$ . By a similar argument as in the case for  $T_u \cong \text{D}_{12}$ , we know that  $X = T = \text{PSL}(2, p)$ , and so  $G = M \times T$ . Note that  $T$  has a subgroup isomorphic to  $\text{D}_{12}$ . It follows that  $p + \epsilon$  is divisible by 12, that is  $p \equiv \pm 11 \pmod{24}$ . Then part (3) of Theorem 2 occurs.

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