Non-jumping Numbers for 5-Uniform Hypergraphs*

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Abstract

Let ℓ and r be integers. A real number $\alpha \in [0, 1)$ is a jump for r if for any $\varepsilon > 0$ and any integer m, $m \geq r$, any r-uniform graph with $n > n_0(\varepsilon, m)$ vertices and at least $(\alpha + \varepsilon)\binom{n}{r}$ edges contains a subgraph with m vertices and at least $(\alpha + c)\binom{m}{r}$ edges, where $c = c(\alpha)$ does not depend on ε and m. It follows from a theorem of Erdős, Stone and Simonovits that every $\alpha \in [0, 1)$ is a jump for $r = 2$. Erdős asked whether the same is true for $r \geq 3$. However, Frankl and Rödl gave a negative answer by showing that $1 - \frac{1}{\ell^{r-1}}$ is not a jump for r if $r \geq 3$ and $\ell > 2r$. Peng gave more sequences of non-jumping numbers for $r = 4$ and $r \geq 3$. However, there are also a lot of unknowns on determining whether a number is a jump for $r \geq 3$. Following a similar approach as that of Frankl and Rödl, we give several sequences of non-jumping numbers for $r = 5$, and extend one of the results to every $r \geq 5$, which generalize the above results.

Keywords: extremal problems in hypergraphs; Erdős jumping constant conjecture; Lagrangians of uniform graphs; non-jumping numbers

1 Introduction

For a given finite set V and a positive integer r, denote by $\binom{V}{r}$ the family of all r-subsets of V. Let $G = (V(G), E(G))$ be a graph with *vertex set* $V(G)$ and *edge set* $E(G)$. We call G an *r-uniform graph* if $E(G) \subseteq \binom{V(G)}{r}$. An r-uniform graph H

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is called a *subgraph* of an r-uniform graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Furthermore, H is called an *induced subgraph* of G if $E(H) = E(G) \cap \binom{V(H)}{r}$.

Let G be an r-uniform graph, we define the *density* of G as $\frac{|E(G)|}{|\binom{V(G)}{r}|}$, which is denoted by $d(G)$. Note that the density of a complete $(\ell+1)$-partite graph with partition classes of size m is greater than $1 - \frac{1}{\ell+1}$ (approaches $1 - \frac{1}{\ell+1}$ when $m \to \infty$). The density of a complete r-partite r-uniform graph with partition classes of size m is greater than $\frac{r!}{r^r}$ (approaches $\frac{r!}{r^r}$ when $m \to \infty$).

In [7], Katona, Nemetz and Simonovits showed that, for any r-uniform graph G, the average of densities of all induced subgraphs of G with $m \geq r$ vertices is $d(G)$. From this result we know that there exists a subgraph of G with m vertices, whose density is at least $d(G)$. A natural question is: for a constant $c > 0$, whether there exists a subgraph of G with m vertices and density at least $d(G) + c$? To be precise, the concept of "jump" was introduced.

Definition 1.1. A real number $\alpha \in [0, 1)$ is a jump for r if there exists a constant $c > 0$ such that for any $\varepsilon > 0$ and any integer m, $m \geq r$, there exists $n_0(\varepsilon, m)$ such that any r-uniform graph with $n > n_0(\varepsilon, m)$ vertices and density $\geq \alpha + \varepsilon$ contains a subgraph with m vertices and density $\geq \alpha + c$.

Erdős, Stone and Simonovits [2, 3] proved that every $\alpha \in [0, 1)$ is a jump for $r = 2$. This result can be easily obtained from the following theorem.

Theorem 1.1 ([3])**.** *Suppose ℓ is a positive integer. For any $\varepsilon > 0$ and any positive integer m, there exists $n_0(m, \varepsilon)$ such that any graph G on $n > n_0(m, \varepsilon)$ vertices with density $d(G) \geq 1 - \frac{1}{\ell} + \varepsilon$ contains a copy of the complete $(\ell+1)$-partite graph with partition classes of size m (i.e., there exists $\ell + 1$ pairwise disjoint sets $V_1, \ldots, V_{\ell+1}$, each of them with size m such that $\{x, y\}$ is an edge whenever $x \in V_i$ and $y \in V_j$ for some $1 \leq i < j \leq \ell + 1$).*

Moreover, from the following theorem, Erdős showed that for $r \geq 3$, every $\alpha \in [0, \frac{r!}{r^r})$ is a jump.

Theorem 1.2 ([1])**.** *For any $\varepsilon > 0$ and any positive integer m, there exists $n_0(\varepsilon, m)$ such that any r-uniform graph G on $n > n_0(\varepsilon, m)$ vertices with density $d(G) \geq \varepsilon$ contains a copy of the complete r-partite r-uniform graph with partition classes of size m (i.e., there exist r pairwise disjoint subsets $V_1, \ldots, V_r$, each of cardinality m such that $\{x_1, x_2, \ldots, x_r\}$ is an edge whenever $x_i \in V_i, 1 \leq i \leq r$).*

Furthermore, Erdős proposed the following jumping constant conjecture.

Conjecture 1.1. Every $\alpha \in [0, 1)$ is a jump for every integer $r \geq 2$.

Unfortunately, Frankl and Rödl [6] disproved this conjecture by showing the following result.

Theorem 1.3 ([6]). *Suppose $r \geq 3$ and $\ell > 2r$, then $1 - \frac{1}{\ell^{r-1}}$ is not a jump for r.*

Using the approach developed by Frankl and Rödl in [6], some other non-jump numbers were given. However, for $r \geq 3$, there are still a lot of unknowns on determining whether a given number is a jump. A well-known open question of Erdős is

whether $\frac{r!}{r^r}$ is a jump for $r \geq 3$ and what is the smallest non-jump?

In [5], another question was raised:

whether there is an interval of non-jumps for some $r \geq 3$?

Both questions seem to be very challenging. Regarding the first question, in [5], it was shown that $\frac{5r!}{2r^r}$ is a non-jump for $r \geq 3$ and it is the smallest known non-jump until now. Some efforts were made in finding more non-jumps for some $r \geq 3$. For $r = 3$, one more infinite sequence of non-jumps (converging to 1) was given in [5]. And for $r = 4$, several infinite sequences of non-jumps (converging to 1) were found in [9, 10, 12, 13]. Every non-jump in the above papers was extended to many sequences of non-jumps (still converging to 1) in [11, 15, 16]. Besides, in [14], Peng found an infinite sequence of non-jumps for $r = 3$ converging to $\frac{7}{12}$.

If a number α is a jump, then there exists a constant $c > 0$ such that every number in $[\alpha, \alpha+c)$ is a jump. As a direct result, we have that if there is a set of non-jumping numbers whose limits form an interval (a number a is a limit of a set A if there is a sequence $\{a_n\}_{n=1}^{\infty}, a_n \in A$ such that $lim_{n\to\infty} a_n = a$), then every number in this interval is not a jump. It is still an open problem whether such a "dense enough" set of non-jumping numbers exists or not.

In this paper, we intend to find more non-jumping numbers in addition to the known non-jumping numbers given in [5, 9, 10, 11, 12, 13, 15, 14, 16, 17]. Our approach is still based on the approach developed by Frankl and Rödl in [6]. We first consider the case $r = 5$ and find a sequence of non-jumping numbers. In Section 3, we prove the following result.

Theorem 1.4. *Let $\ell \geq 2$ be an integer. Then $1 - \frac{5}{\ell^3} + \frac{4}{\ell^4}$ is not a jump for $r = 5$.*

Then we extend Theorem 1.4 to Theorem 1.5 for the case $\ell = 5$ to every $r \geq 5$ in Section 4. When $r = 5$, Theorem 1.5 is exactly Theorem 1.4 for the case $\ell = 5$.

Theorem 1.5. *Let $r \geq 5$, $\frac{151r!}{6r^r}$ is not a jump for r.*

In [15], Peng gave the following result: for positive integers $p \geq r \geq 3$, if $\alpha \cdot \frac{r!}{r^r}$ is a non-jump for r, then $\alpha \cdot \frac{p!}{p^p}$ is a non-jump for p. Combining with the Theorem 1.5, we have the following corollary directly.

Corollary 1.1. Let $p \geq r \geq 5$ be positive integers. Then $\frac{151p!}{6p^p}$ is not a jump for p.

Since in [5], it was shown that $\frac{5r!}{2r^r}$ is a non-jumping number for $r \geq 3$. In [11], it was shown that for integers $r \geq 3$ and p, $3 \leq p \leq r$, $(1 - \frac{1}{p^{p-1}})\frac{p^p}{p!}\frac{r!}{r^r}$ is not a jump for r. In particular, $\frac{12}{125}$ (take $r = 5$ in $\frac{5r!}{2r^r}$), $\frac{96}{625}$ (take $p = 3$ and $r = 5$ in $(1 - \frac{1}{p^{p-1}})\frac{p^p}{p!}\frac{r!}{r^r}$) and $\frac{252}{625}$ (take $p = 4$ and $r = 5$ in $(1 - \frac{1}{p^{p-1}})\frac{p^p}{p!}\frac{r!}{r^r}$) are non-jumping numbers for $r = 5$. In Section 5, we will go back to the case of $r = 5$ and prove the following result.

Theorem 1.6. *Let $\ell \geq 2$ and $q \geq 1$ be integers. Then for $r = 5$, we have*

(a) If $q = 1$ or $q \geq 2\ell^2 + 2\ell$, then $1 - \frac{10}{\ell q} + \frac{35}{\ell^2 q^2} - \frac{50}{\ell^3 q^3} + \frac{4}{\ell^4 q^4} + \frac{10}{\ell q^4} - \frac{35}{\ell^2 q^4} + \frac{45}{\ell^3 q^4}$ is not a jump.

(b) If $q = 1$ or $q \geq 10\ell^3$, then $1 - \frac{10}{\ell q} + \frac{35}{\ell^2 q^2} - \frac{50}{\ell^3 q^3} + \frac{10}{\ell q^4} - \frac{35}{\ell^2 q^4} + \frac{50}{\ell^3 q^4} - \frac{1}{\ell^4 q^4}$ is not a jump.

(c) $1 - \frac{2}{q} + \frac{7}{5q^2} - \frac{2}{5q^3} + \frac{12}{125q^4}$ is not a jump.

(d) $1 - \frac{2}{q} + \frac{7}{5q^2} - \frac{2}{5q^3} + \frac{96}{625q^4}$ is not a jump.

(e) If $q = 1$ or $q \geq 3$, then $1 - \frac{2}{q} + \frac{7}{5q^2} - \frac{2}{5q^3} + \frac{252}{625q^4}$ is not a jump.

When $q = 1$, (a) reduces to Theorem 1.4 for $r = 5$, (b) reduces to Theorem 1.3 for $r = 5$, (c) shows that $\frac{12}{125}$ is not a jump for $r = 5$, (d) shows that $\frac{96}{625}$ is not a jump for $r = 5$, and (e) shows that $\frac{252}{625}$ is not a jump for $r = 5$.

2 Lagrangians and other tools

In this section, we introduce the definition of Lagrangian of an r-uniform graph and some other tools to be applied in the approach.

We first describe a definition of the Lagrangian of an r-uniform graph, which is a helpful tool in the approach. More studies of Lagrangians were given in [4, 6, 8, 18].

Definition 2.1. For an r-uniform graph G with vertex set $\{1, 2, \ldots, m\}$, edge set $E(G)$ and a vector $\vec{x} = \{x_1, \ldots, x_m\} \in R^m$, define

$$\lambda(G, \vec{x}) = \sum_{\{i_1,\ldots,i_r\} \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

x_i is called the weight of vertex i.

Definition 2.2. Let $S = \{\vec{x} = (x_1, x_2, \ldots, x_m) : \sum_{i=1}^m x_i = 1,\ x_i \geq 0 \text{ for } i = 1, 2, \ldots, m\}$. The Lagrangian of G, denoted by $\lambda(G)$, is defined as

$$\lambda(G) = max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.$$

A vector $\vec{x}$ is called an optimal vector for $\lambda(G)$ if $\lambda(G, \vec{x}) = \lambda(G)$.

We note that if G is a subgraph of an r-uniform graph H, then for any vector $\vec{x}$ in S, $\lambda(G, \vec{x}) \leq \lambda(H, \vec{x})$. The following fact is obtained directly.

Fact 2.1. Let G be a subgraph of an r-uniform graph H. Then

$$\lambda(G) \leq \lambda(H).$$

For an r-uniform graph G and $i \in V(G)$ we define G_i to be the $(r-1)$-uniform graph on $V - \{i\}$ with edge set $E(G_i)$ given by $e \in E(G_i)$ if and only if $e \cup \{i\} \in E(G)$.

We call two vertices i, j of an r-uniform graph G *equivalent* if for all $f \in \binom{V(G)-\{i,j\}}{r-1}$, $f \in E(G_i)$ if and only if $f \in E(G_j)$.

The following lemma given in [6] will be useful when calculating Lagrangians of some certain hypergraphs.

Lemma 2.1 ([6])**.** *Suppose G is an r-uniform graph on vertices $\{1, 2, \ldots, m\}$.*

1. If vertices $i_1, i_2, \ldots, i_t$ are pairwise equivalent, then there exists an optimal vector $\vec{y} = (y_1, y_2, \ldots, y_m)$ for $\lambda(G)$ such that $y_{i_1} = y_{i_2} = \cdots = y_{i_t}$.

2. Let $\vec{y} = (y_1, y_2, \ldots, y_m)$ be an optimal vector for $\lambda(G)$ and $y_i > 0$. Let $\hat{y}_i$ be the restriction of $\vec{y}$ on $\{1, 2, \ldots, m\} \backslash \{i\}$. Then $\lambda(G_i, \hat{y}_i) = r\lambda(G)$.

We also note that for an r-uniform graph G with m vertices, if we take $\vec{x} = (x_1, x_2, \ldots, x_m)$, where each $x_i = \frac{1}{m}$, then

$$\lambda(G) \geq \lambda(G_i, \vec{x}) = \frac{|E(G)|}{m^r} \geq \frac{d(G)}{r!} - \varepsilon$$

for $m \geq m'(\varepsilon)$.

On the other hand, we introduce a blow-up of an r-uniform graph G which allow us to construct an r-uniform graph with a large number of vertices and density close to $r!\lambda(G)$.

Definition 2.3. Let G be an r-uniform graph with $V(G) = \{1, 2, \ldots, m\}$ and $\vec{\boldsymbol{n}} = (n_1, \ldots, n_m)$ be a positive integer vector. Define the $\vec{\boldsymbol{n}}$ blow-up of G, $\vec{\boldsymbol{n}} \otimes G$ to be the m-partite r-uniform graph with vertex set $V_1 \cup \cdots \cup V_m$, $|V_i| = n_i$, $1 \leq i \leq m$, and edge set $E(\vec{\boldsymbol{n}} \otimes G) = \{\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\} :\ v_{i_k} \in V_{i_k} \text{ for } 1 \leq k \leq r,\ \{i_1, i_2, \ldots, i_r\} \in E(G)\}$.

In addition, we make the following easy remark given in [9].

Remark 2.1 ([9])**.** Let G be an r-uniform graph with m vertices and $\vec{y} = (y_1, y_2, \dots, y_m)$ be an optimal vector for $\lambda(G)$. Then for any $\varepsilon > 0$, there exists an integer $n_1(\varepsilon)$, such that for any integer $n \geq n_1(\varepsilon)$,

$$d((\lfloor ny_1 \rfloor, \lfloor ny_2 \rfloor, \dots, \lfloor ny_m \rfloor) \otimes G) \geq r!\lambda(G) - \varepsilon. \tag{1}$$

Let us also state a fact relating the Lagrangian of an r-uniform graph to the Lagrangian of its blow-up used in [6] ([5, 9, 10, 12] as well).

Fact 2.2 ([6])**.** If $n \geq 1$ and $\vec{\boldsymbol{n}} = (n, n, \dots, n)$, then $\lambda(\vec{\boldsymbol{n}} \otimes G) = \lambda(G)$ holds for every r-uniform graph G.

First, we state a definition as follows.

Definition 2.4. For $\alpha \in [0, 1)$ and a family $\mathcal{F}$ of r-uniform graphs, we say that α is a threshold for $\mathcal{F}$ if for any $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon)$ such that any r-uniform graph G with $d(G) \geq \alpha + \varepsilon$ and $|V(G)| > n_0$ contains some member of $\mathcal{F}$ as a subgraph. We denote this fact by $\alpha \to \mathcal{F}$.

The following lemma proved in [6] gives a necessary and sufficient condition for a number α to be a jump.

Lemma 2.2 ([6])**.** *The following two properties are equivalent.*

1. α is a jump for r.

2. $\alpha \to \mathcal{F}$ for some finite family $\mathcal{F}$ of r-uniform graphs satisfying $\lambda(F) > \frac{\alpha}{r!}$ for all $F \in \mathcal{F}$.

Lemma 2.3 ([6])**.** *For any $\sigma \geq 0$ and any integer $k \geq r$, there exists $t_0(k, \sigma)$ such that for every $t > t_0(k, \sigma)$, there exists an r-uniform graphs A satisfying:*

1. $|V(A)| = t$.

2. $|E(A)| \geq \sigma t^{r-1}$.

3. For all $V_0 \subset V(A), r \leq |V_0| \leq k$ we have $|E(A) \cap \binom{V_0}{r}| \leq |V_0| - r + 1$.

We sketch the approach in proving Theorems 1.4, 1.5, 1.6 as follows (similar to the proof in [9, 10, 12]): Let α be the non-jumping numbers described in those theorems. Assuming that α is a jump, we will derive a contradiction by the following two steps.

Step 1: Construct an r-uniform graph (in Theorem 1.4, 1.6, $r = 5$) with the Lagrangian close to but slightly smaller than $\frac{\alpha}{r!}$, then use Lemma 2.3 to add an r-uniform graph with a large enough number of edges but spare enough (see properties

2 and 3 in Lemma 2.3) and obtain an r-uniform graph with the Lagrangian $\geq \frac{\alpha}{r!} + \varepsilon$ for some positive ε. Then we "blow up" this r-uniform graph to an new r-uniform graph, say H, with a large enough number of vertices and density $> \alpha + \frac{\varepsilon}{2}$ (see Remark 2.1). By Lemma 2.2, if α is a jump then α is a threshold for some finite family $\mathcal{F}$ of r-uniform graphs with Lagrangian $> \frac{\alpha}{r!}$. So H must contain some member of $\mathcal{F}$ as a subgraph.

Step 2: We show that any subgraph of H with the number of vertices no more than $\max\{|\mathrm{V(F)}|, \mathrm{F} \in \mathcal{F}\}$ has Lagrangian $\leq \frac{\alpha}{r!}$ and derive a contradiction.

3 Proof of Theorem 1.4

In this section, we focus on $r = 5$ and give a proof of Theorem 1.4.

Let $\ell \geq 2$ and $\alpha = 1 - \frac{5}{\ell^3} + \frac{4}{\ell^4}$. Let t be a large enough integer given later. We first define a 5-uniform hypergraph $G(\ell, t)$ on ℓ pairwise disjoint sets $V_1, V_2, \ldots, V_\ell$, each of cardinality t whose density is close to α when t is large enough. The edge set of $G(\ell, t)$ consists of all 5-subsets taking exactly one vertex from each of V_i, V_j, V_k, V_h, V_s ($1 \leq i < j < k < h < s \leq \ell$), all 5-subsets taking two vertices from V_i and one vertex from each of V_j, V_k, V_h ($1 \leq i \leq \ell$, $1 \leq j < k < h \leq \ell$, $j, k, h \neq i$), all 5-subsets taking two vertices from each of V_i, V_j and one vertex from V_k ($1 \leq i < j \leq \ell$, $1 \leq k \leq \ell$, $k \neq i, j$), all 5-subsets taking three vertices from V_i, and one vertex from each of V_j, V_k ($1 \leq i \leq \ell$, $1 \leq j < k \leq \ell$, $j, k \neq i$), all 5-subsets taking three vertices from V_i and two vertices from V_j ($1 \leq i \leq \ell$, $1 \leq j \leq \ell$, $j \neq i$). When $\ell = 2, 3, 4$, some of them are vacant.

Note that

$$|E(G(\ell, t))| = \binom{\ell}{5} t^5 + \ell \binom{\ell - 1}{3} \binom{t}{2} t^3 + \binom{\ell}{2} (\ell - 2) \binom{t}{2} \binom{t}{2} t + \ell \binom{\ell - 1}{2} \binom{t}{3} t^2$$
$$+ \ell(\ell - 1) \binom{t}{3} \binom{t}{2} = \frac{\alpha}{120} \ell^5 t^5 - c_0(\ell) t^4 + o(t^4),$$

where $c_0(\ell)$ is positive (we omit giving the precise calculation here). It is easy to verify that the density of $G(\ell, t)$ is close to α if t is large enough. Corresponding to the ℓt vertices of $G(\ell, t)$, we take the vector $\vec{x} = (x_1, \ldots, x_{\ell t})$, where $x_i = \frac{1}{\ell t}$ for each i, $1 \leq i \leq \ell t$, then

$$\lambda(G(\ell, t)) \geq \lambda(G(\ell, t), \vec{x}) = \frac{|E(G(\ell, t))|}{(\ell t)^5} = \frac{\alpha}{120} - \frac{c_0(\ell)}{\ell^5 t} + o(\frac{1}{t}),$$

which is close to $\frac{\alpha}{120}$ when t is large enough. We will use Lemma 2.3 to add a 5-uniform graph to $G(\ell, t)$ so that the Lagrangian of the resulting graph is $> \frac{\alpha}{120} + \varepsilon(t)$

for some $\varepsilon(t) > 0$. Suppose that α is a jump for $r = 5$. According to Lemma 2.2, there exists a finite collection $\mathcal{F}$ of 5-uniform graphs satisfying:

i) $\lambda(F) > \frac{\alpha}{120}$ for all $F \in \mathcal{F}$, and

ii) α is a threshold for $\mathcal{F}$.

Set $k_0 = max_{F\in\mathcal{F}}|V(F)|$ and $\sigma_0 = 2c_0(\ell)$. Let $r = 5$ and $t_0(k_0, \sigma_0)$ be given as in Lemma 2.3. Take an integer $t > t_0$ and a 5-uniform hypergraph $A(k_0, \sigma_0, t)$ satisfying the three conditions in Lemma 2.3 with $V(A(k_0, \sigma_0, t)) = V_1$. The 5-uniform hypergraph $H(\ell, t)$ is obtained by adding $A(k_0, \sigma_0, t)$ to the 5-uniform hypegraph $G(\ell, t)$. For sufficiently large t, we have

$$\lambda(H(\ell,t)) \geq \frac{|E(H(\ell,t))|}{(\ell t)^5} \geq \frac{|E(G(\ell,t))| + \sigma_0 t^4}{(\ell t)^5} \geq \frac{\alpha}{120} + \frac{c_0(\ell)}{2\ell^5 t}.$$

Now suppose $\vec{y} = (y_1, y_2, \ldots, y_{\ell t})$ is an optimal vector of $\lambda(H(\ell, t))$. Let $\varepsilon = \frac{30c_0(\ell)}{\ell^5 t}$ and $n > n_1(\varepsilon)$ as in Remark 2.1. Then the 5-uniform graph $S_n = (\lfloor ny_1\rfloor, \ldots, \lfloor ny_{\ell t}\rfloor)\otimes H(\ell, t)$ has density not less than $\alpha + \varepsilon$. Since α is a threshold for $\mathcal{F}$, some member F of $\mathcal{F}$ is a subgraph of S_n for $n \geq max\{n_0(\varepsilon), n_1(\varepsilon)\}$. For such $F \in \mathcal{F}$, there exists a subgraph M of $H(\ell, t)$ with $|V(M)| \leq |V(F)| \leq k_0$, such that $F \subset \vec{\boldsymbol{n}} \otimes M$. By Fact 2.2 we have

$$\lambda(F) \leq \lambda(\vec{\boldsymbol{n}} \otimes M) = \lambda(M). \tag{2}$$

Lemma 3.1. *Let M be any subgraph of $H(\ell, t)$ with $|V(M)| \leq k_0$. Then*

$$\lambda(M) \leq \frac{\alpha}{120}$$

holds.

Applying Lemma 3.1 to (2), we have $\lambda(F) \leq \frac{\alpha}{120}$, which contradicts our choice of F, i.e., contradicts the fact that $\lambda(F) > \frac{\alpha}{120}$ for all $F \in \mathcal{F}$.

Proof of Lemma 3.1. By Fact 2.1, we may assume that M is an induced subgraph of $H(\ell, t)$. Let $U_i = V(M) \cap V_i$. Define $M_1 = (U_1, E(M) \cap \binom{U_1}{5})$, i.e., the subgraph of M induced on U_1. In view of Fact 2.1, it is enough to show Lemma 3.1 for the case $E(M_1) \neq \emptyset$. We assume $|V(M_1)| = 4 + d$ with d a positive integer. By Lemma 2.3, M_1 has at most d edges. Let $V(M_1) = \{v_1, v_2, \ldots, v_{4+d}\}$ and $\vec{\xi} = (x_1, x_2, \ldots, x_{4+d})$ be an optimal vector for $\lambda(M)$ where x_i is the weight of vertex v_i. We may assume $x_1 \geq x_2 \geq \ldots \geq x_{4+d}$. The following claim was proved (see Claim 4.4 in [6] there).

Claim 3.1. $\sum_{\{v_i,v_j,v_k,v_h,v_s\}\in E(M_1)} x_{v_i}x_{v_j}x_{v_k}x_{v_h}x_{v_s} \leq \sum_{5\leq i\leq 4+d} x_1x_2x_3x_4x_i.$

By Claim 3.1, we may assume that $E(M_1) = \{\{v_1, v_2, v_3, v_4, v_i\} : 5 \leq i \leq 4 + d\}$. Since v_1, v_2, v_3, v_4 are equivalent, in view of Lemma 2.1, we may assume that $x_1 =$

$x_2 = x_3 = x_4 \overset{\text{def}}{=} \rho$. For each i, let a_i be the sum of the weights of vertices of U_i. Notice that

$$\begin{cases} \sum_{i=1}^{\ell} a_i = 1, \\ a_i \geq 0,\ 1 \leq i \leq \ell \\ 0 \leq \rho \leq \frac{a_1}{4}. \end{cases}$$

Considering different types of edges in M and according to the definition of the Lagrangian, we have

$$\begin{aligned}
\lambda(M) \leq & \sum_{1\le i<j<k<h<s\le \ell} a_i a_j a_k a_h a_s + \frac{1}{2} \sum_{\substack{2\le i\le \ell; 1\le j<k<h\le \ell; \\ j,k,h\neq i}} {a_i}^2 a_j a_k a_h \\
& + \left(\sum_{2\le j<k<h\le \ell} a_j a_k a_h \right) \left[\frac{1}{2}(a_1 - 4\rho)^2 + 4\rho(a_1 - 4\rho) + 6\rho^2 \right] \\
& + \frac{1}{2} \left(\sum_{\substack{2\le j\le \ell; 2\le k\le \ell; \\ k\neq j}} a_j^2 a_k \right) \left[\frac{1}{2}(a_1 - 4\rho)^2 + 4\rho(a_1 - 4\rho) + 6\rho^2 \right] \\
& + \frac{1}{4} \sum_{\substack{2\le i<j\le \ell; 1\le k\le \ell; \\ k\neq i,j}} a_i^2 a_j^2 a_k + \frac{1}{6} \sum_{\substack{2\le i\le \ell; 1\le j<k\le \ell; \\ j,k\neq i}} a_i^3 a_j a_k + \rho^4 (a_1 - 4\rho) \\
& + \left(\sum_{2\le j<k\le \ell} a_j a_k \right) \left[\frac{1}{6}(a_1 - 4\rho)^3 + 2\rho(a_1 - 4\rho)^2 + 6\rho^2(a_1 - 4\rho) + 4\rho^3 \right] \\
& + \frac{1}{12} \sum_{\substack{2\le i\le \ell; 2\le j\le \ell; \\ j\neq i}} a_i^3 a_j^2 + \frac{1}{6} \left(\sum_{2\le i\le \ell} a_i^3 \right) \left[\frac{1}{2}(a_1 - 4\rho)^2 + 4\rho(a_1 - 4\rho) + 6\rho^2 \right] \\
& + \frac{1}{2} \left(\sum_{2\le j\le \ell} a_j^2 \right) \left[\frac{1}{6}(a_1 - 4\rho)^3 + 2\rho(a_1 - 4\rho)^2 + 6\rho^2(a_1 - 4\rho) + 4\rho^3 \right] \\
= & \sum_{1\le i<j<k<h<s\le \ell} a_i a_j a_k a_h a_s + \frac{1}{2} \sum_{\substack{1\le i\le \ell; 1\le j<k<h\le \ell; \\ j,k,h\neq i}} {a_i}^2 a_j a_k a_h \\
& + \frac{1}{4} \sum_{\substack{1\le i<j\le \ell; 1\le k\le \ell; \\ k\neq i,j}} a_i^2 a_j^2 a_k + \frac{1}{6} \sum_{\substack{1\le i\le \ell; 1\le j<k\le \ell; \\ j,k\neq i}} {a_i}^3 a_j a_k + \frac{1}{12} \sum_{\substack{1\le i\le \ell; 1\le j\le \ell; \\ j\neq i}} {a_i}^3 {a_j}^2 \\
& - 2\rho^2 \left(\sum_{2\le j<k<h\le \ell} a_j a_k a_h \right) - \rho^2 \left(\sum_{\substack{2\le j\le \ell; 2\le k\le \ell; \\ k\neq j}} a_j^2 a_k \right) - \frac{1}{3}\rho^2 \left(\sum_{2\le i\le \ell} a_i^3 \right)
\end{aligned}$$

$$
\begin{aligned}
&- a_1\rho^2\left(\sum_{2\le j\le \ell} a_j^2\right) - 2a_1\rho^2\left(\sum_{2\le j<k\le \ell} a_ja_k\right) + \frac{4}{3}\rho^3\left(\sum_{2\le j<k\le \ell} a_ja_k\right)\\
&+ \frac{2}{3}\rho^3\left(\sum_{2\le j\le \ell} a_j^2\right) + \rho^4(a_1-4\rho)\\
=& \sum_{1\le i<j<k<h<s\le \ell} a_ia_ja_ka_ha_s + \frac{1}{2}\sum_{\substack{1\le i\le \ell;1\le j<k<h\le \ell;\\ j,k,h\ne i}} {a_i}^2a_ja_ka_h\\
&+ \frac{1}{4}\sum_{\substack{1\le i<j\le \ell;1\le k\le \ell;\\ k\ne i,j}} a_i^2a_j^2a_k + \frac{1}{6}\sum_{\substack{1\le i\le \ell;1\le j<k\le \ell;\\ j,k\ne i}} {a_i}^3a_ja_k + \frac{1}{12}\sum_{\substack{1\le i\le \ell;1\le j\le \ell;\\ j\ne i}} {a_i}^3{a_j}^2\\
&- \frac{1}{3}\rho^2\left(\sum_{2\le i\le \ell} a_i\right)^3 - a_1\rho^2\left(\sum_{2\le i\le \ell} a_i\right)^2 + \frac{2}{3}\rho^3\left(\sum_{2\le i\le \ell} a_i\right)^2 + \rho^4\left(a_1-4\rho\right)\\
=& \sum_{1\le i<j<k<h<s\le \ell} a_ia_ja_ka_ha_s + \frac{1}{2}\sum_{\substack{1\le i\le \ell;1\le j<k<h\le \ell;\\ j,k,h\ne i}} a_i^2a_ja_ka_h\\
&+ \frac{1}{4}\sum_{\substack{1\le i<j\le \ell;1\le k\le \ell;\\ k\ne i,j}} a_i^2a_j^2a_k + \frac{1}{6}\sum_{\substack{1\le i\le \ell;1\le j<k\le \ell;\\ j,k\ne i}} {a_i}^3a_ja_k + \frac{1}{12}\sum_{\substack{1\le i\le \ell;1\le j\le \ell;\\ j\ne i}} {a_i}^3{a_j}^2\\
&+ \rho^2\left[a_1\rho^2 - 4\rho^3 + \left(\frac{2}{3}\rho - a_1\right)(1-a_1)^2 - \frac{1}{3}(1-a_1)^3\right]\\
\stackrel{\text{def}}{=}& f(a_1,a_2,\ldots,a_\ell,\rho). \qquad (3)
\end{aligned}
$$

Note that

$$f(\frac{1}{\ell},\frac{1}{\ell},\ldots,\frac{1}{\ell},0) = \frac{\alpha}{120}.$$

Therefore, to show Lemma 3.1, we just need to show the following claim:

Claim 3.2.

$$f(a_1,a_2,\ldots,a_\ell,\rho) \le f(\frac{1}{\ell},\frac{1}{\ell},\ldots,\frac{1}{\ell},0) = \frac{\alpha}{120}$$

holds under the constraints

$$\begin{cases} \sum\limits_{i=1}^{\ell} a_i = 1,\\ a_i \ge 0,\ 1\le i\le \ell\\ 0\le \rho \le \frac{a_1}{4}. \end{cases}$$

Claim 3.3. Let c be a positive number and $L \ge 2$ be an integer. Suppose that $\sum\limits_{i=1}^{L} c_i = c$ and each $c_i \ge 0$. Then the function

$$g(c_1,c_2,\ldots,c_L) \stackrel{\text{def}}{=} \sum_{1\le i<j<k<h<s\le L} c_ic_jc_kc_hc_s + \frac{1}{2}\sum_{\substack{1\le i\le L;1\le j<k<h\le L;\\ j,k,h\ne i}} c_i^2c_jc_kc_h$$
$$+\frac{1}{4}\sum_{\substack{1\le i<j\le L;1\le k\le L;\\ k\ne i,j}} c_i^2c_j^2c_k + \frac{1}{6}\sum_{\substack{1\le i\le L;1\le j<k\le L;\\ j,k\ne i}} {c_i}^3c_jc_k + \frac{1}{12}\sum_{\substack{1\le i\le L;1\le j\le L;\\ j\ne i}} {c_i}^3{c_j}^2,$$

reaches the maximum $\frac{1}{120}(1-\frac{5}{L^3}+\frac{4}{L^4})c^5$ when $c_1=c_2=\cdots=c_L=\frac{c}{L}$.

Proof. Since each term in function g has degree 5, we can assume that $c=1$. Suppose that g reaches the maximum at $(c_1,c_2,\ldots,c_L)$, we show that $c_1=c_2=\ldots=c_L=\frac{c}{L}$ must hold. If not, without loss of generality, assume that $c_2>c_1$, we will show that $g(c_1+\varepsilon,c_2-\varepsilon,c_3,\ldots,c_L)-g(c_1,c_2,c_3,\ldots,c_L)>0$ for small enough $\varepsilon>0$ and derive a contradiction. Notice that the summation of the terms in $g(c_1,c_2,\ldots,c_L)$ containing c_1,c_2 is

$$\begin{aligned}
&(c_1+c_2)\sum_{3\le i<j<k<h\le L} c_ic_jc_kc_h + c_1c_2\sum_{3\le i<j<k\le L} c_ic_jc_k\\
&+\frac{1}{2}(c_1^2+c_2^2)\sum_{3\le i<j<k\le L} c_ic_jc_k + \frac{1}{2}(c_1+c_2)\sum_{3\le i\le L;3\le j<k\le L;j,k\ne i} c_i^2c_jc_k\\
&+\frac{1}{2}(c_1^2c_2+c_2^2c_1)\sum_{3\le i<j\le L} c_ic_j + \frac{1}{2}c_1c_2\sum_{3\le i\le L;3\le j\le L;j\ne i} c_i^2c_j + \frac{1}{4}(c_1^2c_2^2)\sum_{3\le i\le L} c_i\\
&+\frac{1}{4}(c_1^2c_2+c_2^2c_1)\sum_{3\le i\le L} c_i^2 + \frac{1}{4}(c_1^2+c_2^2)\sum_{3\le i\le L;3\le j\le L;j\ne i} c_i^2c_j + \frac{1}{4}(c_1+c_2)\sum_{3\le i<j\le L} c_i^2c_j^2\\
&+\frac{1}{6}(c_1^3+c_2^3)\sum_{3\le i<j\le L} c_ic_j + \frac{1}{6}(c_1+c_2)\sum_{3\le i\le L;3\le j\le L;i\ne j} c_i^3c_j + \frac{1}{6}(c_1^3c_2+c_1c_2^3)\sum_{3\le i\le L} c_i\\
&+\frac{1}{6}c_1c_2\sum_{3\le i\le L} c_i^3 + \frac{1}{12}(c_1^3+c_2^3)\sum_{3\le i\le L} c_i^2 + \frac{1}{12}(c_1^2+c_2^2)\sum_{3\le i\le L} c_i^3 + \frac{1}{12}(c_1^3c_2^2+c_1^2c_2^3)\\
=\;&\frac{1}{24}(c_1+c_2)[(\sum_{3\le i\le L} c_i)^4 - \sum_{3\le i\le L} c_i^4] + \frac{1}{12}(c_1+c_2)^2(\sum_{3\le i\le L} c_i)^3\\
&+\frac{1}{12}(c_1+c_2)^3(\sum_{3\le i\le L} c_i)^2 + \frac{1}{12}c_1c_2(2c_1^2+2c_2^2+3c_1c_2)\sum_{3\le i\le L} c_i + \frac{1}{12}(c_1^3c_2^2+c_1^2c_2^3)\\
=\;&\frac{1}{24}(c_1+c_2)(1-c_1-c_2)^4 - \frac{1}{24}(c_1+c_2)\sum_{3\le i\le L} c_i^4\\
&+\frac{1}{12}(c_1+c_2)^2(1-c_1-c_2)^3 + \frac{1}{12}(c_1+c_2)^3(1-c_1-c_2)^2\\
&+\frac{1}{12}c_1c_2(2c_1^2+2c_2^2+3c_1c_2)(1-c_1-c_2) + \frac{1}{12}(c_1^3c_2^2+c_1^2c_2^3).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& g(c_1+\varepsilon, c_2-\varepsilon, c_3, \ldots, c_L) - g(c_1, c_2, c_3, \ldots, c_L) \\
= \ & \frac{1}{12}(c_1+\varepsilon)(c_2-\varepsilon)[2(c_1+\varepsilon)^2 + 2(c_2-\varepsilon)^2 + 3(c_1+\varepsilon)(c_2-\varepsilon)](1-c_1-c_2) \\
& + \frac{1}{12}(c_1+\varepsilon)^2(c_2-\varepsilon)^2(c_1+c_2) - \frac{1}{12}c_1c_2(2c_1^2+2c_2^2+3c_1c_2)(1-c_1-c_2) \\
& - \frac{1}{12}c_1^2c_2^2(c_1+c_2) \\
= \ & \frac{1}{6}(c_2-c_1)(c_1^2+c_2^2+c_1c_2)(1-c_1-c_2)\varepsilon + \frac{1}{6}c_1c_2(c_2-c_1)(c_1+c_2)\varepsilon + o(\varepsilon) > 0.
\end{aligned}$$

Since $c_2 > c_1$ and c_1c_2, $1-c_1-c_2$ cannot be equal to zero simultaneously due to the assumption that g reaches the maximum at $(c_1, c_2, \ldots, c_L)$. Therefore,

$$g(c_1+\varepsilon, c_2-\varepsilon, c_3, \ldots, c_L) - g(c_1, c_2, c_3, \ldots, c_L) > 0$$

for small enough $\varepsilon > 0$. This contradicts the assumption that g reaches the maximum at $(c_1, c_2, \ldots, c_L)$. ∎

Since $0 \le \rho \le \frac{a_1}{4}$, $a_1 - 4\rho \ge 0$, $(1-a_1)^2 \ge 0$, then we have,

$$\begin{aligned}
& \rho^2\left[a_1\rho^2 - 4\rho^3 + \left(\frac{2}{3}\rho - a_1\right)(1-a_1)^2 - \frac{1}{3}(1-a_1)^3\right] \\
\le \ & \rho^2\left[\frac{a_1^3}{16} - \frac{a_1^2}{4}\rho + \left(\frac{2}{3}\times\frac{a_1}{4} - a_1\right)(1-a_1)^2 - \frac{1}{3}(1-a_1)^3\right] \\
= \ & \rho^2\left[\frac{a_1^3}{16} - \frac{a_1^2}{4}\rho - \left(\frac{a_1}{2} + \frac{1}{3}\right)(1-a_1)^2\right] \\
= \ & \rho^2\left[\frac{1}{48}(-21a_1^3 + 32a_1^2 + 8a_1 - 16) - \frac{1}{4}a_1^2\rho\right].
\end{aligned}$$

Let $h(a_1) = -21a_1^3 + 32a_1^2 + 8a_1 - 16$, then, $h'(a_1) = -63a_1^2 + 64a_1 + 8$, $h''(a_1) = -126a_1 + 64$. So $h'(a_1)$ increases when $0 \le a_1 \le \frac{32}{63}$, $h'(a_1)$ decreases when $\frac{32}{63} \le a_1 \le 1$. Hence, $h'(a_1) \ge min\{h'(0), h'(1)\} > 0$, thus, $h(a_1)$ increases when $0 \le a_1 \le 1$. Note that $h(0) < 0$, $h(\frac{11}{15}) < 0$, $h(1) > 0$, when $0 \le a_1 \le \frac{11}{15}$, we have $\rho^2[a_1\rho^2 - 4\rho^3 + (\frac{2}{3}\rho - a_1)(1-a_1)^2 - \frac{1}{3}(1-a_1)^3] \le 0$, by Claim 3.3 and (3), we have $f(a_1, a_2, \ldots, a_\ell, \rho) \le g(a_1, a_2, \ldots, a_\ell) \le \frac{\alpha}{120}$. So Claim 3.2 holds for $0 \le a_1 \le \frac{11}{15}$. Therefore, we can assume that $\frac{11}{15} \le a_1 \le 1$. Since the geometric mean is not greater than the arithmetic mean, we have,

$$\rho^2\left[\frac{a_1^3}{16} - \frac{a_1^2}{4}\rho - \left(\frac{a_1}{2} + \frac{1}{3}\right)(1-a_1)^2\right] = \frac{64}{a_1^4}\left(\frac{a_1^2\rho}{8}\right)^2\left[\frac{a_1^3}{16} - \frac{a_1^2}{4}\rho - \left(\frac{a_1}{2} + \frac{1}{3}\right)(1-a_1)^2\right]$$

$$\leq \frac{64}{a_1^4}\left[\frac{\frac{a_1^3}{16}-\left(\frac{a_1}{2}+\frac{1}{3}\right)(1-a_1)^2}{3}\right]^3$$

$$< \frac{64}{a_1^4}\left(\frac{a_1^3}{16\times 3}\right)^3 \leq \frac{1}{1728}.$$

Combining with (3) we have

$$f(a_1,a_2,\ldots,a_\ell,\rho) \leq f(a_1,a_2,\ldots,a_\ell)$$

$$\stackrel{\text{def}}{=} \sum_{1\leq i<j<k<h<s\leq\ell} a_ia_ja_ka_ha_s + \frac{1}{2}\sum_{\substack{1\leq i\leq\ell;1\leq j<k<h\leq\ell;\\ j,k,h\neq i}} a_i^2a_ja_ka_h$$

$$+\frac{1}{4}\sum_{\substack{1\leq i<j\leq\ell;1\leq k\leq\ell;\\ k\neq i,j}} a_i^2a_j^2a_k + \frac{1}{6}\sum_{\substack{1\leq i\leq\ell;1\leq j<k\leq\ell;\\ j,k\neq i}} a_i^3a_ja_k + \frac{1}{12}\sum_{\substack{1\leq i\leq\ell;1\leq j\leq\ell;\\ j\neq i}} a_i^3a_j^2 + \frac{1}{1728}.$$

Therefore, to show Claim 3.2, it is sufficient to show the following claim:

Claim 3.4.

$$f(a_1,a_2,\ldots,a_\ell) \leq \frac{\alpha}{120}$$

holds under the constraints $\sum_{i=1}^{\ell} a_i = 1$, $a_1 \geq \frac{11}{15}$, and each $a_i \geq 0$.

In order to prove Claim 3.4, we need to prove the following claim first:

Claim 3.5.

$$h(a_2,a_3,\ldots,a_\ell) \stackrel{\text{def}}{=} \sum_{2\leq j<k<h<s\leq\ell} a_ja_ka_ha_s + \frac{1}{2}\sum_{\substack{2\leq j\leq\ell;2\leq k<h\leq\ell;\\ k,h\neq j}} a_j^2a_ka_h + \frac{1}{4}\sum_{2\leq j<k\leq\ell} a_j^2a_k^2$$

$$+\frac{1}{6}\sum_{\substack{2\leq j\leq\ell;2\leq k\leq\ell;\\ k\neq j}} {a_j}^3a_k,$$

reaches maximum $\frac{1}{24}(1-\frac{1}{(\ell-1)^3})c^4$ at $a_2 = a_3 = \ldots = a_\ell = \frac{c}{\ell-1}$ under the constraints $\sum_{i=1}^{\ell} a_i = c$, and each $a_i \geq 0$.

Proof of Claim 3.5. Since $h(a_2,a_3,\ldots,a_\ell)$ is a polynomial with degree 4 for each term, we just need to prove the claim for the case $c = 1$. Suppose that h reaches the maximum at $(c_2,c_3,\ldots,c_\ell)$, we show that $c_2 = c_3 = \ldots = c_\ell = \frac{1}{\ell-1}$. Otherwise, assume that $c_3 > c_2$, we will show that $h(c_2+\varepsilon,c_3-\varepsilon,c_4,\ldots,c_\ell)-h(c_2,c_3,\ldots,c_\ell) > 0$ for small enough $\varepsilon > 0$ and derive a contradiction. Notice that

$$
\begin{aligned}
& h(c_2+\varepsilon, c_3-\varepsilon, c_4, \ldots, c_\ell) - h(c_2, c_3, c_4, \ldots, c_\ell) \\
= \; & [(c_2+\varepsilon)(c_3-\varepsilon) - c_2c_3] \sum_{4\le j<k\le \ell} c_jc_k \\
& + \frac{1}{2}[(c_2+\varepsilon)^2 + (c_3-\varepsilon)^2 - c_2^2 - c_3^2] \sum_{4\le j<k\le \ell} c_jc_k + \frac{1}{2}[(c_2+\varepsilon)(c_3-\varepsilon) - c_2c_3] \sum_{4\le j\le \ell} c_j^2 \\
& + \frac{1}{2}[(c_2+\varepsilon)^2(c_3-\varepsilon) + (c_3-\varepsilon)^2(c_2+\varepsilon) - c_2^2c_3 - c_2c_3^2] \sum_{4\le j\le \ell} c_j \\
& + \frac{1}{4}[(c_2+\varepsilon)^2 + (c_3-\varepsilon)^2 - c_2^2 - c_3^2] \sum_{4\le j\le \ell} c_j^2 + \frac{1}{4}[(c_2+\varepsilon)^2(c_3-\varepsilon)^2 - c_2^2c_3^2] \\
& + \frac{1}{6}[(c_2+\varepsilon)^3 + (c_3-\varepsilon)^3 - c_2^3 - c_3^3] \sum_{4\le j\le \ell} c_j \\
& + \frac{1}{6}[(c_2+\varepsilon)^3(c_3-\varepsilon) + (c_3-\varepsilon)^3(c_2+\varepsilon) - c_2^3c_3 - c_3^3c_2] \\
= \; & \frac{1}{6}(c_3^3 - c_2^3)\varepsilon + o(\varepsilon) > 0,
\end{aligned}
$$

for small enough $\varepsilon > 0$ and get a contradiction. ∎

Proof of Claim 3.4. We will apply Claims 3.3 and 3.5. Separating the terms containing a_1 from the terms not containing a_1, we write function $f(a_1, a_2, \ldots, a_\ell)$ as follows:

$$
\begin{aligned}
& f(a_1, a_2, \ldots, a_\ell) \\
= & \sum_{2\le i<j<k<h<s\le \ell} a_ia_ja_ka_ha_s + \frac{1}{2} \sum_{\substack{2\le i\le \ell; 2\le j<k<h\le \ell;\\ j,k,h\ne i}} a_i^2a_ja_ka_h \\
& + \frac{1}{4} \sum_{\substack{2\le i<j\le \ell; 2\le k\le \ell;\\ k\ne i,j}} a_i^2a_j^2a_k + \frac{1}{6} \sum_{\substack{2\le i\le \ell; 2\le j<k\le \ell;\\ j,k\ne i}} {a_i}^3a_ja_k + \frac{1}{12} \sum_{\substack{2\le i\le \ell; 2\le j\le \ell;\\ j\ne i}} {a_i}^3a_j^2 \\
& + a_1\Big(\sum_{2\le j<k<h<s\le \ell} a_ja_ka_ha_s + \frac{1}{2} \sum_{\substack{2\le j\le \ell; 2\le k<h\le \ell;\\ k,h\ne j}} a_j^2a_ka_h + \frac{1}{4} \sum_{2\le j<k\le \ell} a_j^2a_k^2 \\
& + \frac{1}{6} \sum_{\substack{2\le j\le \ell; 2\le k\le \ell;\\ k\ne j}} {a_j}^3a_k\Big) + \frac{1}{2}a_1^2\Big(\sum_{2\le j<k<h\le \ell} a_ja_ka_h\Big) + \frac{1}{4}a_1^2\Big(\sum_{\substack{2\le j\le \ell; 2\le k\le \ell;\\ k\ne j}} a_j^2a_k\Big) \\
& + \frac{1}{6}a_1^3\Big(\sum_{2\le j<k\le \ell} a_ja_k\Big) + \frac{1}{12}a_1^3\Big(\sum_{2\le j\le \ell} a_j^2\Big) + \frac{1}{12}a_1^2\Big(\sum_{2\le j\le \ell} a_j^3\Big) + \frac{1}{1728} \\
= & \sum_{2\le i<j<k<h<s\le \ell} a_ia_ja_ka_ha_s + \frac{1}{2} \sum_{\substack{2\le i\le \ell; 2\le j<k<h\le \ell;\\ j,k,h\ne i}} a_i^2a_ja_ka_h
\end{aligned}
$$

$$+\frac{1}{4}\sum_{\substack{2\le i<j\le\ell;2\le k\le\ell;\\ k\neq i,j}} a_i^2a_j^2a_k+\frac{1}{6}\sum_{\substack{2\le i\le\ell;2\le j<k\le\ell;\\ j,k\neq i}} {a_i}^3a_ja_k+\frac{1}{12}\sum_{\substack{2\le i\le\ell;2\le j\le\ell;\\ j\neq i}} {a_i}^3a_j^2$$

$$+a_1(\sum_{2\le j<k<h<s\le\ell} a_ja_ka_ha_s+\frac{1}{2}\sum_{\substack{2\le j\le\ell;2\le k<h\le\ell;\\ k,h\neq j}} a_j^2a_ka_h+\frac{1}{4}\sum_{2\le j<k\le\ell} a_j^2a_k^2$$

$$+\frac{1}{6}\sum_{\substack{2\le j\le\ell;2\le k\le\ell;\\ k\neq j}} {a_j}^3a_k)+\frac{1}{12}a_1^3(\sum_{2\le j\le\ell} a_j)^2+\frac{1}{12}a_1^2(\sum_{2\le j\le\ell} a_j)^3+\frac{1}{1728}.$$

Applying Claim 3.3 by taking $L=\ell-1$ variables $a_2,a_3,\ldots,a_\ell$ and $c=1-a_1$, Claim 3.5 and $\frac{1}{12}a_1^2(\sum\limits_{2\le j\le\ell} a_j)^3+\frac{1}{12}a_1^3(\sum\limits_{2\le j\le\ell} a_j)^2=\frac{1}{12}a_1^2(1-a_1)^3+\frac{1}{12}a_1^3(1-a_1)^2=\frac{1}{12}a_1^2(1-a_1)^2$, we have

$$f(a_1,a_2,\ldots,a_\ell)\le f(a_1)\stackrel{\text{def}}{=}\frac{1}{120}\left[1-\frac{5}{(\ell-1)^3}+\frac{4}{(\ell-1)^4}\right](1-a_1)^5$$
$$+\frac{1}{24}\left[1-\frac{1}{(\ell-1)^3}\right](1-a_1)^4a_1+\frac{1}{12}a_1^2(1-a_1)^2+\frac{1}{1728}.$$

Therefore, to show Claim 3.4, we need to show the following claim:

Claim 3.6.

$$f(a_1)\le\frac{\alpha}{120}$$

holds when $\frac{11}{15}\le a_1\le 1$.

Proof. By a direct calculation,

$$f'(a_1)=\frac{1}{6}\left[\frac{1}{(\ell-1)^3}-\frac{1}{(\ell-1)^4}\right](1-a_1)^4+\frac{1}{6(\ell-1)^3}(1-a_1)^3a_1-\frac{1}{6}a_1^3(1-a_1),$$

$$f''(a_1)=\left[\frac{2}{3(\ell-1)^4}-\frac{1}{2(\ell-1)^3}\right](1-a_1)^3-\frac{1}{2(\ell-1)^3}(1-a_1)^2a_1+\frac{2}{3}a_1^3-\frac{1}{2}a_1^2,$$

$$f^{(3)}(a_1)=\left[\frac{1}{(\ell-1)^3}-\frac{2}{(\ell-1)^4}\right](1-a_1)^2+\frac{1}{(\ell-1)^3}(1-a_1)a_1+2a_1^2-a_1,$$

$$f^{(4)}(a_1)=\left[4-\frac{4}{(\ell-1)^4}\right]a_1-1-\frac{1}{(\ell-1)^3}+\frac{4}{(\ell-1)^4},$$

Note that $f^{(4)}(a_1)>0$, when $\frac{11}{15}\le a_1\le 1$, so $f^{(3)}(a_1)$ increases when $\frac{11}{15}\le a_1\le 1$. By a direct calculation, $f^{(3)}(\frac{11}{15})>0$, so $f''(a_1)$ increases when $\frac{11}{15}\le a_1\le 1$. Since we have $f''(\frac{11}{15})<0$, $f''(1)>0$, thus, $f'(a_1)\le max\{f'(\frac{11}{15}),f'(1)\}$. By a direct calculation, $f'(\frac{11}{15})<0$, $f'(1)=0$, so $f(a_1)$ is a decreasing function when $\frac{11}{15}\le a_1\le 1$. When $\ell=2$, $f(\frac{11}{15})=\frac{1}{12}\times\frac{11^2\times 4^2}{15^4}+\frac{1}{1728}<\frac{1}{120}\times\frac{5}{8}=\frac{\alpha}{120}$. If $\ell\ge 3$, since $1-\frac{5}{(\ell-1)^3}+\frac{4}{(\ell-1)^4}\ge 1-\frac{5}{(2)^3}+\frac{4}{(2)^4}$, then we have $f(\frac{11}{15})=\frac{1}{120}[1-\frac{5}{(\ell-1)^3}+\frac{4}{(\ell-1)^4}]-(1-$

$\frac{4^5}{15^5}) \times \frac{1}{120}(1 - \frac{5}{(\ell)^3} + \frac{4}{(\ell)^4}) + \frac{1}{24}[1 - \frac{1}{(\ell-1)^3}] \times \frac{11\times 4^4}{15^5} + \frac{1}{12} \times \frac{11^2\times 4^2}{15^4} + \frac{1}{1728} \leq \frac{1}{120}[1 - \frac{5}{(\ell-1)^3} + \frac{4}{(\ell-1)^4}] - (1 - \frac{4^5}{15^5}) \times \frac{1}{120}(1 - \frac{5}{2^3} + \frac{4}{2^4}) + \frac{1}{24} \times \frac{11\times 4^4}{15^5} + \frac{1}{12} \times \frac{11^2\times 4^2}{15^4} + \frac{1}{1728} \leq \frac{1}{120}[1 - \frac{5}{(\ell-1)^3} + \frac{4}{(\ell-1)^4}]$. So, $f(a_1) \leq f(\frac{11}{15}) \leq \frac{1}{120}[1 - \frac{5}{(\ell-1)^3} + \frac{4}{(\ell-1)^4}] < \frac{1}{120}[1 - \frac{5}{\ell^3} + \frac{4}{\ell^4}] = \frac{\alpha}{120}$. This completes the proof of Claim 3.6. ∎

Applying Claim 3.2 to (3), we have

$$\lambda(M) \leq \frac{\alpha}{120}.$$

This completes the proof of Lemma 3.1. ∎

4 Proof of Theorem 1.5

Theorem 1.5 extends Theorem 1.4 for the case $\ell = 5$ to every integer $r \geq 5$. The proof is based on an extension of the 5-uniform graph $H(\ell, t)$ in Section 3 for the case $\ell = 5$.

Suppose that $\frac{151r!}{6r^r}$ is a jump for $r \geq 5$. In view of Lemma 2.2, there exists a finite collection $\mathcal{F}$ of r-uniform graphs satisfying the following:

i) $\lambda(F) > \frac{151}{6r^r}$ for all $F \in \mathcal{F}$, and

ii) $\frac{151r!}{6r^r}$ is a threshold for $\mathcal{F}$.

Set $k_0 = max_{F\in\mathcal{F}} |V(F)|$ and $\sigma_0 = 2c_0(\ell)$ be the number defined as in the above. Let $r = 5$ and $t_0(k_0, \sigma_0)$ be given as in Lemma 2.3. Take an integer $t > t_0$ and a 5-uniform hypergraph $H(5, t)$ (i.e. $\ell = 5$) the same way as in the above with the new k_0. For simplicity, we write $H(5, t)$ as $H(t)$.

Since Theorem 1.4 holds, we may assume that $r \geq 6$.

Based on the 5-uniform graph $H(t)$, we construct an r-uniform graph $H^{(r)}(t)$ on r pairwise disjoint sets $V_1, V_2, V_3, V_4, V_5, \ldots, V_r$, each with order t by taking the edge set $\{u_1, u_2, u_3, u_4, u_5, \ldots, u_r\}$, where $\{\{u_1, u_2, u_3, u_4, u_5\}$ is an edge in $H(t)$ and for each j, $6 \leq j \leq r$, $u_j \in V_j\}$. Notice that

$$|E(H^{(r)}(t))| = t^{r-5}|E(H(t))|.$$

Take $\ell = 5$, we get

$$|E(H(t))| \geq \frac{151}{6}t^5 + \frac{c_0(\ell)t^4}{2}.$$

Hence, we have

$$\lambda(H^{(r)}(t)) \geq \frac{|E(H^{(r)}(t))|}{(rt)^r} \geq \frac{151}{6r^r} + \frac{c_0(\ell)}{2r^r t}.$$

Similar as the case that Theorem 1.4 follows from Lemma 3.1, we have that Theorem 1.5 follows from the following lemma.

Lemma 4.1. *Let $M^{(r)}$ be a subgraph of $H^{(r)}(t)$ with $|V(M^{(r)})| \leq k_0$. Then*

$$\lambda(M^{(r)}) \leq \frac{151}{6r^r}$$

holds.

Proof. In view of Fact 2.1, we may assume that $M^{(r)}$ is a non-empty induced subgraph of $H^{(r)}(t)$. Define $U_i = V(M) \cap V_i$ for $1 \leq i \leq r$. Let $M^{(5)}$ be the 5-uniform graph defined on $\bigcup_{i=1}^{5} U_i$. The edge set of $M^{(5)}$ consists of all 5-sets of the form of $e \cap (\bigcup_{i=1}^{5} U_i)$, where e is an edge of $M^{(r)}$. Let $\vec{\xi}$ be an optimal vector for $\lambda(M^{(r)})$. Let $\vec{\xi}^{(5)}$ be the restriction of $\vec{\xi}$ to $U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5$. Let a_i be the sum of the weights of vertices of U_i, $1 \leq i \leq r$, respectively.

According to the relationship between $M^{(r)}$ and $M^{(5)}$, we have

$$\lambda(M^{(r)}) = \lambda(M^{(5)}, \vec{\xi}^{(5)}) \times \prod_{i=6}^{r} a_i.$$

Applying Lemma 3.1 with $\ell = 5$ and observing that $\sum_{i=1}^{5} a_i = 1 - \sum_{i=6}^{r} a_i$, we obtain that,

$$\begin{aligned}\lambda(M^{(r)}) &\leq \frac{1}{120} \times \frac{604}{5^4}(1 - \sum_{i=6}^{r} a_i)^5 \prod_{i=6}^{r} a_i \leq \frac{1}{120} \times \frac{604}{5^4} \times 5^5 \times \left(\frac{1 - \sum_{i=6}^{r} a_i}{5}\right)^5 \prod_{i=6}^{r} a_i \\ &= \frac{1}{120} \times \frac{604}{5^4} \times 5^5 \times \left(\frac{1}{r}\right)^r = \frac{151}{6r^r}.\end{aligned}$$

This completes the proof of Lemma 4.1. ∎

5 Proof of Theorem 1.6

In this section, we focus on $r = 5$ and prove the following Theorem, which implies Theorem 1.6.

Theorem 5.1. *Let $\ell \geq 2$, $q \geq 1$ be integers. Let $N(\ell)$ be any of the five numbers given below.*

$$N(\ell) = \begin{cases} 1 - \frac{5}{\ell^3} + \frac{4}{\ell^4}, \ \textit{or} \\ 1 - \frac{1}{\ell^4}, \ \textit{or} \\ \frac{12}{125} \ (\textit{in this case, view } \ell = 5), \ \textit{or} \\ \frac{96}{625} \ (\textit{in this case, view } \ell = 5), \ \textit{or} \\ \frac{252}{625} \ (\textit{in this case, view } \ell = 5). \end{cases} \tag{4}$$

Then

$$N(\ell, q) = 1 - \frac{10}{\ell q} + \frac{35}{\ell^2 q^2} - \frac{50}{\ell^3 q^3} + \frac{10}{\ell q^4} - \frac{35}{\ell^2 q^4} + \frac{50}{\ell^3 q^4} - \frac{1}{q^4} + \frac{N(\ell)}{q^4} \tag{5}$$

is not a jump for 5 provided

$$q = 1 \ \textit{or} \ \ell^3(1 - N(\ell))(q^3 + q^2 + q + 1) - 10\ell^2(q^2 + q + 1) + 35\ell(q + 1) - 50 \geq 0 \tag{6}$$

holds.

Now let us explain why Theorem 5.1 implies Theorem 1.6.

If $N(\ell) = \alpha$, then

$$\begin{aligned} & \ell^3(1 - N(\ell))(q^3 + q^2 + q + 1) - 10\ell^2(q^2 + q + 1) + 35\ell(q + 1) - 50 \\ = \ & \ell^3\left(\frac{5}{\ell^3} - \frac{4}{\ell^4}\right)(q^3 + q^2 + q + 1) - 10\ell^2(q^2 + q + 1) + 35\ell(q + 1) - 50 \\ = \ & \frac{1}{\ell}[(5\ell - 4)q^3 + (5\ell - 10\ell^3 - 4)q^2 + (5\ell - 10\ell^3 + 35\ell^2 - 4)q \\ & + (-45\ell - 10\ell^3 + 35\ell^2 - 4)] \\ \stackrel{\text{def}}{=} \ & f_1(q) \end{aligned}$$

is an increasing function of q when $q \geq 2\ell^2 + 2\ell$ and $f_1(2\ell^2 + 2\ell) > 0$. Therefore, when $q \geq 2\ell^2 + 2\ell$, (6) is satisfied. Applying Theorem 5.1, we get Part (a) of Theorem 1.6.

If $N(\ell) = 1 - \frac{1}{\ell^4}$, then

$$\begin{aligned} & \ell^3(1 - N(\ell))(q^3 + q^2 + q + 1) - 10\ell^2(q^2 + q + 1) + 35\ell(q + 1) - 50 \\ = & \ell^3\left(\frac{1}{\ell^4}\right)(q^3 + q^2 + q + 1) - 10\ell^2(q^2 + q + 1) + 35\ell(q + 1) - 50 \\ = & \frac{1}{\ell}[q^3 - (10\ell^3 - 1)q^2 - (10\ell^3 - 35\ell^2 - 1)q + (1 - 10\ell^3 + 35\ell^2 - 50\ell)] \\ \stackrel{\text{def}}{=} & f_2(q) \end{aligned}$$

is an increasing function of q when $q \geq 7\ell^3$ and $f_2(10\ell^3) > 0$. Therefore, when $q \geq 10\ell^3$, (6) is satisfied. Applying Theorem 5.1, we get Part (b) of Theorem 1.6.

If $\ell = 5$ and $N(\ell) = \frac{12}{125}$, then

$$\begin{aligned}&\ell^3(1-N(\ell))(q^3+q^2+q+1)-10\ell^2(q^2+q+1)+35\ell(q+1)-50\\=&113q^3-137q^2+38q-12\\\stackrel{\text{def}}{=}&f_3(q)\end{aligned}$$

is an increasing function of q when $q \geq 1$ and $f_3(2) > 0$. Therefore, (6) is satisfied. Applying Theorem 5.1, we get Part (c) of Theorem 1.6.

If $\ell = 5$ and $N(\ell) = \frac{96}{625}$, then

$$\begin{aligned}&\ell^3(1-N(\ell))(q^3+q^2+q+1)-10\ell^2(q^2+q+1)+35\ell(q+1)-50\\=&\frac{1}{5}(529q^3-721q^2+154q-96)\\\stackrel{\text{def}}{=}&f_4(q)\end{aligned}$$

is an increasing function of q when $q \geq 1$ and $f_4(2) > 0$. Therefore, (6) is satisfied. Applying Theorem 5.1, we get Part (d) of Theorem 1.6.

If $\ell = 5$ and $N(\ell) = \frac{252}{625}$, then

$$\begin{aligned}&\ell^3(1-N(\ell))(q^3+q^2+q+1)-10\ell^2(q^2+q+1)+35\ell(q+1)-50\\=&\frac{1}{5}(373q^3-877q^2-2q-252)\\\stackrel{\text{def}}{=}&f_5(q)\end{aligned}$$

is an increasing function of q when $q \geq 2$ and $f_5(3) > 0$. Therefore, when $q \geq 3$, (6) is satisfied. Applying Theorem 5.1, we get Part (e) of Theorem 1.6.

Now we give the proof of Theorem 5.1.

Proof of Theorem 5.1. Let integers ℓ, q and numbers $N(\ell)$ and $N(\ell, q)$ be given as in Theorem 5.1. We will show that $N(\ell, q)$ is not a jump for 5. Let t be a fixed large enough integer determined later. We first define a 5-uniform hypergraph $G(\ell, t)$ on ℓ pairwise disjoint sets $V_1, \ldots, V_\ell$, each of them with size t and the density of $G(\ell, t)$ is close to $N(\ell)$ when t is large enough. Each of five choices of $N(\ell)$ corresponds to a construction.

1. If $N(\ell) = \alpha$, then $G(\ell, t)$ is defined in section 3. Notice that

$$d(G(\ell,t))=\frac{\binom{\ell}{5}t^5+\ell\binom{\ell-1}{3}\binom{t}{2}t^3+\binom{\ell}{2}(\ell-2)\binom{t}{2}\binom{t}{2}t+\ell\binom{\ell-1}{2}\binom{t}{3}t^2+\ell(\ell-1)\binom{t}{3}\binom{t}{2}}{\binom{\ell t}{5}}$$

which is close to α if t is large enough.

2. If $N(\ell) = 1 - \frac{1}{\ell^4}$, then $G(\ell, t)$ is defined on ℓ pairwise disjoint sets $V_1, V_2, \ldots, V_\ell$, where $|V_i| = t$, and the edge set of $G(\ell, t)$ is $\binom{\cup_{i=1}^{\ell} V_i}{5} - \cup_{i=1}^{\ell} \binom{V_i}{5}$. Notice that

$$d(G(\ell, t)) = \frac{\binom{\ell t}{5} - \ell \binom{t}{5}}{\binom{\ell t}{5}}$$

which is close to $1 - \frac{1}{\ell^4}$ if t is large enough.

3. If $N(5) = \frac{12}{125}$ (in this case, view $\ell = 5$), then $G(5, t)$ is defined on 5 pairwise disjoint sets V_1, V_2, V_3, V_4, V_5, where $|V_i| = t$, and the edge set of $G(5, t)$ consists of all 5-sets in the form of $\{\{a, b, c, v_4, v_5\}$, where $a \in V_1,\ b \in V_2,\ c \in V_3$ and $v_4 \in V_4,\ v_5 \in V_5\}$, or $\{\{a, b, c, v_4, v_5\}$, where $\{a, b\} \in \binom{V_1}{2},\ c \in V_2$ and $v_4 \in V_4,\ v_5 \in V_5\}$, or $\{\{a, b, c, v_4, v_5\}$, where $\{a, b\} \in \binom{V_2}{2},\ c \in V_3$ and $v_4 \in V_4,\ v_5 \in V_5\}$, or $\{\{a, b, c, v_4, v_5\}$, where $\{a, b\} \in \binom{V_3}{2},\ c \in V_1$ and $v_4 \in V_4,\ v_5 \in V_5\}$. Notice that

$$d(G(5, t)) = \frac{t^5 + 3\binom{t}{2}t^3}{\binom{5t}{5}}$$

which is close to $\frac{12}{125}$ if t is large enough.

4. If $N(5) = \frac{96}{625}$ (in this case, view $\ell = 5$), then $G(5, t)$ is defined on 5 pairwise disjoint sets V_1, V_2, V_3, V_4, V_5, where $|V_i| = t$, and the edge set of $G(5, t)$ consists of all 5-sets in the form of $\{\{v_1, v_2, v_3, v_4, v_5\}$, where $\{v_1, v_2, v_3\} \in \binom{\cup_{i=1}^{3} V_i}{3} - \cup_{i=1}^{3} \binom{V_i}{3}$, and $v_4 \in V_4,\ v_5 \in V_5\}$. Notice that

$$d(G(5, t)) = \frac{(\binom{3t}{3} - 3\binom{t}{3})t^2}{\binom{5t}{5}}$$

which is close to $\frac{96}{625}$ if t is large enough.

5. If $N(5) = \frac{252}{625}$ (in this case, view $\ell = 5$), then $G(5, t)$ is defined on 5 pairwise disjoint sets V_1, V_2, V_3, V_4, V_5, where $|V_i| = t$, and the edge set of $G(5, t)$ consists of all 5-sets in the form of $\{\{v_1, v_2, v_3, v_4, v_5\}$, where $\{v_1, v_2, v_3, v_4\} \in \binom{\cup_{i=1}^{4} V_i}{4} - \cup_{i=1}^{4} \binom{V_i}{4}$, and $v_5 \in V_5\}$. Notice that

$$d(G(5, t)) = \frac{(\binom{4t}{4} - 4\binom{t}{4})t}{\binom{5t}{5}}$$

which is close to $\frac{252}{625}$ if t is large enough.

We also note that

$$\frac{|E(G(\ell, t))| + \frac{1}{12}\ell^4 t^4}{(\ell t)^5} \geq \frac{1}{120}(N(\ell) + \frac{1}{\ell^5 t}) \tag{7}$$

holds for $t \geq t_1$.

The 5-uniform graph $G(\ell, q, t)$ on ℓq pairwise disjoint sets V_i, $1 \le i \le \ell q$, each of them with size t is obtained as follows: for each p, $0 \le p \le q-1$, take a copy of $G(\ell, t)$ on the vertex set $\cup_{p\ell+1 \le j \le (p+1)\ell} V_j$, then add all other edges (not entirely in any copy of $G(\ell, t)$) in the form of $\{\{v_{j_1}, v_{j_2}, v_{j_3}, v_{j_4}, v_{j_5}\}$, where $1 \le j_1 < j_2 < j_3 < j_4 < j_5 \le \ell q$ and $v_{j_k} \in V_{j_k}$ for $1 \le k \le 5\}$. We will use Lemma 2.3 to add a 5-uniform graph to $G(\ell, q, t)$ so that the Lagrangian of the resulting graph is $> \frac{N(\ell,q)}{120} + \varepsilon(t)$ for some $\varepsilon(t) > 0$. The precise argument is given below.

Suppose that $N(\ell, q)$ is a jump for $r = 5$. By Lemma 2.2, there exists a finite collection $\mathcal{F}$ of 5-uniform graphs satisfying the following:

i) $\lambda(F) > \frac{N(\ell,q)}{120}$ for all $F \in \mathcal{F}$, and

ii) $N(\ell, q)$ is a threshold for $\mathcal{F}$.

Assume that $r = 5$ and set $k_1 = max_{F \in \mathcal{F}} |V(F)|$ and $\sigma_1 = \frac{1}{12}\ell^4 q$. Let $t_0(k_1, \sigma_1)$ be given as in Lemma 2.3. Fix an integer $t > \max(\mathrm{t_0}, \mathrm{t_1})$, where t_1 is the number from (7).

Take a 5-uniform graph $A_{k_1,\sigma_1}(t)$ satisfying the conditions in Lemma 2.3 with $V(A_{k_1,\sigma_1}(t)) = V_1$. The 5-uniform hypergraph $H(\ell, q, t)$ is obtained by adding $A_{k_1,\sigma_1}(t)$ to the 5-uniform hypergraph $G(\ell, q, t)$. Now we give a lower bound of $\lambda(H(\ell, q, t))$. Notice that,

$$\lambda(H(\ell, q, t)) \ge \frac{|E(H(\ell, q, t))|}{(\ell q t)^5}.$$

In view of the construction of $H(\ell, q, t)$, we have

$$\frac{|E(H(\ell, q, t))|}{(\ell q t)^5} = \frac{|E(G(\ell, q, t))| + \sigma_1 t^4}{(\ell q t)^5}$$

$$= \frac{q|E(G(\ell, t))| + \frac{1}{12}\ell^4 q t^4 + (\binom{\ell q}{5} - q\binom{\ell}{5})t^5}{(\ell q t)^5}$$

$$= \frac{q|E(G(\ell, t))| + \frac{1}{12}\ell^4 q t^4}{(\ell q t)^5} + \frac{1}{120}(1 - \frac{10}{\ell q} + \frac{35}{\ell^2 q^2} - \frac{50}{\ell^3 q^3} - \frac{1}{q^4} + \frac{10}{\ell q^4} - \frac{35}{\ell^2 q^4} + \frac{50}{\ell^3 q^4})$$

$$\stackrel{(7)}{\ge} \frac{1}{120}(\frac{N(\ell)}{q^4} + \frac{1}{(\ell q)^5 t}) + \frac{1}{120}(1 - \frac{10}{\ell q} + \frac{35}{\ell^2 q^2} - \frac{50}{\ell^3 q^3} - \frac{1}{q^4} + \frac{10}{\ell q^4} - \frac{35}{\ell^2 q^4} + \frac{50}{\ell^3 q^4})$$

$$\stackrel{(5)}{=} \frac{1}{120}(N(\ell, q) + \frac{1}{(\ell q)^5 t}).$$

Hence, we have

$$\lambda(H(\ell, q, t)) \ge \frac{1}{120}(N(\ell, q) + \frac{1}{(\ell q)^5 t}).$$

Now suppose $\vec{y} = \{y_1, y_2, \ldots, y_{\ell q t}\}$ is an optimal vector of $\lambda(H(\ell, q, t))$. Let $\varepsilon = \frac{1}{2(\ell q)^5 t}$ and $n > n_1(\varepsilon)$ as in Remark 2.1. Then 5-uniform graph $S_n = (\lfloor n y_1 \rfloor, \ldots, \lfloor n y_{\ell q t} \rfloor) \otimes H(\ell, q, t)$ has density larger than $N(\ell, q) + \varepsilon$. Since $N(\ell, q)$ is a threshold for

$\mathcal{F}$, some member F of $\mathcal{F}$ is a subgraph of S_n for $n \geq \max\{\mathrm{n}_0(\varepsilon), \mathrm{n}_1(\varepsilon)\}$. For such $F \in \mathcal{F}$, there exists a subgraph M' of $H(\ell, q, t)$ with $|V(M')| \leq k_1$ so that $F \subset \vec{\boldsymbol{n}} \otimes M' \subset \vec{\boldsymbol{n}} \otimes H(\ell, q, t)$.

Theorem 5.1 will follow from the following lemma.

Lemma 5.1. *Let M' be any graph of $H(\ell, q, t)$ with $|V(M')| \leq k_1$. Then*

$$\lambda(M') \leq \frac{1}{120} N(\ell, q) \tag{8}$$

holds.

The proof of Lemma 5.1 will be given as follows. We continue the proof of Theorem 5.1 by applying this Lemma. By Fact 2.2 we have

$$\lambda(F) \leq \lambda(\vec{\boldsymbol{n}} \otimes M') = \lambda(M') \leq \frac{1}{120} N(\ell, q)$$

which contradicts our choice of F, i.e., contradicts the fact that $\lambda(F) > \frac{1}{120} N(\ell, q)$ for all $F \in \mathcal{F}$. This completes the proof of Theorem 5.1. ∎

Proof of Lemma 5.1. Let M' be any subgraph of $H(\ell, q, t)$ with $|V(M')| \leq k_1$ and $\vec{\xi}$ be an optimal vector for $\lambda(M')$. Define $U_i = V(M') \cap V_i$ for $1 \leq i \leq \ell q$. Let a_i be the sum of the weights in U_i, $1 \leq i \leq \ell q$, respectively. Note that $\sum_{i=1}^{\ell q} a_i = 1$ and $a_i \geq 0$ for each i, $1 \leq i \leq \ell q$.

The proof of Lemma 5.1 is based on Lemma 3.1, Claim 3.2, 3.3 and an estimation given in [5] and [11] on the summation of the terms in $\lambda(M')$ corresponding to edges in $E(M') \cap \binom{\cup_{i=1}^{\ell} V_i}{5}$, denoted by $\lambda(M' \cap \cup_{i=1}^{\ell} V_i)$. For our purpose, we formulate Claim 3.2 in Section 3, Lemma 4.2 in [5] and Lemma 3.2 in [11] as follows.

Lemma 5.2. *There exists a function f such that*

$$\lambda(M' \cap \cup_{i=1}^{\ell} V_i) \leq f(a_1, a_2, \ldots, a_\ell, \rho), \tag{9}$$

where the function f satisfies the following property:

$$f(a_1, a_2, \ldots, a_\ell, \rho) \leq f(\frac{1}{\ell}, \frac{1}{\ell}, \ldots, \frac{1}{\ell}, 0) = \frac{1}{120} N(\ell) \tag{10}$$

holds under the constraints $\sum_{j=1}^{\ell} a_j = 1$ and each $a_j \geq 0$, $1 \leq j \leq \ell$ and $0 \leq \rho \leq \frac{a_1}{4}$.

In view of the construction of $H(\ell, q, t)$, for each p, $1 \leq p \leq q-1$, the structure of M' restricted on the vertex set $\cup_{i=p\ell+1}^{(p+1)\ell} V_i$ is similar to the structure of M' restricted on the vertex set $\cup_{i=1}^{\ell} V_i$, but there might be some other extra edges in $\binom{V_1}{5}$ for M'

restricted on the vertex set $\cup_{i=1}^{\ell} V_i$. Therefore, for each p, $1 \le p \le q-1$ the summation of the terms in $\lambda(M')$ corresponding to edges in $E(M') \cap \binom{\cup_{i=p\ell+1}^{(p+1)\ell} V_i}{4}$ denoted by $\lambda(M' \cap \cup_{i=p\ell+1}^{(p+1)\ell} V_i)$. For our purpose, we formulate Claim 3.3 in section 3, Lemma 4.2 in [5] and Lemma 3.2 in [11] as follows.

Lemma 5.3. *There exists a function g such that*

$$\lambda(M' \cap \cup_{i=p\ell+1}^{(p+1)\ell} V_i) \le g(a_{p\ell+1}, a_{p\ell+2}, \ldots, a_{(p+1)\ell}), \tag{11}$$

where the function g satisfies the following property:

$$g(d_{p\ell+1}, d_{p\ell+2}, \ldots, d_{(p+1)\ell}) \le g(\frac{c}{\ell}, \frac{c}{\ell}, \ldots, \frac{c}{\ell}) = \frac{1}{120} N(\ell) c^5 \tag{12}$$

holds under the constraints $\sum_{j=p\ell+1}^{(p+1)\ell} d_j = c$ and each $d_j \ge 0$, $p\ell + 1 \le j \le (p+1)\ell$ for any positive constant c.

Consequently,

$$\begin{aligned}
\lambda(M') \le & f(a_1, a_2, \ldots, a_\ell, \rho) + \sum_{p=1}^{q-1} g(a_{p\ell+1}, a_{p\ell+2}, \ldots, a_{(p+1)\ell}) \\
& + (\sum_{1 \le i_1 < i_2 < i_3 < i_4 < i_5 \le \ell q} a_{i_1} a_{i_2} a_{i_3} a_{i_4} a_{i_5} - \sum_{p=0}^{q-1} \sum_{p\ell+1 \le i_1 < i_2 < i_3 < i_4 < i_5 \le (p+1)\ell} a_{i_1} a_{i_2} a_{i_3} a_{i_4} a_{i_5}) \\
& \stackrel{\text{def}}{=} F(a_1, a_2, \ldots, a_{\ell q}, \rho).
\end{aligned}$$

Note that

$$F(\frac{1}{\ell q}, \frac{1}{\ell q}, \ldots, \frac{1}{\ell q}, 0) = \frac{N(\ell)}{120 q^4} + \frac{\binom{\ell q}{5} - q\binom{\ell}{5}}{(\ell q)^5} = \frac{N(\ell, q)}{120}. \tag{13}$$

Therefore, to show Lemma 5.1, we only need to show the following claim:

Claim 5.1.

$$F(a_1, a_2, \ldots, a_{\ell q}, \rho) \le F(\frac{1}{\ell q}, \frac{1}{\ell q}, \ldots, \frac{1}{\ell q}, 0) \tag{14}$$

holds under the constraints $\sum_{j=1}^{\ell q} a_j = 1$ and each $a_j \ge 0$, $1 \le j \le \ell q$ and $0 \le \rho \le \frac{a_1}{4}$.

Proof. Suppose the function F reaches the maximum at $(a_1, a_2, \ldots, a_\ell, \rho)$. By applying Lemma 5.2, we claim that we can assume that $a_1 = a_2 = \cdots = a_\ell$ and $\rho = 0$. Otherwise, let $c_1 = c_2 = \cdots = c_\ell = \frac{\sum_{j=1}^{\ell} a_j}{\ell}$. Then

$$F(c_1, c_2, \ldots, c_\ell, a_{\ell+1}, \ldots, a_{\ell q}, 0) - F(a_1, a_2, \ldots, a_\ell, a_{\ell+1}, \ldots, a_{\ell q}, \rho)$$

$$
\begin{aligned}
=&f(c_1,c_2,\ldots,c_\ell,0)-f(a_1,a_2,\ldots,a_\ell,\rho)\\
&+(\sum_{1\le i<j<k<h\le\ell}c_ic_jc_kc_h-\sum_{1\le i<j<k<h\le\ell}a_ia_ja_ka_h)(\sum_{s=\ell+1}^{\ell q}a_s)\\
&+(\sum_{1\le i<j<k\le\ell}c_ic_jc_k-\sum_{1\le i<j<k\le\ell}a_ia_ja_k)(\sum_{\ell+1\le h<s\le\ell q}a_ha_s)\\
&+(\sum_{1\le i<j\le\ell}c_ic_j-\sum_{1\le i<j\le\ell}a_ia_j)(\sum_{\ell+1\le k<h<s\le\ell q}a_ka_ha_s)\ge 0
\end{aligned}
$$

holds by combining (10), $\sum_{1\le i<j<k<h\le\ell}c_ic_jc_kc_h-\sum_{1\le i<j<k<h\le\ell}a_ia_ja_ka_h\ge 0$, $\sum_{1\le i<j<k\le\ell}c_ic_jc_k-\sum_{1\le i<j<k\le\ell}a_ia_ja_k\ge 0$ and $\sum_{1\le i<j\le\ell}c_ic_j-\sum_{1\le i<j\le\ell}a_ia_j\ge 0$. This implies that $a_1=a_2=\cdots=a_\ell$ and $\rho=0$ can be assumed. Similarly, by applying Lemma 5.3, for each p, $1\le p\le q-1$, we can assume that $a_{p\ell+1}=a_{p\ell+2}=\cdots=a_{(p+1)\ell}$. Set $b_{p+1}=a_{p\ell+1}=a_{p\ell+2}=\cdots=a_{(p+1)\ell}$ for each $0\le p\le q-1$. In view of Lemma 5.2 and Lemma 5.3, we have

$$
\begin{aligned}
&F(a_1,a_2,\ldots,a_{\ell q},\rho)\le H(b_1,b_2,\ldots,b_q)\\
\overset{\text{def}}{=}\ &\frac{N(\ell)}{120}\sum_{p=1}^{q}\ell^5b_p^5+\sum_{p=1}^{q}\binom{\ell}{4}b_p^4(1-\ell b_p)+\sum_{1\le p_1\le q;1\le p_2\le q;p_2\ne p_1}\binom{\ell}{3}\binom{\ell}{2}b_{p_1}^3b_{p_2}^2\\
&+\sum_{1\le p_1\le q;1\le p_2<p_3\le q;p_2,p_3\ne p_1}\binom{\ell}{3}\ell^2b_{p_1}^3b_{p_2}b_{p_3}+\sum_{1\le p_1<p_2\le q;1\le p_3\le q;p_3\ne p_1,p_2}\binom{\ell}{2}^2\ell b_{p_1}^2b_{p_2}^2b_{p_3}\\
&+\sum_{1\le p_1\le q;1\le p_2<p_3<p_4\le q;p_2,p_3,p_4\ne p_1}\binom{\ell}{2}\ell^3b_{p_1}^2b_{p_2}b_{p_3}b_{p_4}+\sum_{1\le p_1<p_2<p_3<p_4<p_5\le q}\ell^5b_{p_1}b_{p_2}b_{p_3}b_{p_4}b_{p_5}.
\end{aligned}
$$

Note that

$$
H(\frac{1}{\ell q},\frac{1}{\ell q},\ldots,\frac{1}{\ell q})=F(\frac{1}{\ell q},\frac{1}{\ell q},\ldots,\frac{1}{\ell q},0)\overset{(11)}{=}\frac{N(\ell,q)}{120}. \tag{15}
$$

Therefore, to show Claim 5.1, it is sufficient to show the following claim

Claim 5.2.

$$
H(b_1,b_2,\ldots,b_q)\le H(\frac{1}{\ell q},\frac{1}{\ell q},\ldots,\frac{1}{\ell q})
$$

holds under the constraints

$$
\begin{cases}\sum_{i=1}^q b_i=\frac{1}{\ell},\\ b_i\ge 0,\quad 1\le i\le q.\end{cases} \tag{16}
$$

Suppose that function H reaches the maximum at $(b_1,b_2,\ldots,b_q)$. We will apply Claim 5.3 and 5.4 stated below.

Claim 5.3. Let $i,\ j,\ 1 \le i < j \le q$ be a pair of integers and ε be a real number. Let $c_i = b_i + \varepsilon,\ c_j = b_j - \varepsilon$, and $c_k = b_k$ for $k \ne i, j$. Let $(b_j - b_i)A(b_1, b_2, \ldots, b_q)$ and $B(b_1, b_2, \ldots, b_q)$ be the coefficients of ε and ε^2 in $H(c_1, c_2, \ldots, c_q) - H(b_1, b_2, \ldots, b_q)$, respectively, i.e.,

$$\begin{aligned} &H(c_1, c_2, \ldots, c_q) - H(b_1, b_2, \ldots, b_q) \\ =&(b_j - b_i)A(b_1, b_2, \ldots, b_q)\varepsilon + B(b_1, b_2, \ldots, b_q)\varepsilon^2 + o(\varepsilon^2). \end{aligned}$$

If $b_i \ne b_j$, then

$$A(b_1, b_2, \ldots, b_q) + B(b_1, b_2, \ldots, b_q) \ge 0.$$

Proof. Without loss of generality, we take $i = 1$ and $j = 2$. By the definition of the function $H(b_1, b_2, \ldots, b_q)$, we have

$$\begin{aligned} &H(b_1+\varepsilon, b_2-\varepsilon, \ldots, b_q) - H(b_1, b_2, \ldots, b_q) \\ =\ & \frac{N(\ell)}{120}\ell^5[(b_1+\varepsilon)^5 + (b_2-\varepsilon)^5 - b_1^5 - b_2^5] \\ &+ \binom{\ell}{4}[(b_1+\varepsilon)^4(1-\ell b_1-\ell\varepsilon) + (b_2-\varepsilon)^4(1-\ell b_2+\ell\varepsilon) - b_1^4(1-\ell b_1) - b_2^4(1-\ell b_2)] \\ &+ \binom{\ell}{3}\binom{\ell}{2}[(b_1+\varepsilon)^3 + (b_2-\varepsilon)^3 - b_1^3 - b_2^3](\sum_{3\le p_1\le q} b_{p_1}^2) \\ &+ \binom{\ell}{3}\binom{\ell}{2}[(b_1+\varepsilon)^2 + (b_2-\varepsilon)^2 - b_1^2 - b_2^2](\sum_{3\le p_1\le q} b_{p_1}^3) \\ &+ \binom{\ell}{3}\binom{\ell}{2}[(b_1+\varepsilon)^3(b_2-\varepsilon)^2 + (b_2-\varepsilon)^3(b_1+\varepsilon)^2 - b_1^3b_2^2 - b_2^3b_1^2] \\ &+ \binom{\ell}{3}\ell^2[(b_1+\varepsilon)^3 + (b_2-\varepsilon)^3 - b_1^3 - b_2^3](\sum_{3\le p_1<p_2\le q} b_{p_1}b_{p_2}) \\ &+ \binom{\ell}{3}\ell^2[(b_1+\varepsilon)^3(b_2-\varepsilon) + (b_2-\varepsilon)^3(b_1+\varepsilon) - b_1^3b_2 - b_2^3b_1](\sum_{3\le p_1\le q} b_{p_1}) \\ &+ \binom{\ell}{3}\ell^2[(b_1+\varepsilon)(b_2-\varepsilon) - b_1b_2](\sum_{3\le p_1\le q} b_{p_1}^3) \\ &+ \binom{\ell}{2}^2\ell[(b_1+\varepsilon)^2 + (b_2-\varepsilon)^2 - b_1^2 - b_2^2](\sum_{3\le p_1\le q;3\le p_2\le q;p_2\ne p_1} b_{p_1}^2b_{p_2}) \\ &+ \binom{\ell}{2}^2\ell[(b_1+\varepsilon)^2(b_2-\varepsilon)^2 - b_1^2b_2^2](\sum_{3\le p_1\le q} b_{p_1}) \\ &+ \binom{\ell}{2}^2\ell[(b_1+\varepsilon)^2(b_2-\varepsilon) + (b_2-\varepsilon)^2(b_1+\varepsilon) - b_1^2b_2 - b_2^2b_1](\sum_{3\le p_1\le q} b_{p_1}^2) \end{aligned}$$

$$+\binom{\ell}{2}\ell^3[(b_1+\varepsilon)^2+(b_2-\varepsilon)^2-b_1^2-b_2^2](\sum_{3\le p_1<p_2<p_3\le q} b_{p_1}b_{p_2}b_{p_3})$$
$$+\binom{\ell}{2}\ell^3[(b_1+\varepsilon)^2(b_2-\varepsilon)+(b_2-\varepsilon)^2(b_1+\varepsilon)-b_1^2b_2-b_2^2b_1](\sum_{3\le p_1<p_2\le q} b_{p_1}b_{p_2})$$
$$+\binom{\ell}{2}\ell^3[(b_1+\varepsilon)(b_2-\varepsilon)-b_1b_2](\sum_{3\le p_1\le q;3\le p_2\le q;p_2\ne p_1} b_{p_1}^2b_{p_2})$$
$$+\ell^5[(b_1+\varepsilon)(b_2-\varepsilon)-b_1b_2](\sum_{3\le p_1<p_2<p_3\le q} b_{p_1}b_{p_2}b_{p_3}).$$

By a direct calculation, we obtain that

$$\begin{aligned}
& A(b_1,b_2,\ldots,b_q)+B(b_1,b_2,\ldots,b_q)\\
= \ & -\frac{N(\ell)}{24}\ell^5(b_1+b_2)(b_1^2+b_2^2)+5\ell\binom{\ell}{4}(b_1+b_2)(b_1^2+b_2^2)-4\binom{\ell}{4}(b_1^2+b_2^2+b_1b_2)\\
& +2\binom{\ell}{3}\binom{\ell}{2}b_1b_2(b_1+b_2)+\binom{\ell}{3}\ell^2(b_1-b_2)^2(\sum_{3\le p_1\le q} b_{p_1})+2\binom{\ell}{2}^2\ell b_1b_2(\sum_{3\le p_1\le q} b_{p_1})\\
& +\frac{N(\ell)}{12}\ell^5(b_1^3+b_2^3)+\binom{\ell}{4}(6b_1^2+6b_2^2-10\ell b_1^3-10\ell b_2^3)+\binom{\ell}{3}\binom{\ell}{2}(b_1^3+b_2^3-3b_1b_2^2-3b_1^2b_2)\\
& -3\binom{\ell}{3}\ell^2(b_1-b_2)^2(\sum_{3\le p_1\le q} b_{p_1})+\binom{\ell}{2}^2\ell(b_1^2+b_2^2-4b_1b_2)(\sum_{3\le p_1\le q} b_{p_1})\\
= \ & [2\ell\binom{\ell}{4}-2\binom{\ell}{3}\ell^2+\binom{\ell}{2}^2\ell]\frac{1}{\ell}(b_1-b_2)\\
& +[\frac{N(\ell)}{24}\ell^5-5\ell\binom{\ell}{4}+\binom{\ell}{3}\binom{\ell}{2}+2\binom{\ell}{3}\ell^2-\binom{\ell}{2}^2\ell](b_1+b_2)(b_1-b_2)^2\\
& \ge [2\ell\binom{\ell}{4}-2\binom{\ell}{3}\ell^2+\binom{\ell}{2}^2\ell](b_1+b_2)(b_1-b_2)^2\\
& +[\frac{N(\ell)}{24}\ell^5-5\ell\binom{\ell}{4}+\binom{\ell}{3}\binom{\ell}{2}+2\binom{\ell}{3}\ell^2-\binom{\ell}{2}^2\ell](b_1+b_2)(b_1-b_2)^2\\
= \ & [\frac{N(\ell)}{24}\ell^5-3\ell\binom{\ell}{4}+\binom{\ell}{3}\binom{\ell}{2}](b_1+b_2)(b_1-b_2)^2\\
= \ & \begin{cases}
(\frac{5}{12}\ell^4-\frac{23}{24}\ell^3+\frac{3}{8}\ell^2+\frac{1}{6}\ell)(b_1+b_2)(b_1-b_2)^2 \ when\ N(\ell)=\alpha\\
(\frac{5}{12}\ell^4-\frac{23}{24}\ell^3+\frac{7}{12}\ell^2-\frac{1}{24}\ell)(b_1+b_2)(b_1-b_2)^2 \ when\ N(\ell)=1-\frac{1}{\ell^4}\\
\frac{75}{2}(b_1+b_2)(b_1-b_2)^2 \ when\ \ell=5\ and\ N(5)=\frac{12}{125}\\
45(b_1+b_2)(b_1-b_2)^2 \ when\ \ell=5\ and\ N(5)=\frac{96}{625}\\
\frac{155}{2}(b_1+b_2)(b_1-b_2)^2 \ when\ \ell=5\ and\ N(5)=\frac{252}{625}
\end{cases}\\
> \ & 0
\end{aligned}$$

if $b_1\ne b_2$ and since $2\ell\binom{\ell}{4}-2\binom{\ell}{3}\ell^2+\binom{\ell}{2}^2\ell=\frac{\ell^2(\ell-1)}{2}>0$ and $\frac{1}{\ell}\ge(b_1+b_2)$. This completes the proof of Claim 5.3. ∎

We will apply Claim 5.3 to prove the following claim.

Claim 5.4. Let $i,\ j,\ 1 \le i < j \le q$ be a pair of integers. Let $A(b_1, b_2, \dots, b_q)$ and $B(b_1, b_2, \dots, b_q)$ be given as in Claim 5.3.

Case 1. If $A(b_1, b_2, \dots, b_q) > 0$ then $b_i = b_j$;

Case 2. If $A(b_1, b_2, \dots, b_q) \le 0$, then either $b_i = b_j$, or $min\{b_i, b_j\} = 0$.

The proof of Claim 5.4 (based on Claim 5.3) can be given by exactly the same lines as in the proof of Claim 4.5 in [9] and is omitted here. ∎

Proof of Claim 5.2. By Claim 5.4, either $b_1 = b_2 = \dots = b_q = \frac{1}{\ell q}$ or for some integer $p < q,\ b_{i_1} = b_{i_2} = \dots = b_{i_p} = \frac{1}{\ell p}$ and other $b_i = 0$.

Now we compare $H(\frac{1}{\ell q}, \frac{1}{\ell q}, \dots, \frac{1}{\ell q}) = \frac{N(\ell,q)}{120}$ and $H(\frac{1}{\ell p}, \frac{1}{\ell p}, \dots, \frac{1}{\ell p}, 0, \dots, 0) = \frac{N(\ell,p)}{120}$. It sufficient to show that $N(\ell, p) \le N(\ell, q)$ when $1 \le p \le q$. Note that condition (6) implies that $N(\ell, 1) \le N(\ell, q)$. Hence it is sufficient to show that $N(\ell, p) \le N(\ell, q)$ when $2 \le p \le q$ for each of the five choices of $N(\ell)$. In each case, we view $N(\ell, q)$ as a function with one variable q.

Case a. $N(\ell) = \alpha$ and $q \ge 2\ell^2 + 2\ell$.

In this case, the derivative of $N(\ell, q)$ with respect to q is

$$\begin{aligned}\frac{d(N(\ell,q))}{dq} &= \frac{10}{\ell q^2} - \frac{70}{\ell^2 q^3} + \frac{150}{\ell^3 q^4} - \frac{16}{\ell^4 q^5} - \frac{40}{\ell q^5} + \frac{140}{\ell^2 q^5} - \frac{180}{\ell^3 q^5} \\ &= \frac{1}{\ell^4 q^5}(10\ell^3 q^3 - 70\ell^2 q^2 + 150\ell q - 16 - 40\ell^3 + 140\ell^2 - 180\ell).\end{aligned}$$

Let $h_1(q) = 10\ell^3 q^3 - 70\ell^2 q^2 + 150\ell q - 16 - 40\ell^3 + 140\ell^2 - 180\ell$, then $h_1'(q) = 30\ell^3 q^2 - 140\ell^2 q + 150\ell$, $h_1''(q) = 60\ell^3 q - 140\ell^2$. Note that $h_1''(q) > 0$ when $q \ge 2,\ \ell \ge 2$, so $h_1'(q)$ increases when $q \ge 2,\ \ell \ge 2$. By a direct calculation, $h_1'(2) > 0$ when $\ell \ge 2$, thus, $h_1(q)$ increases when $q \ge 2,\ \ell \ge 2$. Since, $h_1(2) = 40\ell^3 - 140\ell^2 + 120\ell - 16 > 0$ when $q \ge 2,\ \ell \ge 3$, we know that $N(\ell, q)$ increases when $q \ge 2,\ \ell \ge 3$. When $\ell = 2$, by a direct calculation, $h_1(3) > 0$, so $N(2, q)$ increases when $q \ge 3$. Also we calculate that $N(2, 2) \le N(2, q)$ since $q \ge 2\ell^2 + 2\ell$. So $N(\ell, p) \le N(\ell, q)$ for $2 \le p \le q$.

Case b. $N(\ell) = 1 - \frac{1}{\ell^4}$ and $q \ge 10\ell^3$.

In this case, the derivative of $N(\ell, q)$ with respect to q is

$$\begin{aligned}\frac{d(N(\ell,q))}{dq} &= \frac{10}{\ell q^2} - \frac{70}{\ell^2 q^3} + \frac{150}{\ell^3 q^4} + \frac{4}{\ell^4 q^5} - \frac{40}{\ell q^5} + \frac{140}{\ell^2 q^5} - \frac{200}{\ell^3 q^5} \\ &= \frac{1}{\ell^4 q^5}(10\ell^3 q^3 - 70\ell^2 q^2 + 150\ell q + 4 - 40\ell^3 + 140\ell^2 - 200\ell).\end{aligned}$$

Let $h_2(q) = 10\ell^3 q^3 - 70\ell^2 q^2 + 150\ell q + 4 - 40\ell^3 + 140\ell^2 - 200\ell$, then $h_2'(q) = 30\ell^3 q^2 - 140\ell^2 q + 150\ell$, $h_2''(q) = 60\ell^3 q - 140\ell^2$. Note that $h_2''(q) > 0$ when $q \geq 2,\ \ell \geq 2$, so $h_2'(q)$ increases when $q \geq 2,\ \ell \geq 2$. By a direct calculation, $h_2'(2) > 0$ when $\ell \geq 2$, thus, $h_2(q)$ increases when $q \geq 2,\ \ell \geq 2$. Since, $h_2(2) = 40\ell^3 - 140\ell^2 + 100\ell + 4 > 0$ when $q \geq 2,\ \ell \geq 3$, we know that $N(\ell, q)$ increases when $q \geq 2,\ \ell \geq 3$. When $\ell = 2$, by a direct calculation, $h_2(3) > 0$, so $N(2, q)$ increases when $q \geq 3$. Also we calculate that $N(2, 2) \leq N(2, q)$ since $q \geq 10\ell^3$. So $N(\ell, p) \leq N(\ell, q)$ for $2 \leq p \leq q$.

Case c. $N(\ell) = \frac{12}{125}$ and $\ell = 5$.

In this case, the derivative of $N(5, q)$ with respect to q is

$$\frac{d(N(\ell, q))}{dq} = \frac{2}{q^2} - \frac{14}{5q^3} + \frac{6}{5q^4} - \frac{48}{125q^5} = \frac{1}{125q^5}(250q^3 - 350q^2 + 150q - 48) \geq 0$$

when $q \geq 2$. This proves that $N(5, q)$ increases as $q \geq 2$ increases. So $N(5, p) \leq N(5, q)$ for $2 \leq p \leq q$.

Case d. $N(\ell) = \frac{96}{625}$ and $\ell = 5$.

In this case , the derivative of $N(5, q)$ with respect to q is

$$\frac{d(N(\ell, q))}{dq} = \frac{2}{q^2} - \frac{14}{5q^3} + \frac{6}{5q^4} - \frac{384}{625q^5} = \frac{1}{625q^5}(1250q^3 - 1750q^2 + 750q - 384) \geq 0$$

when $q \geq 2$. This proves that $N(5, q)$ increases as $q \geq 2$ increases. So $N(5, p) \leq N(5, q)$ for $2 \leq p \leq q$.

Case e. $N(\ell) = \frac{252}{625}$ and $\ell = 5$.

In this case , the derivative of $N(5, q)$ with respect to q is

$$\frac{d(N(\ell, q))}{dq} = \frac{2}{q^2} - \frac{14}{5q^3} + \frac{6}{5q^4} - \frac{1008}{625q^5} = \frac{1}{625q^5}(1250q^3 - 1750q^2 + 750q - 1008) \geq 0$$

when $q \geq 2$. This proves that $N(5, q)$ increases as $q \geq 2$ increases. So $N(5, p) \leq N(5, q)$ for $2 \leq p \leq q$.

The proof is thus complete. ∎

References

[1] P. Erdős, On extremal problems of graphs and generalized graphs, *Israel J. Math.* **2**(1964), 183–190.

[2] P. Erdős, M. Simonovits, A limit theorem in graph theory, *Studia Sci. Mat. Hung. Acad.* **1**(1966), 51–57.

[3] P. Erdős, A.H. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* **52**(1946), 1087–1091.

[4] P. Frankl, Z. Füredi, Extremal problems whose solutions are the blow-ups of the small Witt-designs, *J. Combin. Theory (A)* **52**(1989), 129–147.

[5] P. Frankl, Y. Peng, V. Rödl, J. Talbot, A note on the jumping constant conjecture of Erdős, *J. Combin. Theory (B)* **97**(2007), 204–216.

[6] P. Frankl, V. Rödl, Hypergrpahs do not jump, *Combinatorica* **4**(1984), 149–159.

[7] G. Katona, T. Nemetz and M. Simonovits, On a graph problem of Turán, *Mat. Lapok* **15**(1964), 228–238.

[8] T.S. Motzkin, E.G. Straus, maxima for graphs and a new proof of a theorem of Turán, *Canad. J. Math.* **17**(1965), 533–540.

[9] Y. Peng, Non-jumping numbers for 4-uniform hypergraphs, *Graphs Combin.* **23(1)**(2007), 97–110.

[10] Y. Peng, Using Lagrangians of hypergraphs to find non-jumping numbers (II), *Discrete Math.* **307**(2007), 1754–1766.

[11] Y. Peng, A note on non-jumping numbers, *Australasian J. Combin.* **41**(2008), 3–13.

[12] Y. Peng, Using Lagrangians of hypergraphs to find non-jumping numbers (I), *Ann. Comb.* **12**(2008), 307–324.

[13] Y. Peng, A note on the Structure of Turán Densities of Hypergraphs, *Graphs and Combin.* **24**(2008), 113–125.

[14] Y. Peng, On Substructure Densities of Hypergraphs, *Graphs and Combin.* **25**(2009), 583–600.

[15] Y. Peng, On Jumping Densities of Hypergraphs, *Graphs and Combin.* **25**(2009), 759–766.

[16] Y. Peng, C. Zhao, Generating non-jumping numbers recursively, *Discrete Appl. Math.* **156**(2008), 1856–1864.

[17] O. Pikhurko, On possible Turán densities, arXiv:1204.4423v3.

[18] J. Talbot, Lagrangians of hypergraphs, *Combin. Probab. Comput.* **11**(2002), 199–216.