Enumeration for spanning forests of complete bipartite graphs

Yinglie Jin, Chunlin Liu
The Key Laboratory of Pure Mathematics and Combinatorics,
Ministry of Education, China
Center for Combinatorics, Nankai University, Tianjin 300071, P.R. China
E-mail: yljin@public.tpt.tj.cn, lchliu@eyou.com

Abstract. This paper discusses the enumeration of rooted labelled spanning forests of the complete bipartite graph $K_{m,n}$.
MSC: primary 05C30; secondary 05C05
Keywords: forest, complete bipartite graph

1 Introduction

In this paper we consider the enumeration problem of rooted labelled spanning forests of the complete bipartite graph. Labelled spanning forest of the complete graph K_n has been researched by ([5], [6]), but the spanning forest of the complete bipartite graph seems to appear occasionally. A component of forest consisting of only a vertex is also viewed as a rooted tree in this paper. For convenience call a forest of $l+k$ labelled rooted trees as spanning subgraphs of $K_{m,n}$ ($V(K_{m,n}) = A\bigcup B, |A| = m, |B| = n$) with l roots in A and k roots in B as a $[m,l;n,k]$-forest ($l \leq m, k \leq n$). We assume there is no order between the l trees or the k trees, unless otherwise stated. Vertices in A and B will be labelled on the set $\{1', 2', \ldots, m'\}$ and $\{1, 2, \ldots, n\}$ respectively. From the definition we know that $[m,0;n,1]$-forest and $[m,1;n,0]$-forest are rooted spanning trees of $K_{m,n}$ in fact. This paper first concerns the $[m,0;n,k]$-forest ($k > 1$) by adding some new vertices to them and then establishing a bijective correspondence between them and $[m,0;n,1]$-forests; and later solves the general case of $[m,l;n,k]$-forests by a further discussion of the method contained in [1].

ARS COMBINATORIA 70(2004), pp. 135-138

2 The number of labelled $[m, l; n, k]$-forests

Let T be a $[m, 0; n, 1]$-forest with root v_0. To a vertex v ($v \in V(T)$), we will let $OV(v)$ denote the vertex subset $\{u | u \in V(T), d(u, v_0) = d(u, v) + d(v, v_0)\}$, where $d(u, v)$ is the distance between u and v. Heights of vertex v ($v \in V(T)$) and T are defined by $d(v, v_0)$ and $max\{d(u, v_0) | u \in V(T)\}$ respectively. Obviously in a $[m, 0; n, k]$-forest, there are m odd height vertices and n even height vertices. Denote the number of labelled $[m, l; n, k]$-forests by $f(m, l; n, k)$. First we calculate $f(m, 0; n, k)$, where $f(0, 0; 1, 1)$ is defined to be 1.

Lemma 2.1 ([1]–[4]) *The number of spanning trees of $K_{m,n}$ is $n^{m-1}m^{n-1}$.*

From above Lemma we can get

$$f(m, 0; n, 1) = n^m m^{n-1}. \tag{1}$$

Now suppose $k > 1$ and $n > 1$.

Theorem 2.2

$$f(m, 0; n, k) = k\binom{n}{k} m^{n-k} n^{m-1}. \tag{2}$$

Proof. Denote by F_1 and F_k the set of all $[m, 0; n, 1]$-forests and $[m, 0; n, k]$-forests, respectively. From (1) it is sufficient for us to show

$$|F_1|\binom{n-1}{k-1} = |F_k| m^{k-1}.$$

where $|F_1| = n^m m^{n-1}$ and $|F_k| = f(m, 0; n, k)$.

To any forest in F_k, there are m odd height vertices altogether. We add $k-1$ new vertices labelled by $1^*, 2^*, \ldots, (k-1)^*$ into the forest and link an edge between $i^*(1 \le i \le k-1)$ and some odd height vertex, there are m^{k-1} ways. Then we get a new forest of k rooted trees with m odd height vertices and $n+k-1$ even height vertices, let F_k' be the set of all these new forests and we have $|F_k'| = |F_k| m^{k-1}$. The procedure to construct a $[m, 0; n, 1]$-forest from a forest F in F_k' is as follows:

(1) Find tree T_0 in F with the smallest root such that there is not any vertex assigned * in T_0. Let i be the root.

(2) Find tree T_1 in F containing the smallest vertex assigned *. Let j^* be this assigned vertex.

(3) Merge T_0 and T_1 by identifying i and j^* and keeping i as the new vertex.

(4) Repeat (1), (2) and (3) until there is no vertex assigned *.

Without question we then get a $[m,0;n,1]$-forest. On the contrary, to get a forest in F_k' from a $[m,0;n,1]$-forest T, we can select out $k-1$ even height vertices in T (root of T can't be selected), there being $\binom{n-1}{k-1}$ ways. Suppose these selected vertices are $i_1,\ldots,i_{k-1}$ $(1 \le i_1 < \cdots < i_{k-1} \le n)$.
(1) Select out the vertex i_j with the biggest height in T. If we meet some vertices with the same heights, then we choose the vertex with the least label.
(2) Remove the subgraph of T induced by vertex set $OV(i_j)$ which is also a rooted tree with root i_j, and relabel the original vertex i_j by j^*.
(3) Repeat (1) and (2) until the heights of those $k-1$ selected vertices are all 0. Then we get a forest in F_k'. □

Now we consider the number $f(m,l;n,k)$ of $[m,l;n,k]$-forests, where $(1 \le l \le m, 1 \le k \le n)$. Here we solve the general case through another method similar to that in [1].

Theorem 2.3

$$f(m,l;n,k) = \binom{m}{l}\binom{n}{k} n^{m-l-1} m^{n-k-1}(km + ln - kl). \tag{3}$$

Proof. Suppose any edge in a rooted tree has a direction leading toward the root. Then the out-degree of any vertex (with root as exception) is 1. We say an directed edge $e = \overrightarrow{uv}$ is determined by vertex u, and call u link e or e is linked by u in the following proof. Since there are l roots in A and k roots in B, there are $\binom{m}{l}\binom{n}{k}$ ways to select these roots. For convenience, let $\{a_1,\ldots,a_l\}$ and $\{b_1,\ldots,b_k\}$ are chosen in A and B respectively. Denote $A' = A/\{a_1,\ldots,a_l\}$ and $B' = B/\{b_1,\ldots,b_k\}$, then the out-degree of any vertex in A' and B' is 1. Suppose in B' there are s vertices linking edges into A' and $n-k-s$ vertices linking edges into A/A'. There are two cases as to s:

Case 1. $s=0$, every vertex in B' links an edge into A/A'. Obviously any vertex in A' can link an edge to an vertex in B, so there are $l^{n-k}n^{m-l}$ ways altogether.

Case 2. $s>0$, there are $\binom{n-k}{s}$ ways to choose s vertices out from B', denoted by $B'' = \{b_{k+1},\ldots,b_{k+s}\}$. Then vertices in B'' link edges into A' and those left $n-k-s$ vertices in B'/B'' link edges into A/A', there being $(m-l)^s l^{n-k-s}$ ways. To vertices in A' there are also two ways to link edges-(into B'' or B/B''). Suppose the number of vertices in A' linking edges into B'' is t, we have $0 \le t \le m-l-1$. To avoid producing any cycle after linking edges, there are $\frac{s(m-l-1)s(m-l-2)\cdots s(m-l-t)}{t!} = s^t\binom{m-l-1}{t}$ ways to link those t edges. Each of the left $m-l-t$ vertices in A' link an edge into B/B'', there are $(n-s)^{m-l-t}$ ways. Therefore when $s>0$ the ways is

$$\sum_{s=1}^{n-k}\binom{n-k}{s}(m-l)^s l^{n-k-s}\sum_{t=0}^{m-l-1}\binom{m-l-1}{t}s^t(n-s)^{m-l-t}$$
$$=n^{m-l-1}l^{n-k}\sum_{s=1}^{n-k}\binom{n-k}{s}(\frac{m}{l}-1)^s(n-s).$$

From above two cases we get that $f(m,l;n,k)$ equals

$$\binom{m}{l}\binom{n}{k}\left(l^{n-k}n^{m-l}+n^{m-l-1}l^{n-k}\sum_{s=1}^{n-k}\binom{n-k}{s}(\frac{m}{l}-1)^s(n-s)\right)$$
$$=\binom{m}{l}\binom{n}{k}n^{m-l-1}m^{n-k-1}(km+ln-kl).\square$$

Acknowledgements

The authors wish to express their sincere thanks to the referees for their very helpful comments which greatly contributed to this paper.

References

[1] M. Z. Abu-Sbeih, On the number of spanning trees of K_n and $K_{m,n}$, Discrete math. 84 (1990) 205-207.

[2] T. L. Austin, The enumeration of point labelled chromatic graphs and trees, Canad. J. Math. 12 (1960) 535-545.

[3] L. H. Clark, J. E. McCanna and L. A. Székely, A survey of counting bicoloured trees, Bull. Inst. Combin. Appl. 21 (1997) 33-45.

[4] M. Fieldler and J. Sedlaček, O W-basich orientovanych grafu, Časopis Pro Pěst. Mat. 83 (1958) 214-225.

[5] C. J. Liu and Y. Chow, Enumeration of forests in a graph, Proc. A. M. S., Vol. 83, No. 3 (1981), 659-662.

[6] J. Riordan, Inverse relations and combinatorial identities, Amer. Math. Monthly 71 (1964) 485-498.