ON THE INTEGRITY OF GRAPHS

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ABSTRACT

The integrity of a graph $G = (V, E)$ is defined as $I(G) = min\{|S| + m(G - S) : S \subseteq V(G)\}$, where $m(G - X)$ denotes the order of the largest component in the graph $G - X$. This is a better parameter to measure the stability of a network G, as it takes into account both the amount of work done to damage the network and how badly the network is damaged. In this paper, the maximum networks are obtained with prescribed order and integrity, and a method for constructing this sort of networks is also presented. Finally, we give the trees of minimum integrity with given order and maximum degree.

KEY WORDS

Integrity, Maximum network, Nonlinear integer programming, Saturated tree.

1 Introduction

In an analysis of the vulnerability of a communication network to disruption, two qualities that come to mind are the number of elements that are not functioning and the size of the largest remaining subnetwork within which mutual communication can still occur. In particular, in an adversarial relationship, it would be desirable for an opponent's network to be such that the two qualities can be made to be simultaneously small.

The *integrity* of a graph $G = (V, E)$, which was introduced in [1] as a useful measure of the vulnerability of the graph, is defined as follows:

$$I(G) = min\{|S| + m(G - S) : S \subseteq V(G)\},$$

where $m(G - S)$ denotes the order of the largest component in $G - S$.

Unlike the connectivity measures, integrity shows not only the difficulty to break down the network but also the damage that has been caused. In [1] Barefoot *et al* gave some basic results on integrity and Clark *et al* [4] proved that the determination of the integrity is NP-complete. In [5] Moazzami *et al* compared the integrity, connectivity, binding number, toughness, and tenacity for several classes

of graphs. To know more about integrity, one can see a survey of integrity in [2].

A vertex subset S of a graph G is called an *I-set* of G if it satisfies that $I(G) = |S| + m(G - S)$.

A network is called the *maximum (minimum) network* if it has maximum (minimum) number of edges with prescribed order and some properties.

As a useful parameter to measure the stability of networks, what we are interested in it is the following question: For any given two integers n and I such that $2 \leq I \leq n$, how can we construct a network with order n and integrity I? The paper is organized as follows: In Section 2, the maximum network with prescribed order and integrity is given, and a method for constructing such maximum network is presented in Section 3. In Section 4, we give the minimum integrity of trees with given order and maximum degree.

Throughout this paper, a graph $G = (V, E)$ always means a simple connected graph with vertex set V and edge set E. We use Bondy and Murty [3] for terminology and notations not defined here. E_n and K_n, respectively, denotes the *null graph* and *complete graph* of order n. If S is a nonempty subset of V, we use $G[S]$ denotes the *induced subgraph* of G. We shall use $\lfloor x \rfloor$ for the largest integer not larger than a real number x. Δ denotes the maximum degree of a graph. A Δ-*edge* is an edge which joins two vertices of degree Δ. A *leaf* means a vertex of degree 1. An edge incident with a leaf is called a *leaf-edge*. An edge is said to be *subdivided* when it is replaced by a path of length two connecting its ends, and the internal vertex in this path is a new vertex.

2 The maximum network with prescribed order and integrity

In this section, we give the maximum network with given order and integrity. The following lemma is used in the proof of our main theorem.

Lemma 2.1 ([2]) *If G is a connected graph with order n and integrity I, then $2 \leq I(G) \leq n$, and $I(G) = n$ if and*

only if $G = K_n$.

In the following, we denote by $G[n, I]$ the set of all connected graphs with order n and integrity I. Then, we have

Theorem 2.1

$$max_{G \in G[n,I]}|E(G)| = \frac{1}{2}I(I - 1) + (I - 1)(n - I).$$

Proof. Let $S \subseteq V(G)$ be an *I-set* of G, i.e.,

$$I(G) = |S| + m(G - S).$$

Suppose that the components of $G - S$ are $G_1, G_2, \cdots G_p$, and let $|S| = x$, $|V(G_i)| = n_i$ $(i = 1, 2, \cdots, p)$. Then we have

$$G - S = G_1 \cup G_2 \cup \cdots \cup G_p$$

and

$$I(G) = x + m(G - S), \sum_{i=1}^{p} n_i = n - x.$$

If we want the number of edges of a graph G, $|E(G)|$ to achieve the maximum, the following statements must hold:

(1) $G[S]$ is a complete subgraph of G,

(2) All G_i $(i = 1, 2, \cdots, p)$ are complete subgraphs of G,

(3) All vertices in S must be adjacent to all vertices in G_i $(i = 1, 2, \cdots, p)$.

If the above three conditions are satisfied, let $f(n_1, n_2, \cdots, n_p, x)$ denotes the number of edges of graph G, then we have,

$$f(n_1, n_2, \cdots, n_p, x)$$

$$= \sum_{i=1}^{p} \binom{n_i}{2} + \binom{x}{2} + x \sum_{i=1}^{p} n_i$$

$$= \frac{1}{2} \sum_{i=1}^{p} n_i^2 - \frac{1}{2} \sum_{i=1}^{p} n_i + \binom{x}{2} + x \sum_{i=1}^{p} n_i$$

$$= \frac{1}{2}(\sum_{i=1}^{p} n_i)^2 + (x - \frac{1}{2}) \sum_{i=1}^{p} n_i + \binom{x}{2} - \sum_{1 \leq i < j \leq p} n_i n_j.$$

So, by $\sum_{i=1}^{p} n_i = n - x$ we have

$$f(n_1, n_2, \cdots, n_p, x) =$$

$$\frac{1}{2}(n - x)^2 + (x - \frac{1}{2})(n - x) + \binom{x}{2} - \sum_{1 \leq i < j \leq p} n_i n_j.$$

To get the maximum value of $f(n_1, n_2, \cdots, n_p, x)$, it is necessary to make

$$\sum_{1 \leq i < j \leq p} n_i n_j$$

minimum. And it is easy to see that

$$1 \leq n_i \leq n - x - (p - 1) \ (i = 1, 2, \cdots, p).$$

Now let us determine the minimum value of

$$\sum_{1 \leq i < j \leq p} n_i n_j,$$

i.e., solve the following nonlinear integer programming

$$ming(N) = \sum_{1 \leq i < j \leq p} n_i n_j$$

$$s.t \begin{cases} 1 \leq n_i \leq n - x - (p - 1) \\ i = 1, 2, \cdots, p \\ \sum_{i=1}^{p} n_i = n - x \\ n_i \in Z \end{cases}$$

where $N = (n_1, n_2, \cdots, n_p)$, Z is the set of positive integers.

To solve this problem, we first suppose that $N^0 = (n_1^0, n_2^0, \cdots, n_p^0)$ is an arbitrary feasible solution of the above nonlinear integer programming. Let n_j^0 be the first number larger than 1 among $n_1^0, n_2^0, \cdots, n_p^0$, i.e.,

$$N^0 = (\underbrace{1, 1, \cdots, 1}_{j-1}, n_j^0, \cdots, n_p^0).$$

Construct a new feasible solution

$$N^1 = (\underbrace{1, 1, \cdots, 1}_{j}, n_{j+1}^0 + n_j^0 - 1, n_{j+2}^0, \cdots, n_p^0).$$

Then we have $g(N^1) \leq g(N^0)$. Repeating the above process, we can finally get a feasible solution

$$N' = (\underbrace{1, 1, \cdots, 1}_{p-1}, \cdots, n - x - p + 1).$$

Since N^0 is an arbitrary feasible solution, we know that N' is optimal. That's to say, when $n_1 = n_2 = \cdots = n_{p-1} = 1$ and $n_p = n - x - p + 1$,

$$\sum_{1 \leq i < j \leq p} n_i n_j$$

is minimized. Now substitute these values into $f(n_1, n_2, \cdots, n_p, x)$, we have

$$f(1, 1, \cdots, 1, n - x - p + 1, x) =$$

$$\binom{n - x - p + 1}{2} + \binom{x}{2} + x(n - x).$$

On the other hand, from above we know that $m(G - S) = n_p = I - x$, and so we get

$$f_1(x) = f(1, 1, \cdots, 1, n - x - p + 1, x)$$

$$= \binom{I - x}{2} + \binom{x}{2} + x(n - x).$$

It is easy to see that $|S| = x \geq 1$ and $n_p = I - x \geq 1$, i.e., $1 \leq x \leq I - 1$.

In order to get the maximum value of $f_1(x)$, we solve the following nonlinear integer programming.

$$max f_1(x) = \binom{I - x}{2} + \binom{x}{2} + x(n - x)$$

$$s.t \begin{cases} 1 \leq x \leq I - 1 \\ x \in Z, \end{cases}$$

where Z is the set of positive integers.

Since $f_1'(x) = n - I$, by Lemma 2.1 we know that $f_1'(x) \geq 0$, and so $f_1(x)$ is an increasing function in the interval $1 \leq x \leq I - 1$. Then we have

$$max f_1(x) = f(I - 1) = \frac{1}{2}I(I - 1) + (I - 1)(n - I),$$

i.e.,

$$max|E(G)| = \frac{1}{2}I(I - 1) + (I - 1)(n - I).$$

This completes the proof. ∎

It is easy to see that, when $I(G) = n$, the maximum network G is just the complete graph K_n.

3 Construction and examples

From Section 2 we know the size of the maximum network with prescribed order and integrity. In the following, we introduce a method for constructing such maximum network $G = (V, E)$ with order n and integrity $I(G) = I$.

Construction:

(1) *If $I = n$, construct the complete graph K_n.*

(2) *If $2 \leq I \leq n - 1$, then the construction is as follows:*

Step 1 *Construct the complete graph K_I,*

Step 2 *Construct the null graph E_{n-I}, such that*

$$V(E_{n-I}) \cap V(K_I) = \Phi,$$

Step 3 *Arbitrarily select any $I - 1$ vertices in graph K_I and join these vertices to all the vertices in E_{n-I}.* ∎

Thus we get the graphs satisfying the requirements. Clearly, when $2 \leq I \leq n - 1$, the maximum network obtained by the above method is not unique. In the following we give two examples.

Example 3.1 Consider a graph G with $n = |V(G)| = 6$, $I(G) = 6$. Then we use (1) of the Construction to construct a complete graph with $max|E(G)| = 15$, as shown in (a) of Figure 1.

Example 3.2 Consider a graph G with $n = |V(G)| = 6$, $I(G) = 4$. Then we use (2) of the Construction to construct a graph with $max|E(G)| = 12$, as shown in (b) of Figure 1.
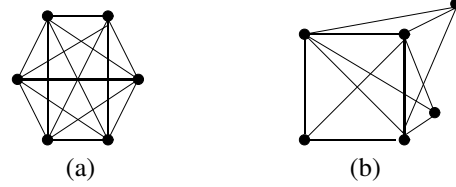

(a)　　　　　(b)

Figure 1. Maximum networks with given order and integrity

4 The trees of minimum integrity with given order and maximum degree

In this section, we determine the minimum integrity of trees with given order and maximum degree. Meanwhile, a method for constructing such trees is presented. The following lemmas are used later.

Lemma 4.1([6]) *For three positive integers m, n $(m \geq n)$ and s, there exists an integer $r(0 \leq r \leq n - 1)$ such that $m = sn + r$.*

Lemma 4.2([1]) *If H is a connected spanning subgraph of a connected graph G, then $I(G) \geq I(H)$.*

Lemma 4.3 *If T is a tree with maximum degree Δ and order n, then T has at most $\lfloor \frac{n-2}{\Delta-1} \rfloor$ vertices of degree Δ.*

Proof. Let $V' = \{v : d_T(v) = \Delta\}$ and $|V'| = x$. In order to have the number of vertices of degree Δ achieve the maximum, without loss of generality, we assume that $T[V']$ is a path, and so we have $(n-x) \geq \Delta x - 2(x-1)$, i.e., $x(\Delta - 1) \leq n - 2$. Noticing that x is an positive integer, we have that $x \leq \lfloor \frac{n-2}{\Delta-1} \rfloor$. ∎

Definition 4.1 *A tree is called a saturated tree with p vertices of maximum degree Δ if the following conditions are satisfied:*

(1) T has p vertices of degree Δ,

(2) besides all the above vertices, the other vertices in T are all leaves.

Example 4.1 Consider a tree T with order $n = 14$ and $\Delta = 5$. So we have $\lfloor \frac{n-2}{\Delta-1} \rfloor = \lfloor \frac{14-2}{5-1} \rfloor = 3$, and thus the saturated tree is constructed as follows:

Step 1 Construct a path P_3 of length 2 such that its vertices are labelled as v_1, v_2, v_3 from left to right.

Step 2 Join v_1 to four new vertices v_4, v_5, v_6, v_7, and join v_3 to another four new vertices v_8, v_9, v_{10}, v_{11}, then join v_2 to three new vertices v_{12}, v_{13}, v_{14}. The tree is shown in Figure 2.
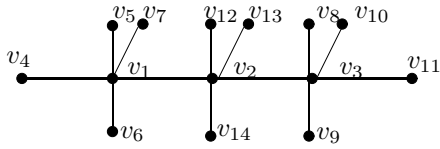

Figure 2. A saturated tree with three vertices of degree $\Delta = 5$

Theorem 4.1 *Let T be a tree with order n $(n \geq 3)$ and maximum degree Δ $(\Delta \geq 2)$. Then the minimum integrity of T is*

$$
min_{T \in T[n,\Delta]} I(T) = \begin{cases} \lfloor \frac{n-2}{\Delta-1} \rfloor + 1, \\ if\ d(\frac{n-2}{\Delta-1}) < \lfloor \frac{n-2}{\Delta-1} \rfloor \\ \lfloor \frac{n-2}{\Delta-1} \rfloor + 2, \\ if\ d(\frac{n-2}{\Delta-1}) \geq \lfloor \frac{n-2}{\Delta-1} \rfloor \end{cases}
$$

where $T[n, \Delta] = \{T : |V(T)| = n, \Delta(T) = \Delta\}$, $d(\frac{n-2}{\Delta-1})$ denotes the remainder of $n - 2$ divided by $\Delta - 1$.

Proof. We distinguish two cases:

Case 1. If $d(\frac{n-2}{\Delta-1}) < \lfloor \frac{n-2}{\Delta-1} \rfloor$, then we construct a tree as follows:

(1) Construct a saturated tree T' with $n - d(\frac{n-2}{\Delta-1})$ vertices and with $\lfloor \frac{n-2}{\Delta-1} \rfloor$ vertices of degree Δ.

(2) It is easy to see that T' has $(\lfloor \frac{n-2}{\Delta-1} \rfloor - 1)$ Δ-edges. Arbitrarily select $d(\frac{n-2}{\Delta-1})$ Δ-edges and subdivide them, and we thus get a tree T with order n and $\lfloor \frac{n-2}{\Delta-1} \rfloor$ vertices of degree Δ, such that $d(\frac{n-2}{\Delta-1}) < \lfloor \frac{n-2}{\Delta-1} \rfloor$. From the construction we know that the vertex subset $S = \{v : d_T(v) = \Delta\}$ is an *I-set* of tree T such that $m(T - S) = 1$ and $|S| = \lfloor \frac{n-2}{\Delta-1} \rfloor$. Hence

$$
I(T) = \lfloor \frac{n-2}{\Delta-1} \rfloor + 1.
$$

The minimality is obvious.

Case 2. If $d(\frac{n-2}{\Delta-1}) \geq \lfloor \frac{n-2}{\Delta-1} \rfloor$, we distinguish two subcases:

Subcase 2.1 If $d(\frac{n-2}{\Delta-1}) = \lfloor \frac{n-2}{\Delta-1} \rfloor$, we construct a tree as follows:

(1) Construct a saturated tree T_1 with $n - d(\frac{n-2}{\Delta-1})$ vertices and with $\lfloor \frac{n-2}{\Delta-1} \rfloor$ vertices of degree Δ.

(2) It is easy to see that there exist $(\lfloor \frac{n-2}{\Delta-1} \rfloor - 1)$ Δ-edges in T_1. First, subdivide them, then randomly select one leaf-edge and also subdivide it. We thus get a new tree T with order n and $\lfloor \frac{n-2}{\Delta-1} \rfloor$ vertices of degree Δ. So, by the construction of the tree T we know that the vertex subset $S = \{v : d_T(v) = \Delta\}$ is an *I-set* of T such that $|S| = \lfloor \frac{n-2}{\Delta-1} \rfloor$ and $m(T - S) = 2$. Thus we have

$$
I(T) = \lfloor \frac{n-2}{\Delta-1} \rfloor + 2.
$$

Subcase 2.2 If $d(\frac{n-2}{\Delta-1}) > \lfloor \frac{n-2}{\Delta-1} \rfloor$, we construct a tree as follows:

(1) Construct a saturated tree T_1 with order $n - d(\frac{n-2}{\Delta-1})$ and with $\lfloor \frac{n-2}{\Delta-1} \rfloor$ vertices of degree Δ. It is obvious that T has $(\Delta - 2)\lfloor \frac{n-2}{\Delta-1} \rfloor + 2$ leaves.

(2) Subdivide every Δ-edge of T_1, and we get a new tree T_2 with

$$
n - d(\frac{n-2}{\Delta-1}) + \lfloor \frac{n-2}{\Delta-1} \rfloor - 1
$$

vertices and with $(\Delta - 2)\lfloor \frac{n-2}{\Delta-1} \rfloor + 2$ leaves. By Lemma 4.1 we know that

$$
d(\frac{n-2}{\Delta-1}) \leq \Delta - 2.
$$

So, we have

$$d(\frac{n-2}{\Delta-1}) - \lfloor\frac{n-2}{\Delta-1}\rfloor + 1 < (\Delta-2)\lfloor\frac{n-2}{\Delta-1}\rfloor + 2.$$

Then we arbitrarily select $d(\frac{n-2}{\Delta-1}) - \lfloor\frac{n-2}{\Delta-1}\rfloor + 1$ leaf-edges. By subdividing them, we get a new tree T with order n and with $\lfloor\frac{n-2}{\Delta-1}\rfloor$ vertices of degree Δ. From the construction we know that the vertex subset $S = \{v : d_T(v) = \Delta\}$ is an *I-set* of T such that $|S| = \lfloor\frac{n-2}{\Delta-1}\rfloor$ and $m(T - S) = 2$. So we have

$$I(T) = \lfloor\frac{n-2}{\Delta-1}\rfloor + 2.$$

∎

It is easy to see that this theorem gives a method to construct trees with minimum integrity when its order and maximum degree are given.

Remark From the proof of theorem 4.1, it is easy to see that the tree of the minimum integrity with given order and maximum degree is not unique.

Example 4.2 Denote by $T[n, I]$ the set of trees of order n and integrity I.

(1) Construct a tree T with order 15 and $\Delta = 4$ such that T has minimum integrity in $T[15, 4]$. Since $n = 15$, $\Delta = 4$ and $d(\frac{n-2}{\Delta-1}) = d(\frac{13}{3}) = 1 < 4 = \lfloor\frac{13}{3}\rfloor = \lfloor\frac{n-2}{\Delta-1}\rfloor$, we know that the tree T is constructed as shown in Figure 3.

(2) Construct a tree T with order 21 and $\Delta = 6$ such that T has minimum integrity in $T[21, 6]$. Since $n = 21$, $\Delta = 6$ and $d(\frac{n-2}{\Delta-1}) = d(\frac{19}{5}) = 4 > 3 = \lfloor\frac{19}{5}\rfloor = \lfloor\frac{n-2}{\Delta-1}\rfloor$, we know that the tree T is constructed as shown in Figure 4.

(3) Construct a tree T with order 23 and $\Delta = 7$ such that T has minimum integrity in $T[23, 7]$. Since $n = 23$, $\Delta = 7$ and $d(\frac{n-2}{\Delta-1}) = d(\frac{21}{6}) = 3 = \lfloor\frac{21}{6}\rfloor = \lfloor\frac{n-2}{\Delta-1}\rfloor$, we know that the tree T is constructed as shown in Figure 5.
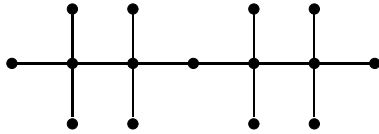

Figure 3. A tree of minimum integrity with order 15 and $\Delta = 4$

The above theorem gives the minimum integrity of a tree T with given order and Δ. It is well-known that any connected graph has a spanning tree, and so we have the following corollary.
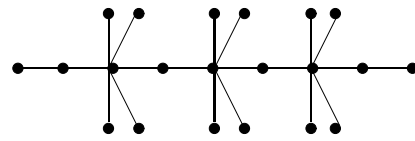

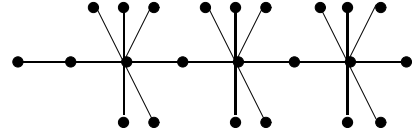Figure 4. A tree of minimum integrity with order 21 and $\Delta = 6$

Figure 5. A tree of minimum integrity with order 23 and $\Delta = 7$

Corollary 4.1 *If $G = (V, E)$ is a connected graph such that $|V(G)| = n \geq 3$ and $\Delta(G) = \Delta \geq 2$, then we have*

$$min_{G \in G[n,\Delta]} I(G) = \begin{cases} \lfloor\frac{n-2}{\Delta-1}\rfloor + 1, \\ if\ d(\frac{n-2}{\Delta-1}) < \lfloor\frac{n-2}{\Delta-1}\rfloor \\ \lfloor\frac{n-2}{\Delta-1}\rfloor + 2, \\ if\ d(\frac{n-2}{\Delta-1}) \geq \lfloor\frac{n-2}{\Delta-1}\rfloor \end{cases}$$

where $G[n, \Delta] = \{G : |V(G)| = n, \Delta(G) = \Delta\}$, $d(\frac{n-2}{\Delta-1})$ denotes the remainder of $n - 2$ divided by $\Delta - 1$.

Proof. From Lemma 4.2 we know that $I(G) \geq I(T)$, where T is the spanning tree of G. It follows from Theorem 4.1 that

$$min_{G \in G[n,\Delta]} I(G) = \begin{cases} \lfloor\frac{n-2}{\Delta-1}\rfloor + 1, \\ if\ d(\frac{n-2}{\Delta-1}) < \lfloor\frac{n-2}{\Delta-1}\rfloor \\ \lfloor\frac{n-2}{\Delta-1}\rfloor + 2, \\ if\ d(\frac{n-2}{\Delta-1}) \geq \lfloor\frac{n-2}{\Delta-1}\rfloor \end{cases}$$

where $G[n, \triangle] = \{G : |V(G)| = n, \Delta(G) = \Delta\}, d(\frac{n-2}{\Delta-1})$ denotes the number of remainder of $n - 2$ divided by $\Delta - 1$. ∎

5 Conclusion

The robustness of a distributed system of computers can be represented by the integrity of the graph describing the network. The authors present and prove a formula to calculate the maximum number of edges in a network of given integrity. Two construction methods for such networks are given, respectively to construct the maximum network with a given integrity and the minimum integrity network when the network graph is a tree. As a useful

parameter, integrity has been studied extensively, but there are many problems remaining unsolved. One interesting problem is: If $G = (V, E)$ is a connected graph with order n and integrity $I(\lceil 2\sqrt{n+1} \rceil - 1 \leq I \leq n - 1)$, then $min_{G \in G[n,I]}|E(G)| = ?$ Another interesting problem is how to construct such networks.

Acknowledgement

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