

# Semi-Finite Forms of Bilateral Basic Hypergeometric Series

William Y. C. Chen<sup>1</sup> and Amy M. Fu<sup>2</sup>

Center for Combinatorics, LPMC  
Nankai University, Tianjin 300071, P.R. China

Email: <sup>1</sup>chen@nankai.edu.cn, <sup>2</sup>fu@nankai.edu.cn

**Abstract.** We show that several classical bilateral summation and transformation formulas have semi-finite forms. We obtain these semi-finite forms from unilateral summation and transformation formulas. Our method can be applied to derive Ramanujan's  ${}_1\psi_1$  summation, Bailey's  ${}_2\psi_2$  transformations, and Bailey's  ${}_6\psi_6$  summation.

**Corresponding Author:** William Y. C. Chen, Email: chen@nankai.edu.cn

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## 1. Introduction

We follow the terminology for basic hypergeometric series in [6]. Assuming  $|q| < 1$ , let

$$(a; q)_\infty = (1 - a)(1 - aq)(1 - aq^2) \cdots .$$

For any integer  $n$ , the  $q$ -shifted factorial  $(a; q)_n$  is given by

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}.$$

For  $n \geq 0$ , we have the following relation which is crucial for this paper:

$$(a; q)_{-n} = \frac{1}{(aq^{-n}; q)_n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a; q)_n}. \quad (1.1)$$

For convenience, we employ the following usual notation:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n.$$

The (unilateral) basic hypergeometric series  ${}_{r+1}\phi_r$  is defined by

$${}_{r+1}\phi_r \left[ \begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix} ; q, z \right] = \sum_{k=0}^{\infty} A(k), \quad (1.2)$$

where

$$A(k) = \frac{(a_1, a_2, \dots, a_{r+1}; q)_k}{(b_1, b_2, \dots, b_r, q; q)_k} z^k.$$

The bilateral basic hypergeometric series  ${}_s\psi_s$  is defined as follows,

$${}_s\psi_s \left[ \begin{matrix} a_1, a_2, \dots, a_s \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right] = \sum_{k=-\infty}^{\infty} B(k), \quad (1.3)$$

where

$$B(k) = \frac{(a_1, a_2, \dots, a_s; q)_k}{(b_1, b_2, \dots, b_s; q)_k} z^k.$$

In this paper, we propose the following method of deriving bilateral summation and transformation formulas using *semi-finite forms*. For a bilateral series  ${}_s\psi_s$  as given in (1.3), we construct a summand  $G(k, m)$  which implies a unilateral series  ${}_{r+s+1}\phi_{r+s}$ , where  $r$  is a nonnegative integer, such that

$$\lim_{m \rightarrow \infty} G(k, m) = B(k)$$

for all  $k$ , and the summation

$$\sum_{k=-m}^{\infty} G(k, m) \quad (1.4)$$

can be easily accomplished as a Laurent extension of the summation

$$\sum_{k=0}^{\infty} G(k-m, m) = G(-m, m) \sum_{k=0}^{\infty} A(k), \quad (1.5)$$

where  $G(k, m)$  can be written as

$$G(k-m, m) = G(-m, m)A(k)$$

for some  $A(k)$ . The bilateral series (1.3) is then obtained from (1.4) as  $m \rightarrow \infty$ , subject to suitable convergence conditions. We apply this procedure to derive bilateral series identities from suitable unilateral ones. The above summation (1.4) is called the *semi-finite form* of the bilateral summation (1.3). A method similar to ours was recently used by Schlosser [9], and Jouhet and Schlosser [8], who derived summations for bilateral series from *finite forms*. We also note that another method, which uses a similar factorization as above, for deriving bilateral series identities from unilateral ones was used by Ismail [7], and Askey and Ismail [2]. Rather than taking limits, they apply analytic continuation as the main ingredient.

In this paper, we present semi-finite forms of several classical bilateral summation and transformation formulas such as Ramanujan's  ${}_1\psi_1$  formula, Bailey's  ${}_2\psi_2$  transformations, and Bailey's  ${}_6\psi_6$  summation.

## 2. From ${}_2\phi_1$ to ${}_1\psi_1$

Using the well known Gauss summation formula

$${}_2\phi_1 \left[ \begin{matrix} a, b \\ c \end{matrix}; q, c/ab \right] = \frac{(c/a, c/b; q)_{\infty}}{(c, c/ab; q)_{\infty}}, \quad (2.1)$$

where  $|c/ab| < 1$ , we get a semi-finite form of Ramanujan's summation of the general  ${}_1\psi_1$ ,

$${}_1\psi_1 \left[ \begin{matrix} a \\ b \end{matrix} ; q, z \right] = \sum_{k=-\infty}^{\infty} \frac{(a; q)_k}{(b; q)_k} z^k = \frac{(q; q)_{\infty} (b/a; q)_{\infty} (az; q)_{\infty} (q/az; q)_{\infty}}{(b; q)_{\infty} (q/a; q)_{\infty} (z; q)_{\infty} (b/az; q)_{\infty}}, \quad (2.2)$$

where  $|b/a| < |z| < 1$ .

**Proposition 2.1** For  $|z| < 1$ , the following identity holds:

$$\sum_{k=-m}^{\infty} \frac{(a; q)_k (bq^m/az; q)_k}{(q^{1+m}; q)_k (b; q)_k} z^k = \frac{(q; q)_m (q/az; q)_m}{(q/a; q)_m (b/az; q)_m} \frac{(b/a; q)_{\infty} (az; q)_{\infty}}{(b; q)_{\infty} (z; q)_{\infty}}. \quad (2.3)$$

*Proof.* The left hand side of (2.3) can be rewritten as

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(a; q)_{k-m} (bq^m/az; q)_{k-m}}{(q^{1+m}; q)_{k-m} (b; q)_{k-m}} z^{k-m} \\ &= z^{-m} \frac{(a; q)_{-m} (bq^m/az; q)_{-m}}{(q^{1+m}; q)_{-m} (b; q)_{-m}} \sum_{k=0}^{\infty} \frac{(aq^{-m}; q)_k (b/az; q)_k}{(q; q)_k (bq^{-m}; q)_k} z^k \\ &\stackrel{(2.1)}{=} z^{-m} \frac{(a; q)_{-m} (bq^m/az; q)_{-m}}{(q^{1+m}; q)_{-m} (b; q)_{-m}} \frac{(b/a; q)_{\infty} (azq^{-m}; q)_{\infty}}{(bq^{-m}; q)_{\infty} (z; q)_{\infty}} \\ &\stackrel{(1.1)}{=} z^{-m} \frac{(q; q)_m (azq^{-m}; q)_m}{(aq^{-m}; q)_m (b/az; q)_m} \frac{(az; q)_{\infty} (b/a; q)_{\infty}}{(b; q)_{\infty} (z; q)_{\infty}}, \end{aligned}$$

which equals the right hand side of (2.3). ■

Taking the limit  $m \rightarrow \infty$  in Proposition 2.1 while assuming  $|b/az| < 1$ , we immediately obtain (2.2).

We remark that our method is different from the method of M. Jackson's elementary proof of (2.2) (see the exposition of Schlosser [9]) in the sense that Jackson's proof does not give a semi-finite form although the Gauss summation is also the basic ingredient. We should also note that a finite form of Ramanujan's  ${}_1\psi_1$  summation has been given by Schlosser [10] using the terminating  $q$ -Pfaff-Saalschütz summation.

### 3. From ${}_3\phi_2$ to ${}_2\psi_2$

In this section, we use two  ${}_3\phi_2$  summation and transformation formulas to give the semi-finite forms of  ${}_2\psi_2$  formulas due to Bailey. We begin with the following  ${}_2\psi_2$  transformation formula [6, Ex. 5.20(i)] valid for  $|z|, |cd/abz|, |d/a|, |c/b| < 1$ :

$${}_2\psi_2 \left[ \begin{matrix} a, b \\ c, d \end{matrix} ; q, z \right] = \frac{(az, d/a, c/b, dq/abz; q)_{\infty}}{(z, d, q/b, cd/abz; q)_{\infty}} {}_2\psi_2 \left[ \begin{matrix} a, abz/d \\ az, c \end{matrix} ; q, \frac{d}{a} \right]. \quad (3.1)$$

Using a  $q$ -analogue of the Kummer-Thomae-Whipple formula [6, Eq. (3.2.7)]:

$${}_3\phi_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc} \right] = \frac{(e/a, de/bc; q)_\infty}{(e, de/abc; q)_\infty} {}_3\phi_2 \left[ \begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; q, \frac{e}{a} \right], \quad (3.2)$$

where  $|de/abc| < 1$  and  $|e/a| < 1$ , we get a semi-finite form of (3.1).

**Proposition 3.1** For  $|z| < 1$  and  $|d/a| < 1$ , we have

$$\sum_{k=-m}^{\infty} \frac{(a, b; q)_k (cdq^m/abz; q)_k}{(c, d; q)_k (q^{1+m}; q)_k} z^k = \frac{(az, d/a; q)_\infty (c/b, dq/abz; q)_m}{(z, d; q)_\infty (q/b, cd/abz; q)_m} \cdot \sum_{k=-m}^{\infty} \frac{(a, cq^m/b, abz/d; q)_k}{(c, q^{1+m}, az; q)_k} (d/a)^k. \quad (3.3)$$

*Proof.* The left hand side of (3.3) equals

$$\begin{aligned} & z^{-m} \frac{(a, b, cdq^m/abz; q)_{-m}}{(c, d, q^{1+m}; q)_{-m}} \sum_{k=0}^{\infty} \frac{(aq^{-m}, bq^{-m}, cd/abz; q)_k}{(cq^{-m}, dq^{-m}, q; q)_k} z^k \\ \stackrel{(3.2)}{=} & z^{-m} \frac{(a, b, cdq^m/abz; q)_{-m}}{(c, d, q^{1+m}; q)_{-m}} \frac{(d/a, azq^{-m}; q)_\infty}{(dq^{-m}, z; q)_\infty} \\ & \cdot \sum_{k=0}^{\infty} \frac{(aq^{-m}, c/b, abzq^{-m}/d; q)_k}{(q, cq^{-m}, azq^{-m}; q)_k} \left( \frac{d}{a} \right)^k \\ \stackrel{(1.1)}{=} & \frac{(d/a, az; q)_\infty (c/b, abzq^{-m}/d; q)_m}{(d, z; q)_\infty (bq^{-m}, cd/abz; q)_m} \left( \frac{d}{az} \right)^m \\ & \cdot \sum_{k=0}^{\infty} \frac{(a, cq^m/b, abz/d; q)_{k-m}}{(c, q^{1+m}, az; q)_{k-m}} (d/a)^{k-m}, \end{aligned}$$

which can be rewritten in the form of the right hand side of (3.3). ■

The next  ${}_2\psi_2$  transformation formula we consider is the following [6, Ex. 5.20(ii)]:

$${}_2\psi_2 \left[ \begin{matrix} a, b \\ c, d \end{matrix}; q, z \right] = \frac{(az, bz, cq/abz, dq/abz; q)_\infty}{(q/a, q/b, c, d; q)_\infty} {}_2\psi_2 \left[ \begin{matrix} abz/c, abz/d \\ az, bz \end{matrix}; q, \frac{cd}{abz} \right]. \quad (3.4)$$

Using a summation of Hall [6, Eq. (3.2.10)]:

$${}_3\phi_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc} \right] = \frac{(b, de/ab, de/bc; q)_\infty}{(d, e, de/abc; q)_\infty} {}_3\phi_2 \left[ \begin{matrix} d/b, e/b, de/abc \\ de/ab, de/bc \end{matrix}; q, b \right], \quad (3.5)$$

where  $|de/abc| < 1$  and  $|b| < 1$ , we obtain the following semi-finite form of (3.4).

**Proposition 3.2** For  $|z| < 1$  and  $|cd/abz| < 1$ , we have

$$\sum_{k=-m}^{\infty} \frac{(a, b; q)_k (cdq^m/abz; q)_k}{(c, d; q)_k (q^{1+m}; q)_k} z^k = \frac{(az, bz, cd/abz; q)_\infty (cq/abz, dq/abz, z; q)_m}{(c, d, z; q)_\infty (q/a, q/b, cd/abz; q)_m} \cdot \sum_{k=-m}^{\infty} \frac{(abz/c, abz/d, zq^m; q)_k}{(az, bz, q^{1+m}; q)_k} (cd/abz)^k.$$

#### 4. From nonterminating ${}_8\phi_7$ to ${}_6\psi_6$

In this section, we give a semi-finite form of Bailey's  ${}_6\psi_6$  summation formula by using Bailey's 3-term transformation formula for a nonterminating very-well-poised  ${}_8\phi_7$  series [6, Eq. (2.11.1)]:

$$\begin{aligned}
& {}_8\phi_7 \left[ \begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, f \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix} ; q, \frac{a^2q^2}{bcdef} \right] \\
&= \frac{(aq, aq/de, aq/df, aq/ef, eq/c, fq/c, b/a, bef/a; q)_\infty}{(aq/d, aq/e, aq/f, aq/def, q/c, eq/c, be/a, bf/a; q)_\infty} \\
&\quad \cdot {}_8\phi_7 \left[ \begin{matrix} ef/c, q(ef/c)^{\frac{1}{2}}, -q(ef/c)^{\frac{1}{2}}, aq/bc, aq/cd, ef/a, e, f \\ (ef/c)^{\frac{1}{2}}, -(ef/c)^{\frac{1}{2}}, bef/a, def/a, aq/c, fq/c, eq/c \end{matrix} ; q, \frac{bd}{a} \right] \\
&\quad + \frac{b}{a} \frac{(aq, bq/a, bq/c, bq/d, bq/e, bq/f, d, e, f; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, bd/a, be/a, bf/a, def/a; q)_\infty} \\
&\quad \cdot \frac{(aq/bc, bdef/a^2, a^2q/bdef; q)_\infty}{(aq/def, q/c, b^2q/a; q)_\infty} \\
&\quad \cdot {}_8\phi_7 \left[ \begin{matrix} b^2/a, qba^{-\frac{1}{2}}, -qba^{-\frac{1}{2}}, b, bc/a, bd/a, be/a, bf/a \\ ba^{-\frac{1}{2}}, -ba^{-\frac{1}{2}}, bq/a, bq/c, bq/d, bq/e, bq/f \end{matrix} ; q, \frac{a^2q^2}{bcdef} \right], \quad (4.1)
\end{aligned}$$

where  $|bd/a| < 1$  and  $|a^2q^2/bcdef| < 1$ .

**Proposition 4.1** *When  $|bd/a| < 1$  and  $|a^2q^2/bcdef| < 1$ , we have*

$$\begin{aligned}
& \sum_{k=-m}^{\infty} \frac{(q^{m-k+1}/a, fq^m; q)_k}{(q^{m-k}f/a, q^{1+m}; q)_k} \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_k}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_k} \left( \frac{qa^2}{bcde} \right)^k \\
&= \frac{1 - efq^{2m}/c}{1 - efq^m/c} \frac{(q/a, df/a, ef/a, aq/bc, aq/cd, efq^m/a; q)_m}{(f/a, q/b, q/c, q/d, def/a, fq^{1+m}/c; q)_m} \\
&\quad \times \frac{(aq, aq/de, aq/df, aq/ef, eq^{1+m}/c, fq^{1+m}/c, b/a, befq^m/a; q)_\infty}{(aq/d, aq/e, aq/f, aq/def, q^{1+m}/c, eq^{1+m}/c, be/a, bfq^m/a; q)_\infty} \\
&\quad \times \sum_{k=-m}^{\infty} \frac{(efq^m/c, q^{1+m}(ef/c)^{\frac{1}{2}}, -q^{1+m}(ef/c)^{\frac{1}{2}}, aq^{1+m}/bc; q)_k}{(q^{1+m}, q^m(ef/c)^{\frac{1}{2}}, -q^m(ef/c)^{\frac{1}{2}}, befq^m/a; q)_k} \\
&\quad \cdot \frac{(aq^{1+m}/cd, efq^{2m}/a, e, fq^m; q)_k}{(defq^m/a, aq/c, fq^{1+2m}/c, eq^{1+m}/c; q)_k} \left( \frac{bd}{a} \right)^k \\
&\quad + \frac{b}{a} \frac{1 - b^2q^{2m}/a}{1 - b^2q^m/a} \left( \frac{a^2q}{bcde} \right)^m \frac{(q/a, bc/a; q)_m}{(f/a; q)_m} \frac{(aq, bq^{1+2m}/a, bq^{1+m}/c; q)_\infty}{(aq/b, aq/c, aq/d; q)_\infty} \\
&\quad \times \frac{(bq^{1+m}/d, bq^{1+m}/e, bq/f, d, e, fq^m, aq/bc, bdef/a^2, a^2q/bdef; q)_\infty}{(aq/e, aq/f, bdq^m/a, beq^m/a, bfq^{2m}/a, def/a, aq/def, q/c, b^2q^{1+m}/a; q)_\infty} \\
&\quad \times \sum_{k=-m}^{\infty} \frac{(b^2q^m/a, q^{1+m}ba^{-\frac{1}{2}}, -q^{1+m}ba^{-\frac{1}{2}}; q)_k}{(q^{1+m}, q^mba^{-\frac{1}{2}}, -q^mba^{-\frac{1}{2}}; q)_k} \\
&\quad \cdot \frac{(b, bcq^m/a, bdq^m/a, beq^m/a, bfq^{2m}/a; q)_k}{(bq^{1+2m}/a, bq^{1+m}/c, bq^{1+m}/d, bq^{1+m}/e, bq/f; q)_k} \left( \frac{a^2q^2}{bcdef} \right)^k. \quad (4.2)
\end{aligned}$$

*Proof.* The left hand side of (4.2) equals

$$\begin{aligned}
& \sum_{k=-m}^{\infty} \frac{(aq^{-m}, fq^m; q)_k}{(aq^{1-m}/f, q^{1+m}; q)_k} \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_k}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_k} \left( \frac{q^2 a^2}{bcdef} \right)^k \\
& \stackrel{(1.5)}{=} \frac{(aq^{-m}, fq^m, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_{-m}}{(aq^{1-m}/f, q^{1+m}, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_{-m}} \left( \frac{q^2 a^2}{bcdef} \right)^{-m} \\
& \quad \times \sum_{k=0}^{\infty} \frac{(aq^{-2m}, q^{1-m} a^{\frac{1}{2}}, -q^{1-m} a^{\frac{1}{2}}; q)_k}{(q, q^{-m} a^{\frac{1}{2}}, -q^{-m} a^{\frac{1}{2}}; q)_k} \\
& \quad \cdot \frac{(bq^{-m}, cq^{-m}, dq^{-m}, eq^{-m}, f; q)_k}{(aq^{1-m}/b, aq^{1-m}/c, aq^{1-m}/d, aq^{1-m}/e, aq^{1-2m}/f; q)_k} \left( \frac{q^2 a^2}{bcdef} \right)^k \\
& \stackrel{(4.1)}{=} \frac{(aq^{-m}, fq^m, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_{-m}}{(aq^{1-m}/f, q^{1+m}, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_{-m}} \left( \frac{q^2 a^2}{bcdef} \right)^{-m} \\
& \quad \times \frac{(aq^{1-2m}, aq/de, aq^{1-m}/df, aq^{1-m}/ef, eq/c, fq^{1+m}/c, bq^m/a, bef/a; q)_{\infty}}{(aq^{1-m}/d, aq^{1-m}/e, aq^{1-2m}/f, aq/def, q^{1+m}/c, eq/c, be/a, bfq^m/a; q)_{\infty}} \\
& \quad \times {}_8\phi_7 \left[ \begin{matrix} ef/c, q(ef/c)^{\frac{1}{2}}, -q(ef/c)^{\frac{1}{2}}, aq/bc, aq/cd, efq^m/a, eq^{-m}, f \\ (ef/c)^{\frac{1}{2}}, -(ef/c)^{\frac{1}{2}}, bef/a, def/a, aq^{1-m}/c, fq^{1+m}/c, eq/c \end{matrix} ; q, \frac{bd}{a} \right] \\
& \quad + \frac{(aq^{-m}, fq^m, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_{-m}}{(aq^{1-m}/f, q^{1+m}, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_{-m}} \left( \frac{q^2 a^2}{bcdef} \right)^{-m} \\
& \quad \times \frac{bq^m}{a} \frac{(aq^{1-2m}, bq^{1+m}/a, bq/c, bq/d, bq/e, bq^{1-m}/f, dq^{-m}; q)_{\infty}}{(aq^{1-m}/b, aq^{1-m}/c, aq^{1-m}/d, aq^{1-m}/e, aq^{1-2m}/f, bd/a, be/a; q)_{\infty}} \\
& \quad \times \frac{(eq^{-m}, f, aq/bc, bdefq^m/a^2, a^2 q^{1-m}/bdef; q)_{\infty}}{(bfq^m/a, def/a, aq/def, q^{1+m}/c, b^2 q/a; q)_{\infty}} \\
& \quad \times {}_8\phi_7 \left[ \begin{matrix} b^2/a, qba^{-\frac{1}{2}}, -qba^{-\frac{1}{2}}, bq^{-m}, bc/a, bd/a, be/a, bfq^m/a \\ ba^{-\frac{1}{2}}, -ba^{-\frac{1}{2}}, bq^{1+m}/a, bq/c, bq/d, bq/e, bq^{1-m}/f \end{matrix} ; q, \frac{a^2 q^2}{bcdef} \right],
\end{aligned}$$

which equals to the right hand side of (4.2). ■

The above proposition can be viewed as a semi-finite form of Bailey's  ${}_6\psi_6$  summation formula. By taking  $f = b$  and  $m \rightarrow \infty$  in (4.2) while assuming  $|a^2 q/bcde| < 1$ , we get

$$\begin{aligned}
& {}_6\psi_6 \left[ \begin{matrix} qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/d, aq/e \end{matrix} ; q, \frac{a^2q}{bcde} \right] \\
&= \frac{(q/a, bd/a, aq/bc, aq/cd, aq, aq/de, aq/bd, aq/be; q)_\infty}{(q/b, q/c, q/d, aq/b, aq/d, aq/e, aq/bde, bde/a; q)_\infty} \\
&\quad \times \sum_{k=-\infty}^{\infty} \frac{(e; q)_k}{(aq/c; q)_k} (bd/a)^k \\
(2.2) \quad &= \frac{(q/a, bd/a, aq/bc, aq/cd, aq, aq/de, aq/bd, aq/be; q)_\infty}{(q/b, q/c, q/d, aq/b, aq/d, aq/e, aq/bde, bde/a; q)_\infty} \\
&\quad \times \frac{(q, aq/ce, bde/a, aq/bde; q)_\infty}{(aq/c, q/e, bd/a, a^2q/bcde; q)_\infty} \\
&= \frac{(aq, aq/bc, aq/bd, aq/be, aq/ce, aq/cd, aq/de, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde; q)_\infty}.
\end{aligned}$$

Many proofs of above identity have been found, see, for example, Slater and Lakin [11], Andrews [1], Askey and Ismail [2], Askey [3], Chen and Liu [5], Schlosser [9] and Jouhet and Schlosser [8]. Our proof shows that the semi-finite form of the  ${}_6\psi_6$  summation is in essence a shifted version of the  ${}_8\phi_7$  summation. This proof utilizes Ramanujan's  ${}_1\psi_1$  summation (2.2). It would be interesting to find a proof using a semi-finite (or even finite) form which yields Bailey's  ${}_6\psi_6$  summation in a direct limit, without the need to invoke another summation formula as above.

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