On some q-identities related to divisor functions

Jiang Zeng

Institut Girard Desargues, Université Claude Bernard (Lyon I)
21 Avenue Claude Bernard, 69622 Villeurbanne Cedex, France
and
Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, People's Republic of China
e-mail: `zeng@igd.univ-lyon1.fr`

Abstract: We give generalizations and simple proofs of some q-identities of Dilcher, Fu and Lascoux related to divisor functions.

Let $a_1, \ldots, a_N$ be N indeterminates. It is easy to see that

$$\frac{1}{(1-a_1z)(1-a_2z)\ldots(1-a_Nz)} = \sum_{k=1}^{N} \frac{\prod_{j=1,j\neq k}^{N}(1-a_j/a_k)^{-1}}{1-a_kz}. \tag{1}$$

The coefficient of z^τ ($\tau \geq 0$) in the left side of (1) is usually called the τ-th *complete symmetric function* $h_\tau(a_1, \ldots, a_N)$ of $a_1, \ldots, a_N$. Clearly, we have $h_0(a_1, \ldots, a_N) = 1$ and equating the coefficients of z^τ ($\tau \geq 1$) in two sides of (1) yields

$$h_\tau(a_1, \ldots, a_N) := \sum_{1\leq i_1\leq i_2\leq\cdots\leq i_\tau\leq N} a_{i_1}a_{i_2}\ldots a_{i_\tau} = \sum_{k=1}^{N}\prod_{j=1,j\neq k}^{N}(1-a_j/a_k)^{-1}\, a_k^\tau. \tag{2}$$

In particular, if $a_k = \frac{a-bq^{k+i-1}}{c-zq^{k+i-1}}$ ($1 \leq k \leq N$) for a fixed integer i ($1 \leq i \leq n$), then formula (2) with $N = n-i+1$ reads

$$\begin{aligned} h_\tau\left(\frac{a-bq^i}{c-zq^i}, \frac{a-bq^{i+1}}{c-zq^{i+1}}, \ldots, \frac{a-bq^n}{c-zq^n}\right) &= \frac{c^{n-i+1}(zq^i/c)_{n-i+1}}{(q)_{n-i+1}(az-bc)^{n-i}} \\ &\cdot \sum_{k=i}^{n}(-1)^{k-i}\begin{bmatrix} n-i+1 \\ n-k \end{bmatrix} q^{\binom{k-i+1}{2}-k(n-i)}\frac{(1-q^{k-i+1})(a-bq^k)^{\tau+n-i}}{(c-zq^k)^{\tau+1}}, \end{aligned} \tag{3}$$

where $(x)_n = (1-x)(1-xq)\ldots(1-xq^{n-1})$ and $\begin{bmatrix} n \\ i \end{bmatrix} = (q^{n-i+1})_i/(q)_i$ with $(x)_0 = 1$.

The aim of this note is to show that (3) turns out to be a common source of several q-identities surfacing recently in the literature.

First of all, the $i = 1$ case of formula (3) with $\tau = m-n+1$ corresponds to an identity of Fu and Lascoux [3, Prop. 2.1]:

$$\begin{aligned} &h_\tau\left(\frac{a-bq}{c-zq}, \frac{a-bq^2}{c-zq^2}, \ldots, \frac{a-bq^n}{c-zq^n}\right) \\ &= \frac{c^n(zq/c)_n}{(q)_n(az-bc)^{n-1}}\sum_{k=1}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^{k-1}q^{\binom{k+1}{2}-nk}\frac{(1-q^k)(a-bq^k)^m}{(c-zq^k)^{\tau+1}}. \end{aligned} \tag{4}$$

Next, for $i = 1, \ldots, n$ and $m \geq 1$ set

$$A_i(z) := \frac{q^i (zq)_{i-1}(q)_n}{(q)_i (zq)_n} h_{m-1}\left(\frac{q^i}{1-zq^i}, \ldots, \frac{q^n}{1-zq^n}\right). \tag{5}$$

Then we have the following polynomial identity in x:

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(x-1)\cdots(x-q^{k-1})}{(1-zq^k)^m} q^{mk} = \sum_{k=1}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\binom{k}{2}+mk}}{(1-zq^k)^m} + \sum_{i=1}^{n} A_i(z) x^i. \tag{6}$$

Indeed, using the q-binomial formula [1, p. 36]:

$$(x-1)(x-q)\ldots(x-q^{N-1}) = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^{N-j} x^j q^{\binom{N-j}{2}},$$

we see that the coefficient of x^i $(1 \leq i \leq n)$ in the left side of (6) is equal to

$$\sum_{k=i}^{n} (-1)^{k-i} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \frac{q^{mk+\binom{k-i}{2}}}{(1-zq^k)^m} = \frac{q^i (zq)_{i-1}(q)_n}{(q)_i (zq)_n} h_{m-1}\left(\frac{q^i}{1-zq^i}, \ldots, \frac{q^n}{1-zq^n}\right), \tag{7}$$

where the last equality follows from (3) with $a = 0$, $c = 1$, $b = -1$ and $\tau = m - 1$.

Now, with $z = i = 1$ and m shifted to $m + 1$, formula (7) reduces to Dilcher's identity [2]:

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^{k-1} q^{\binom{k}{2}+mk}}{(1-q^k)^m} = h_m\left(\frac{q}{1-q}, \ldots, \frac{q^n}{1-q^n}\right) = \sum_{i=1}^{n} A_i(1).$$

On the other hand, formula (1) with $N = n + 1$ and $a_i = q^{i-1}$ $(1 \leq i \leq N)$ yields

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\binom{k}{2}+k}}{1-zq^k} = \frac{(q)_n}{(z)_{n+1}}.$$

Hence, setting, respectively, $z = 1$ and $m = 1$ in formula (6) we recover two recent formulae of Fu and Lascoux [4] (see also [5]):

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(x-1)\cdots(x-q^{k-1})}{(1-q^k)^m} q^{mk} = \sum_{i=1}^{n} (x^i - 1) A_i(1), \tag{8}$$

and

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(x-1)\cdots(x-q^{k-1})}{1-zq^k} q^k = \frac{(q)_n}{(z)_{n+1}} \sum_{i=0}^{n} \frac{(z)_i}{(q)_i} x^i q^i. \tag{9}$$

Acknowledgement

The author is partially supported by EC's IHRP Programme, within the Research Training Network "Algebraic Combinatorics in Europe", grant HPRN-CT-2001-00272.

References

[1] G. Andrews, The theory of partitions, Encyclopedia of mathematics and its applications, G.-C. Rota, editor, vol. 2, 1976, Addison-Wesley Publishing Company.

[2] K. Dilcher, Some q-series identities related to divisor functions, Discrete Math., 145(1995), 83-93.

[3] A.M. Fu and A. Lascoux, q-identities from Lagrange and Newton interpolation, Adv. Appl. Math., 31(2003), 527-531.

[4] A.M. Fu and A. Lascoux, q-identities related to overpartitions and divisor functions, preprint, available at arXiv:math.CO/0404065, 2004.

[5] H. Prodinger, q-Identities of Fu and Lascoux proved by the q-Rice formula, arXiv:math.CO/0404209, 10 April 2004.