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CIRCUMFERENCE OF GRAPHS WITH BOUNDED DEGREE*

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Abstract. Karger, Motwani, and Ramkumar [*Algorithmica*, 18 (1997), pp. 82–98] have shown that there is no constant approximation algorithm to find a longest cycle in a Hamiltonian graph, and they conjectured that this is the case even for graphs with bounded degree. On the other hand, Feder, Motwani, and Subi [*SIAM J. Sci. Comput.*, 31 (2002), pp. 1596–1607] have shown that there is a polynomial time algorithm for finding a cycle of length $n^{\log_3 2}$ in a 3-connected cubic n-vertex graph. In this paper, we show that if G is a 3-connected n-vertex graph with maximum degree at most d, then one can find, in $O(n^3)$ time, a cycle in G of length at least $\Omega(n^{\log_b 2})$, where $b = 2(d-1)^2 + 1$.

Key words. bounded degree, 3-connected components, long cycles and paths, circumference

AMS subject classifications. 05C38, 05C45, 05C85

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1. Introduction and notation.

The *circumference* of a graph is the length of a longest cycle in that graph. The problem of approximating the circumference of a graph is NP-hard [15]. For many canonical NP-hard problems, either good approximation algorithms have been devised, or strong negative results have been established, leading to better understanding of the approximability of these problems. However, not much is known for finding longest paths and cycles, positive or negative. For example, there is no known algorithm which guarantees an approximation ratio better than $n/\text{polylog}(n)$, where n denotes the number of vertices. This is true even for graphs which are Hamiltonian or have bounded degree. Karger, Motwani, and Ramkumar [15] showed that unless $\mathcal{P} = \mathcal{NP}$, it is impossible to find, in polynomial time, a path of length $n - n^\epsilon$ in an n-vertex Hamiltonian graph for any $\epsilon < 1$. They conjectured that it is just as hard for graphs with bounded degree.

On the positive side, if a graph has a path of length L, then one can find a path of length $\Omega((\log L/\log\log L)^2)$ [1] (also see [20]). Feder, Motwani, and Subi [6] showed that there is a polynomial time algorithm for finding a cycle of length at least $n^{\log_3 2}$ in a 3-connected cubic n-vertex graph. They also showed that if a graph has maximum degree at most three and has a path or cycle of length L, then one can find a path or cycle of length at least $L^{(\log_3^2)/2}$. Therefore, an intermediate problem is to find long paths or cycles in graphs of bounded degree that have a Hamilton cycle. Specifically, Feder, Motwani, and Subi (see [6], p. 1605) asked (1) whether there exists some constant $0 < c < 1$ such that if G is a 3-connected planar n-vertex graph, then the circumference of G is at least $\Omega(n^c)$, and (2) whether there exists some constant $0 < c < 1$ such that if G is a 3-connected n-vertex graph with bounded degree, then the circumference of G is at least $\Omega(n^c)$. There are known results showing that such

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a constant c exists in both cases ([3], [14]); however, none addresses the algorithmic issue. The main goal of this paper is to establish a cubic algorithm that produces a long cycle in a 3-connected graph with bounded degree.

The work on circumferences of planar graphs dates back to 1931, when Whitney [21] proved that every 4-connected planar triangulation contains a Hamilton cycle (and, hence, its faces are 4-colorable). This result is generalized to all 4-connected planar graphs in [18]. A linear time algorithm is given in [4] for finding a Hamilton cycle in a 4-connected planar graph. There are many 3-connected planar graphs which do not contain Hamilton cycles (see [9]). On the other hand, the following conjecture of Barnette (see [16]) remains open: every bipartite, cubic, 3-connected, planar graph contains a Hamilton cycle. When studying paths in polytopes, Moon and Moser [17] implicitly conjectured that if G is a 3-connected planar n-vertex graph then G contains a cycle of length at least $\Omega(n^{\log_3 2})$. (Grünbaum and Walther [8] made the same conjecture for a class of 3-connected cubic planar graphs.) Jackson and Wormald [13] gave the first polynomial lower bound $\Omega(n^c)$, where c is approximately 0.2, which was improved to $\Omega(n^{0.4})$ by Gao and Yu [7]. Chung [5] further improved this lower bound to $\Omega(n^{0.5})$. Recently, Chen and Yu [3] fully established the Moon–Moser conjecture; their proof implies a quadratic algorithm for finding a cycle of length at least $\Omega(n^{\log_3 2})$ in a 3-connected planar n-vertex graph. We conjecture that such a cycle may be found in linear time.

The work on circumferences of 3-connected graphs with bounded degree dates back to 1980, when Bondy and Simonovits [2] conjectured that there exists a constant $0 < c < 1$ such that the circumference of any 3-connected cubic n-vertex graph is at least $\Omega(n^c)$. This conjecture was verified by Jackson [12]. In 1993, Jackson and Wormald [14] proved that if G is a 3-connected n-vertex graph with maximum degree at most d, then the circumference of G is at least $\frac{1}{2}n^{\log_b 2} + 1$, where $b = 6d^2$. The argument in [14] is technical, and Jackson and Wormald did not address the algorithmic issue.

In this paper, we improve the lower bound of Jackson and Wormald, for both the exponent and the constant coefficient. Our argument makes efficient use of two results: a convexity result of a function and a decomposition result of 2-connected graphs. Our proof gives rise to a cubic algorithm for finding a long cycle in 3-connected graphs with bounded degree. More precisely, we prove the following result.

THEOREM 1.1. *Let $n \geq 4$ and $d \geq 3$ be integers. Let G be a 3-connected graph on n vertices such that the maximum degree of G is at most d. Then G contains a cycle of length at least $n^{\log_b 2} + 2$, where $b = 2(d-1)^2 + 1$. Moreover, such a cycle can be found in $O(n^3)$ time.*

It is conjectured in [14] that, for $d \geq 4$, the lower bound in Theorem 1.1 may be replaced by $\Omega(n^{\log_{d-1} 2})$. We are hopeful that our approach will eventually lead to a resolution of this conjecture.

To prove Theorem 1.1, we will need to deal with graphs which result from a 3-connected graph by deleting one vertex. Such graphs are 2-connected but not necessarily 3-connected. Our technique is to decompose such a graph into "3-connected components." This can be done in linear time by a result of Hopcroft and Tarjan [10]. (A similar idea is used in [14], but our decomposition is done once for each graph in a single iteration of the algorithm, and we make more efficient use of such a decomposition.) In most situations, we will not use all 3-connected components of a graph. Instead, we will pick some large 3-connected components and find long cycles in such components. We will then use a convexity property of the function $f(x) = x^{\log_b 2}$ to account for the unused components. These two ideas will be made more precise in the next two sections.

This paper is organized as follows. In section 2, we will state a technical result, consisting of three statements about (a) the existence of a long cycle through a given edge and avoiding a given vertex, (b) the existence of a long cycle through two given edges, and (c) the existence of a long cycle through a given edge. (We will see that (c) implies Theorem 1.1.) We will also describe the decomposition of a 2-connected graph into 3-connected components. In section 3, we will prove useful properties of the convex function $f(x) = x^{\log_b 2}$, for $b = 3$ and $b \geq 4$. We will also prove several lemmas to be used in the proof of our main result. In sections 4–6, we will show that each of (a), (b), and (c) can be reduced in linear time to (a), (b), and/or (c) for smaller graphs. In section 7, we will complete the proof of our main result and give a cubic algorithm that finds a long cycle in a 3-connected graph with bounded degree.

We end this section with notation and terminology to be used throughout this paper. Let G be a graph. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively, and we write $G = (V(G), E(G))$. For convenience, we write $|G|$ instead of $|V(G)|$. If $e \in E(G)$ and x, y are the vertices of G incident with e, then we write $e = xy$. For any $S \subseteq V(G) \cup E(G)$, $G - S$ denotes the graph obtained from G by deleting S and all edges of G with an incident vertex in S. If $S = \{x\}$, then we simply write $G - x$ instead of $G - S$.

Let G and H be two graphs. By $H \subseteq G$ we mean that H is a subgraph of G. We use $G \cup H$ and $G \cap H$ to denote the union and intersection, respectively, of G and H. For any $S \subseteq V(G) \cup E(G)$ and for any $H \subseteq G$, we use $H + S$ to denote the graph with vertex set $V(H) \cup (S \cap V(G))$ and edge set $E(H) \cup \{uv \in S : \{u, v\} \subseteq V(H) \cup (S \cap V(G))\}$.

We say that a graph G is k-connected if $|G| \geq k + 1$ and, for any $S \subseteq V(G)$ with $|S| \leq k - 1$, $G - S$ is connected. Let G be a graph. If $S \subseteq V(G)$ for which $G - S$ is not connected, then S is a *cut* of G, and if, in addition, $|S| = k$, then S is a *k-cut*. If $x \in V(G)$ for which $G - x$ is not connected, then x is called a *cut vertex* of G. If $e \in E(G)$ for which $G - e$ is not connected, then e is called a *cut edge* of G.

2. 3-connected components. We begin this section by stating a technical result which implies Theorem 1.1. To motivate that statement, let G be a 3-connected graph. In order to find a long cycle in G, we will try to find a cycle through a specific edge $e = xy$ (for induction purposes). To reduce the problem to smaller graphs, we consider $G - y$. Clearly $G - y$ is 2-connected but not necessarily 3-connected. In the case when $G - y$ is not 3-connected, y is contained in a 3-cut T of G. Let $T := \{y, a, b\}$, and let G_1, G_2 be subgraphs of G such that $E(G_1) \cap E(G_2) = \emptyset$, $V(G_1) \cap V(G_2) = T$, and $G_1 \cup G_2 = G$. See Figure 1 for an illustration. Assume $x \in V(G_1) - T$. We could find a long cycle C_1 through both e and ab in $G_1 + ab$ and a long cycle C_2 through ab in $(G_2 + ab) - y$, and then $C := (C_1 - ab) \cup (C_2 - ab)$ would give a long cycle in G. Note that C_1 is a cycle through two given edges, C_2 is a cycle through one given edge and avoiding a given vertex, and C is a cycle through one given edge. This suggests that we prove three statements simultaneously. Indeed, we will prove the following.

THEOREM 2.1. *Let $n \geq 5$ and $d \geq 3$ be integers, let $r = \log_{2(d-1)^2+1} 2$, and let G be a 3-connected graph on n vertices. Then the following statements hold:*

(a) *Let $xy \in E(G)$ and $z \in V(G) - \{x, y\}$, and let t denote the number of neighbors of z distinct from x and y. Assume that the maximum degree of G is at most $d + 1$, and every vertex of degree $d + 1$ (if any) is incident with the edge zx or zy. Then there is a cycle C through xy in $G - z$ such that $|C| \geq (\frac{n}{2t})^r + 2$.*

(b) *Suppose the maximum degree of G is at most d. Then, for any distinct $e, f \in$*

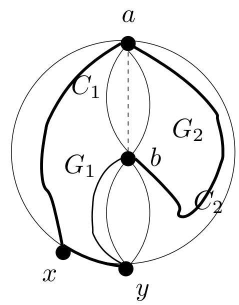

FIG. 1. *An illustration.*

$E(G)$, there is a cycle C through e and f in G such that $|C| \geq (\frac{n}{2(d-1)})^r + 3$.

(c) *Suppose that the maximum degree of G is at most d. Then, for any $e \in E(G)$, there is a cycle C through e in G such that $|C| \geq n^r + 3$.*

Clearly, Theorem 2.1 (c) implies Theorem 1.1 when $n \geq 5$, and Theorem 1.1 is obvious when $n = 4$. Note the condition in (a) about the maximum degree; it is due to the addition of edges in order to maintain 3-connectivity.

To prove Theorem 2.1, we need to decompose a 2-connected graph (such as $G - z$ in (a) above) into 3-connected components. This is similar to the decomposition of a connected graph into 2-connected components. Let G be a connected graph. A *block* of G is a subgraph of G which is either a maximal 2-connected subgraph of G or a subgraph of G induced by a cut edge of G. A block of G is also called a 2-*connected component of* G. It is easy to see that the intersection of any two blocks of G either is empty or consists of only one vertex (which is a cut vertex). Also any noncut vertex of G occurs in exactly one block of G. This implies that the blocks and cut vertices of G form a tree structure.

Now let G be a 2-connected graph. We describe the 3-connected components of G, following Hopcroft and Tarjan [10]. For this purpose, we allow multiple edges (and hence $E(G)$ is a multiset). We say that $\{a, b\} \subseteq V(G)$ is a *separation pair* in G if there are subgraphs G_1, G_2 of G such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{a, b\}$, $E(G_1) \cap E(G_2) = \emptyset$, and $|E(G_i)| \geq 2$ for $i = 1, 2$. Let $G'_i := (V(G_i), E(G_i) \cup \{ab\})$ for $i = 1, 2$. See Figure 2 for an example. Then G'_1 and G'_2 are called *split graphs* of G with respect to the separation pair $\{a, b\}$, and the new edge ab added to G_i is called a *virtual* edge. Virtual edges are illustrated with dashed edges in Figures 2–4. It is easy to see that since G is 2-connected, G'_i is 2-connected or G'_i consists of two vertices and at least three multiple edges between them.

Suppose that a multigraph is split, and the split graphs are split, and so on, until no more splits are possible. Then each remaining graph is called a *split component*. See Figure 3 for a graph G and its split components. No split component contains a separation pair, and therefore each split component must be one of the following: a triangle, a *triple bond* (two vertices with three multiple edges between), or a 3-connected graph.

It is not hard to see that if a split component of a 2-connected graph is 3-connected, then it is unique. It is also easy to see that, for any two split components G_1, G_2 of a 2-connected graph, we have $|V(G_1) \cap V(G_2)| = 0$ or 2, and if $|V(G_1) \cap V(G_2)| = 2$, then either G_1 and G_2 share a virtual edge between vertices in $V(G_1) \cap V(G_2)$ or there is a sequence of triple bonds such that the first shares a virtual edge with G_1, any two consecutive triple bonds in the sequence share a virtual

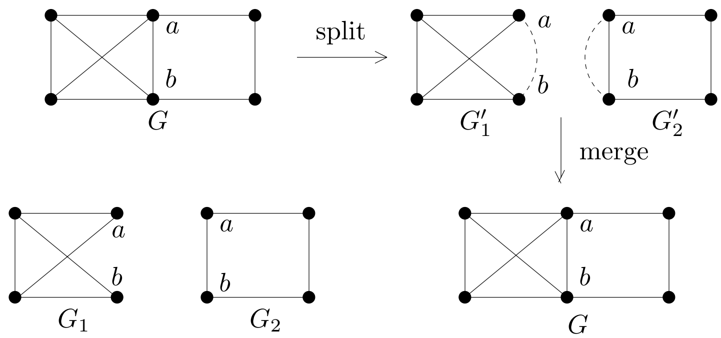

FIG. 2. *Split and merge.*

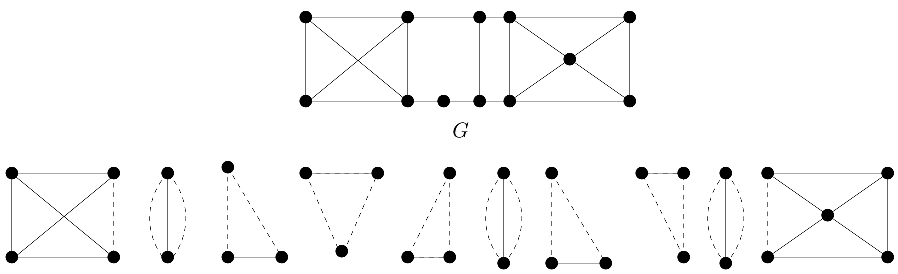

FIG. 3. *Split components of G.*

edge, and the last triple bond shares a virtual edge with G_2.

In order to get unique 3-connected components, we need to merge some triple bonds and to merge some triangles. Let $G'_i = (V'_i, E'_i)$, $i = 1, 2$, be two split components, both containing a virtual edge ab. Let $G' = (V'_1 \cup V'_2, (E'_1 - \{ab\}) \cup (E'_2 - \{ab\}))$. Then the graph G' is called the *merge graph* of G_1 and G_2. See Figure 2 for an example of a merge graph. Clearly, a merge of triple bonds gives a graph consisting of two vertices and multiple edges, which is called a *bond*. Also a merge of triangles gives a cycle, and a merge of cycles also gives a cycle.

Let $\mathcal{D}$ denote the set of 3-connected split components of a 2-connected graph G. We merge the other split components of G as follows: the triple bonds are merged as much as possible to give a set of bonds $\mathcal{B}$, and the triangles are merged as much as possible to give a set of cycles $\mathcal{C}$. Then $\mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ is the set of the 3-*connected components* of G. Figure 4 gives the 3-connected components of the graph in Figure 3. Note that any two 3-connected components either are edge disjoint or share exactly one virtual edge.

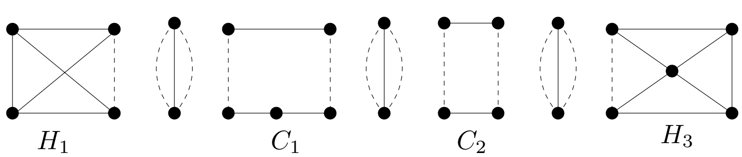

FIG. 4. 3-*connected components of the graph G in Figure* 3.

Tutte [19] proved that the above decomposition of a 2-connected graph into 3-connected components is unique. Hopcroft and Tarjan [10] gave a linear time algorithm for finding the 3-connected components of a graph.

THEOREM 2.2. *For any 2-connected graph, the 3-connected components are unique and can be found in $O(|E|)$ time. Moreover, the total number of edges in the 3-connected components is at most $3|E|-6$.*

We define a graph whose vertices are the 3-connected components of G, and two vertices are adjacent if the corresponding 3-connected components share a virtual edge. Then it is easy to see that such a graph is a tree, and we call it the *block-bond tree* of G.

For convenience, 3-connected components that are not bonds are called 3-*blocks.* An *extreme* 3-block is a 3-block that contains at most one virtual edge. That is, either it is the only 3-connected component, or it corresponds to a degree one vertex in the block-bond tree.

We will make use of cycle chains. Intuitively, a cycle chain in a 2-connected graph G is a sequence $C_1C_2\dots C_k$ of 3-blocks of G for which each C_i is a cycle and there exist bonds $B_1, B_2, \dots, B_{k-1}$ of G such that $C_1B_1C_2B_2\dots B_{k-1}C_k$ is a path in the block-bond tree of G. More precisely, we have the following.

DEFINITION 2.3. *Let G be a 2-connected graph. By a cycle chain in G we mean a sequence $C_1\dots C_k$ with the following properties:*

(i) *for each $1 \le i \le k$, C_i is a 3-block of G and C_i is a cycle;*
(ii) *$|V(C_i)\cap V(C_{i+1})| = 2$, and C_i and C_{i+1} each contain a virtual edge between the vertices in $V(C_i)\cap V(C_{i+1})$; and*
(iii) *$|V(C_i)\cap V(C_j)| \le 1$ when $j \ge i+2$, and if $i<j$ and $|V(C_i)\cap V(C_j)| = 1$, then, for all $i \le t \le j$, $V(C_i)\cap V(C_j) \subseteq V(C_t)\cap V(C_j)$.*

For convenience, we sometimes write $H := C_1\dots C_k$ and view H as the graph $\bigcup_{i=1}^k C_i$. Hence $V(H) := \bigcup_{i=1}^k V(C_i)$. Note that H is a multigraph, with two virtual edges between the vertices in $V(C_i)\cap V(C_{i+1})$, $1 \le i \le k-1$.

As an example, take the graph G in Figure 3 and its 3-connected components in Figure 4; we see that C_1C_2 is a cycle chain.

Remark. We choose not to include bonds in our definition of cycle chains because those bonds do not contribute the vertex count in our arguments.

It is easy to see that if $C_1\dots C_k$ is a cycle chain, then deleting all virtual edges with both ends in $V(C_i)\cap V(C_{i+1})$, $1 \le i \le k-1$, results in a cycle. We state it as follows.

PROPOSITION 2.4. *Let G be a 2-connected graph, let $C_1\dots C_k$ be a cycle chain in G, let $uv \in E(C_1)$ with $\{u,v\} \ne V(C_1)\cap V(C_2)$ when $k \ne 1$, and let $xy \in E(C_k)$ with $\{x,y\} \ne V(C_{k-1})\cap V(C_k)$ when $k\ne 1$. Then $\bigcup_{i=1}^k C_i$ contains a Hamilton cycle through uv and xy. Moreover, such a cycle can be found in $O(|\bigcup_{i=1}^k V(C_i)|)$ time.*

For later applications, we need several facts about paths in cycle chains. We say that a path P in a graph G is from a vertex $x \in V(G)$ to a set $S \subseteq V(G)-\{x\}$ if one end of P is x, the other end of P is in S, and P is otherwise disjoint from S.

PROPOSITION 2.5. *Let G be a 2-connected graph, let $C_1\dots C_k$ be a cycle chain in G, let $uv \in E(C_1)$ with $\{u,v\} \ne V(C_1)\cap V(C_2)$ when $k \ne 1$, and let $xy \in E(C_k)$ with $\{x,y\} \ne V(C_{k-1})\cap V(C_k)$ when $k \ne 1$. Then there is a path in $(\bigcup_{i=1}^k C_i)-\{uv,xy\}$ which is from u to $\{x,y\}$ and contains $(\bigcup_{i=1}^{k-1}(V(C_i)\cap V(C_{i+1}))-(\{x,y\}\cup\{u,v\}$. Moreover, such a path can be found in $O(|\bigcup_{i=1}^k V(C_i)|)$ time.*

Proof. If $k=1$, this is obvious. So assume that $k \ge 2$. Let $x'y'$ denote the virtual edge in C_{k-1} such that $\{x',y'\} = V(C_{k-1})\cap V(C_k)$. By induction, $(\bigcup_{i=1}^{k-1} C_i) - \{uv, x'y'\}$ contains a path P' that is from u to $\{x',y'\}$ and contains $(\bigcup_{i=1}^{k-2}(V(C_i)\cap$

$V(C_{i+1})) - \{x', y', u, v\}$. By symmetry, we may assume that P' ends at x'. If $y' \in V(P')$, then let Q' denote the path in $C_k - \{xy, y'\}$ from x' to $\{x, y\}$. If $y' \notin V(P')$, then let Q' denote the path in $C_k - xy$ that is from x' to $\{x, y\}$ and through the virtual edge $x'y'$. Clearly, $P := P' \cup Q'$ gives the desired path.

It is easy to see that such a path can be found in $O(|\bigcup_{i=1}^{k} V(C_i)|)$ time. □

By a similar argument, we can prove the following.

PROPOSITION 2.6. *Let G be a 2-connected graph, let $C_1 \dots C_k$ be a cycle chain in G, let $uv \in E(C_1)$ with $\{u, v\} \neq V(C_1) \cap V(C_2)$ when $k \neq 1$, and let $x \in V(C_k)$ with $x \notin V(C_{k-1})$ when $k \neq 1$. Then there is a path in $(\bigcup_{i=1}^{k} C_i) - uv$ which is from u to x and contains $(\bigcup_{i=1}^{k-1}(V(C_i) \cap V(C_{i+1})) - \{u, v, x\}$. Moreover, such a path can be found in $O(|\bigcup_{i=1}^{k} V(C_i)|)$ time.*

The next two facts about cycle chains are slightly more complicated. We only prove the first; the other can be proved similarly.

PROPOSITION 2.7. *Let G be a 2-connected graph, let $C_1 \dots C_k$ be a cycle chain in G, let $uv \in E(C_1)$ with $\{u, v\} \neq V(C_1) \cap V(C_2)$ when $k \neq 1$, $ab \in E(C_k)$ with $\{a, b\} \neq V(C_{k-1}) \cap V(C_k)$ when $k \neq 1$, and $cd \in E(\bigcup_{i=1}^{k} C_i) - \{ab\}$. Suppose $ab \neq uv$ when $k = 1$. Then there is a path P in $(\bigcup_{i=1}^{k} C_i) - ab$ from $\{a, b\}$ to $\{c, d\}$ such that $uv \in E(P)$, $cd \notin E(P)$ unless $cd = uv$, and $(\bigcup_{i=1}^{k-1}(V(C_i) \cap V(C_{i+1})) \subseteq V(P)$. Moreover, such a path can be found in $O(|\bigcup_{i=1}^{k} V(C_i)|)$ time.*

Proof. We apply induction on k. If $k = 1$, then since $ab \neq uv$, the result is obvious. So assume that $k \geq 2$.

First, assume that $cd \in E(C_k)$ and $\{c, d\} \neq V(C_{k-1}) \cap V(C_k)$. Let $a'b'$ denote the virtual edge in C_k with $\{a', b'\} = V(C_{k-1}) \cap V(C_k)$. In $C_k - \{ab, cd\}$, we find a path P' from $\{a, b\}$ to $\{c, d\}$ through $a'b'$. In $\bigcup_{i=1}^{k-1} C_i$, we apply Proposition 2.4 to find a Hamilton cycle C through uv and $a'b'$. Now $P := (P' - a'b') \cup (C - a'b')$ gives the desired path.

Thus we may assume that there is some $1 \leq t < k$ such that $cd \in E(C_t)$. We may choose t so that $\{c, d\} \neq V(C_{t-1}) \cap V(C_t)$ when $t \neq 1$.

Suppose $\{c, d\} = V(C_t) \cap V(C_{t+1})$. By applying Proposition 2.4, we find a Hamilton cycle C in $\bigcup_{i=1}^{t} C_i$ such that $uv, cd \in E(C)$. Now $P' := C - cd$ is a path in $\bigcup_{i=1}^{t} C_i$ from c to d through uv. By Proposition 2.5, we find a path P'' in $(\bigcup_{i=t+1}^{k} C_i) - \{ab, cd\}$ that is from d to $\{a, b\}$ and contains $(\bigcup_{i=t+1}^{k-1}(V(C_i) \cap V(C_{i+1})) - (\{a, b\} \cup \{c, d\})$. Now $P := P' \cup P''$ gives the desired path.

So assume that $\{c, d\} \neq V(C_t) \cap V(C_{t+1})$. By applying induction, there is a path P' from $V(C_t) \cap V(C_{t+1})$ to $\{c, d\}$ in $\bigcup_{i=1}^{t} C_i$ such that $uv \in E(P')$, $cd \notin E(P')$ unless $cd = uv$, and $(\bigcup_{i=1}^{t-1}(V(C_i) \cap V(C_{i+1})) \subseteq V(P')$. Let e' denote the virtual edge of C_{t+1} between the vertices in $V(C_t) \cap V(C_{t+1})$, and let $u \in V(C_t) \cap V(C_{t+1})$ be an end of P'. Now apply Proposition 2.5 to $C_{t+1} \dots C_k$, we find a path P'' from u to $\{a, b\}$ in $(\bigcup_{i=t+1}^{k} C_i) - \{e', ab\}$ such that $(\bigcup_{i=t}^{k-1}(V(C_i) \cap V(C_{i+1})) - (V(C_t) \cap V(C_{t+1})) \subseteq V(P'')$. Clearly, $P := P' \cup P''$ gives the desired path.

Since finding P' and P'' takes $O(|\bigcup_{i=1}^{k} V(C_i)|)$ time, P can also be found in $O(|\bigcup_{i=1}^{k} V(C_i)|)$ time. □

By a similar argument, we can prove the following.

PROPOSITION 2.8. *Let G be a 2-connected graph, let $C_1 \dots C_k$ be a cycle chain in G, let $uv \in E(C_1)$ with $\{u, v\} \neq V(C_1) \cap V(C_2)$ when $k \neq 1$, $x \in V(C_k)$ with $x \notin V(C_{k-1})$ when $k \neq 1$, and $cd \in E(\bigcup_{i=1}^{k} C_i)$. Then there is a path P in $(\bigcup_{i=1}^{k} C_i) - uv$ from x to $\{c, d\}$ such that $uv \in E(P)$, $cd \notin E(P)$ unless $cd = uv$, and $(\bigcup_{i=1}^{k-1}(V(C_i) \cap$

$V(C_{i+1})) \subseteq V(P)$. Moreover, such a path can be found in $O(|\bigcup_{i=1}^{k} V(C_i)|)$ time.

We conclude this section by generalizing the concept of a cycle chain to a block chain. Intuitively, a *block chain* in a 2-connected graph G is a sequence $H_1 \ldots H_k$ for which (1) each H_i is either a 3-connected 3-block of G or a cycle chain in G and (2) there exist bonds $B_1, \ldots, B_{h-1}$ of G such that $H_1 B_1 H_2 B_2 \ldots B_{h-1} H_h$ form a path in the block-bond tree of G (by also patching the tree paths corresponding to H_i when H_i is a cycle chain). More precisely, we have the following.

DEFINITION 2.9. *Let G be a 2-connected graph. By a block chain in G we mean a sequence $H_1 \ldots H_h$ with the following properties:*

- (i) *For each $1 \leq i \leq h$, either H_i is a 3-connected component of G or H_i is a cycle chain in G, and for $1 \leq i \leq h-1$, H_i and H_{i+1} cannot both be cycle chains.*
- (ii) *For each $1 \leq i \leq h-1$, $|V(H_i) \cap V(H_{i+1})| = 2$ and both H_i and H_{i+1} contain a virtual edge between the vertices in $V(H_i) \cap V(H_{i+1})$.*
- (iii) *$|V(H_i) \cap V(H_j)| \leq 1$ if $3 \leq i+2 \leq j \leq h$, and if $1 \leq i < j \leq h$ and $|V(H_i) \cap V(H_j)| = 1$, then, for all $i \leq t \leq j$, $V(H_i) \cap V(H_j) \subseteq V(H_t) \cap V(H_j)$.*
- (iv) *Suppose $H_i = C_1 C_2 \ldots C_k$ is a cycle chain. If $i < h$, then $V(H_{i+1}) \cap V(H_i) \subseteq V(C_k)$ and $V(H_{i+1}) \cap V(H_i) \neq V(C_{k-1}) \cap V(C_k)$, and if $i > 1$, then $V(H_{i-1}) \cap V(H_i) \subseteq V(C_1)$ and $V(H_{i-1}) \cap V(H_i) \neq V(C_1) \cap V(C_2)$.*

For convenience, we denote $\mathcal{H} = H_1 \ldots H_h$ and view $\mathcal{H}$ as the graph $\bigcup_{i=1}^{h} H_i$. Thus $V(\mathcal{H}) := \bigcup_{i=1}^{h} V(H_i)$. Note that $\mathcal{H}$ is a multigraph, with two virtual edges between vertices in $V(H_i) \cap V(H_{i+1})$, $1 \leq i \leq h-1$.

In Figure 4, $\mathcal{H} = H_1 H_2 H_3$ is a block chain in G, where H_2 is the cycle chain $C_1 C_2$. In a block chain, we do not include bonds, because bonds do not contribute to the vertex count in our arguments.

3. Technical lemmas.

In this section we prove several lemmas to be used in the proof of Theorem 2.1. Notice $G = G_1 \cup G_2$ in the illustration of Figure 1. If, instead, for some $k \geq 3$, $G = \bigcup_{i=1}^{k} G_i$, $E(G_i) \cap E(G_j) = \emptyset$, and $|V(G_i) \cap V(G_j)| = 3$ for $1 \leq i < j \leq k$, then the following lemma will enable us to conclude that if $|G_1|$ and $|G_2|$ are the largest among all $|G_i|$'s, then the cycle C produced by finding long cycles C_i in G_i (as in the first paragraph in section 2), $i = 1, 2$, will be long as well.

LEMMA 3.1. *Let $b = 3$ or $b \geq 4$ be an integer, and let m, n be positive integers with $m \geq n$. Then $m^{\log_b 2} + n^{\log_b 2} \geq (m + (b-1)n)^{\log_b 2}$.*

Proof. By dividing both sides of the above inequality by $m^{\log_b 2}$, it suffices to show that, for any s with $0 \leq s \leq 1$,

$$1 + s^{\log_b 2} \geq (1 + (b-1)s)^{\log_b 2}.$$

Let $f(s) = 1 + s^{\log_b 2} - (1 + (b-1)s)^{\log_b 2}$. Clearly, $f(0) = f(1) = 0$. Taking the derivative about s, we have

$$f'(s) = (\log_b 2)(s^{(\log_b 2)-1} - (b-1)(1 + (b-1)s)^{(\log_b 2)-1}).$$

A simple calculation shows that $f'(s) = 0$ has a unique solution. Therefore, since $f(0) = f(1) = 0$, either 0 is the absolute maximum of $f(s)$ over $[0,1]$ or 0 is the absolute minimum of $f(s)$ over $[0,1]$. That is, either $f(s) \geq 0$ for all $s \in [0,1]$ or $f(s) \leq 0$ for all $s \in [0,1]$. Note that $0 < \frac{1}{b} < 1$ (since $b \geq 3$) and

$$f\left(\frac{1}{b}\right) = \left(1 + \frac{1}{2}\right) - \left(1 + \frac{b-1}{b}\right)^{\log_b 2}$$

$$= \frac{3}{2} - \frac{(2b-1)^{\log_b 2}}{2}.$$

We claim that $f(\frac{1}{b}) > 0$. If $b = 3$, then $f(\frac{1}{b}) = \frac{1}{2}(3 - 5^{\log_3 2}) > 0$. So assume $b \geq 4$. Then $f(\frac{1}{b}) > \frac{3}{2} - \frac{(2b)^{\log_b 2}}{2} = \frac{3}{2} - 2^{\log_b 2}$. Since $b \geq 4$, $2^{\log_b 2} \leq 2^{\log_4 2} = \sqrt{2} < \frac{3}{2}$. $f(\frac{1}{b}) > 0$ for $b \geq 4$.

Therefore, we have $f(s) \geq 0$ for all $s \in [0, 1]$. □

We remark that Lemma 3.1 holds for $b \geq 3$. We choose to state it for $b = 3$ and $b \geq 4$ for simplicity in calculations.

The observations in the following lemma will be convenient in the proof of Theorem 2.1.

LEMMA 3.2. *Let m be an integer, $b \geq 4$, and $d \geq 3$. If $m \geq 4$, then $m \geq m^{\log_b 2}+2$. If $m \geq 3$, then $m > (\frac{m}{2(d-1)})^{\log_b 2}+2$. If $m \geq 2$, then $m > (\frac{m}{2(d-1)})^{\log_b 2}+1$.*

Proof. Let $f(x) = x - x^{\log_b 2}$. We can show that $f'(x) > 0$ for $x \geq 4$. Hence $f(x)$ is an increasing function when $x \geq 4$. Thus, $f(x) \geq f(4) = 4 - 4^{\log_b 2} \geq 2$ (since $b \geq 4$). Thus when $m \geq 4$, we have $m \geq m^{\log_b 2} + 2$.

Next, let $f(x) = x - (\frac{x}{2(d-1)})^{\log_b 2}$. We can show that $f(x)$ is an increasing function when $x \geq 2$. The second inequality follows from $f(x) \geq f(3) > 2$, and the third inequality follows from $f(x) \geq f(2) > 1$. □

After we decompose a 2-connected graph into 3-connected components, we need to find long cycles in certain 3-connected components. This will be done by inductively applying (a), (b), or (c) of Theorem 2.1 to 3-connected components or to graphs obtained from 3-connected components by an "H-transform" or "T-transform."

Let G be a graph and let e, f be distinct edges of G. An *H-transform* of G *at* $\{e, f\}$ is an operation that subdivides e and f by vertices x and y, respectively, and then adds the edge xy. See Figure 5. Let G be a graph, let $e \in E(G)$, and let $x \in V(G)$, which is not incident with e. A *T-transform* of G *at* $\{x, e\}$ is an operation that subdivides e with a vertex y and then adds the edge xy. If there is no need to specify e, f, x, we will simply speak of an H-transform or a T-transform. The following result is easy to prove.

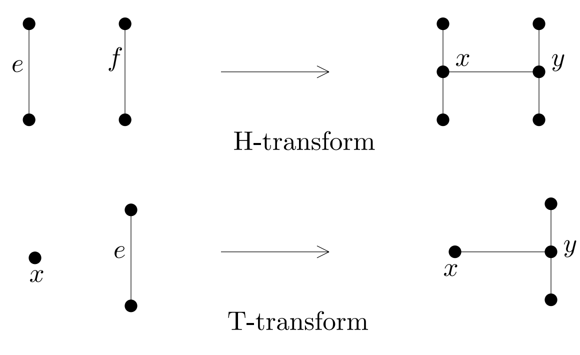

FIG. 5. *H-transform and T-transform.*

LEMMA 3.3. *Let $d \geq 3$ be an integer, and let G be a 3-connected graph with maximum degree at most d. Let G' be a graph obtained from G by an H-transform or a T-transform. Then G' is 3-connected graph, the vertex of G involved in the T-transform has degree at most $d+1$, and all other vertices of G' have degree at most d.*

Next, we state two results from [11]. The first says that any k-connected graph contains a sparse k-connected spanning subgraph.

LEMMA 3.4. *Let G be a k-connected graph, where k is a positive integer. Then G contains a k-connected spanning subgraph with $O(|G|)$ edges, and such a subgraph can be found in $O(|E(G)|)$ time.*

The next result is an easy consequence of a result in [11], which states that, in a 2-connected graph G, one can find, in $O(|G|)$ time, two disjoint paths between two given vertices.

LEMMA 3.5. *Let G be a 2-connected graph and let $e, f \in E(G)$. Then there is a cycle through e and f in G, and such a cycle can be found in $O(|G|)$ time.*

The final two results of this section deal with the existence of certain paths in a 3-connected graph. Since such paths need to be produced (when finding a long cycle), we also show that they can be found in linear time. The proofs of these two results are almost identical, so we omit the details of the second proof.

LEMMA 3.6. *Let G be a 3-connected graph, let $f \in E(G)$, let $ab, cd, vw \in E(G) - \{f\}$, and assume that $\{c, d\} \neq \{v, w\}$. Then there exists a path P from $\{a, b\}$ to some $z \in \{c, d\} \cup \{v, w\}$ in G such that*

(i) *$f \in E(P)$,*
(ii) *$cd \in E(P)$ or $vw \in E(P)$, and*
(iii) *if $cd \in E(P)$, then $z \in \{v, w\}$ and $vw \notin E(P)$, and if $vw \in E(P)$, then $z \in \{c, d\}$ and $cd \notin E(P)$.*

Moreover, such a path can be found in $O(|G|)$ time.

Note that in (iii) above when $vw \notin E(P)$, it is possible that $v \in V(P)$ and/or $w \in V(P)$.

Proof. First, we find a cycle C through both ab and f. This can be done in $O(|G|)$ time using Lemma 3.5. Next we distinguish three cases. Note that checking these cases can be done in $O(|G|)$ time.

Case 1. $cd, vw \in E(C)$. In this case one of the following holds: f and vw are contained in a component P of $C - \{ab, cd\}$, or f and cd are contained in a component P of $C - \{ab, vw\}$. In either case, P gives the desired path and can be found in $O(|G|)$ time.

Case 2. $cd \notin E(C)$ and $vw \in E(C)$, or $cd \in E(C)$ and $vw \notin E(C)$. By symmetry, assume that $cd \notin E(C)$ and $vw \in E(C)$. Let Q_1 and Q_2 denote the components of $C - \{ab, f\}$, and assume that $vw \in E(Q_1)$. By Lemma 3.5, we can find, in $O(|G|)$ time, disjoint paths P_1 and P_2 from c, d to some vertices c', d', respectively, of C which are also disjoint from $C - \{c', d'\}$.

If $c' \in V(Q_2)$ or $d' \in V(Q_2)$, then $(C \cup P_1 \cup P_2) - \{ab, cd\}$ contains a path P from $\{a, b\}$ to $\{c, d\}$ though f and vw. So assume that $c', d' \in V(Q_1)$.

If vw is contained in the subpath of Q_1 between c' and d', then $(C \cup P_1 \cup P_2) - \{ab, vw\}$ contains a path P from $\{a, b\}$ to $\{v, w\}$ through f and cd.

If $\{c', d'\}$ is contained in the subpath of Q_1 between vw and ab, then $(C \cup P_1 \cup P_2) - \{ab, cd\}$ contains a path P from $\{a, b\}$ to $\{c, d\}$ through f and vw.

Assume that $\{c', d'\}$ is contained in the subpath of Q_1 between vw and f. Then $(C \cup P_1 \cup P_2) - \{ab, vw\}$ contains a path P from $\{a, b\}$ to $\{v, w\}$ through f and cd.

Note that the above cases can be checked in constant time, and in each case, P can be found in $O(|G|)$ time.

Case 3. $cd \notin E(C)$ and $vw \notin E(C)$. Let Q_1 and Q_2 denote the components of $C - \{ab, f\}$. By Lemma 3.5, there are disjoint paths P_1 and P_2 in G from c, d to $c', d' \in V(C)$, respectively, which are also disjoint from $C - \{c', d'\}$ (and can be found in $O(|G|)$ time). We may assume that $\{c', d'\} \not\subseteq V(Q_i)$ for $i = 1, 2$; otherwise, $C \cup P_1 \cup P_2$ contains a cycle through ab, cd, f and, as in Cases 1 and 2, we can find the desired path in $O(|G|)$ time. Thus, by symmetry, we may assume that $c' \in V(Q_1)$ and $d' \in V(Q_2)$.

If $vw \in E(P_1 \cup P_2)$, then $(C \cup P_1 \cup P_2) - \{ab, vw\}$ contains a path P from $\{a, b\}$ to $\{v, w\}$ through f and cd, which can be found in $O(|G|)$ time. So assume that $vw \notin E(P_1 \cup P_2)$. Therefore, by Lemma 3.5, we can find, in $O(|G|)$ time, disjoint paths R_1, R_2 from v, w to $v', w' \in V(C \cup P_1 \cup P_2)$, respectively, which are also disjoint from $(C \cup P_1 \cup P_2) - \{v', w'\}$.

By similar arguments, we may assume that $\{v', w'\} \not\subseteq V(Q_i)$ (or we go back to Case 1 or Case 2) and $\{v', w'\} \not\subseteq V(P_i)$ for any $i \in \{1, 2\}$ (or we could have chosen P_1, P_2 to include vw and have gone back to the case in the previous paragraph).

Subcase 3.1. $\{v', w'\} \neq \{c', d'\}$. First assume $\{v', w'\} \subseteq V(P_1 \cup P_2)$. Then cd belongs to the subpath of $(P_1 \cup P_2) + cd$ between v' and w'. We see that there is a path P in $((C - ab) \cup P_1 \cup P_2 \cup R_1 \cup R_2) + cd \subseteq G - \{ab, vw\}$ from $\{a, b\}$ to $\{v, w\}$ through both f and cd.

Now assume $\{v', w'\} \subseteq V(Q_1 \cup Q_2)$. Then by symmetry, we may assume that $v' \in V(Q_1)$, $w' \in V(Q_2)$, and $w' \neq d'$. If f, w', d', ab occur on C in cyclic order, then there is a path P from $\{a, b\}$ to $\{v, w\}$ through f and cd in $((C - ab) \cup P_1 \cup P_2 \cup R_1 \cup R_2) + cd \subseteq G - \{ab, vw\}$. If f, d', w', ab occur on C in cyclic order, then there is a path P from $\{a, b\}$ to $\{c, d\}$ through both f and vw in $((C - ab) \cup P_1 \cup P_2 \cup R_1 \cup R_2) + vw \subseteq G - \{ab, cd\}$.

Thus we may assume by symmetry that $v' \in (V(P_1) \cup V(P_2)) - \{c', d'\}$ and $w' \in (V(Q_1) \cup V(Q_2)) - \{c', d'\}$. It is easy to see that $((C - ab) \cup P_1 \cup P_2 \cup R_1 \cup R_2) + vw \subseteq G - \{ab, cd\}$ contains a path P from $\{a, b\}$ to $\{c, d\}$ through both f and vw.

The above three cases can be checked in $O(|G|)$ time, and in all cases, P can be found in $O(|G|)$ time.

Subcase 3.2. $\{v', w'\} = \{c', d'\}$. Let S_1 and S_2 denote the paths between c' and d' in C containing f and ab, respectively. Since G is 3-connected, there is a path S from some $s \in V(R_1 \cup R_2 \cup S_2) - \{c', d'\}$ to $s' \in V(P_1 \cup P_2 \cup S_1) - \{c', d'\}$, which is also disjoint from $(C \cup P_1 \cup P_2 \cup R_1 \cup R_2) - \{s, s'\}$. Note that S can be found in $O(|G|)$ time.

If $s \in V(S_2)$, then $((C - ab) \cup P_1 \cup P_2 \cup R_1 \cup R_2 \cup S) + cd \subseteq G - \{ab, vw\}$ contains a path P from $\{a, b\}$ to $\{v, w\}$ through both f and cd. If $s' \in V(S_1)$, then $((C - ab) \cup P_1 \cup P_2 \cup R_1 \cup R_2 \cup S) + vw \subseteq G - \{ab, cd\}$ contains a path P from $\{a, b\}$ to $\{c, d\}$ through both f and vw, or $((C - ab) \cup P_1 \cup P_2 \cup R_1 \cup R_2 \cup S) + cd \subseteq G - \{ab, vw\}$ contains a path P from $\{a, b\}$ to $\{v, w\}$ through both f and cd.

Assume that $s \in V(R_1 \cup R_2) - \{c', d'\}$ and $s' \in V(P_1 \cup P_2) - \{v', w'\}$. Then $((C - ab) \cup P_1 \cup P_2 \cup R_1 \cup R_2 \cup S) + vw \subseteq G - \{ab, cd\}$ contains a path P from $\{a, b\}$ to $\{c, d\}$ through both f and vw.

The above three cases can be checked in constant time, and in all cases, P can be found in $O(|G|)$ time. □

LEMMA 3.7. *Let G be a 3-connected graph, let $f \in E(G)$, let $x \in V(G)$ which is not incident with f, let $cd, vw \in E(G) - \{f\}$, and assume that $\{c, d\} \neq \{v, w\}$. Then there exists a path P in G from x to some $z \in \{c, d\} \cup \{v, w\}$ such that*

(i) *$f \in E(P)$,*
(ii) *$cd \in E(P)$ or $vw \in E(P)$, and*
(iii) *if $cd \in E(P)$ then $z \in \{v, w\}$ and $vw \notin E(P)$, and if $vw \in E(P)$, then $z \in \{c, d\}$ and $cd \notin E(P)$.*

Moreover, such a path can be found in $O(|G|)$ time.

Proof. The proof is the same as for Lemma 3.6, with ab replaced by x, and when finding paths P_i, R_i, we apply Lemma 3.5 to $G - x$ (which is 2-connected). □

4. Cycles avoiding a vertex. In this section, we show how to reduce Theorem 2.1 (a) to (b) and/or (c) of the same theorem in linear time. First, we state the reduction as a lemma.

LEMMA 4.1. *Let $n \geq 6$ and $d \geq 3$ be integers, let $r = \log_{2(d-1)^2+1} 2$, and assume that Theorem 2.1 holds for graphs with at most $n-1$ vertices. Let G be a 3-connected graph with n vertices, let $xy \in E(G)$ and $z \in V(G) - \{x, y\}$, and let t denote the number of neighbors of z distinct from x and y. Assume that the maximum degree of G is at most $d+1$, and every vertex of degree $d+1$ in G (if any) is incident with the edge zx or zy. Then there is a cycle C through xy in $G-z$ such that $|C| \geq (\frac{n}{2t})^r + 2$.*

Proof. We consider $G - z$. Since the vertices of G with degree $d + 1$ must be incident with the edge yz or xz, the maximum degree of $G - z$ is at most d. Since G is 3-connected and by the assumption on degrees, we see that $1 \leq t \leq d - 1$.

First, assume that $G - z$ is 3-connected. Since $n \geq 6$, $|G - z| \geq 5$, and hence Theorem 2.1 holds for $G - z$. By Theorem 2.1 (c), $G - z$ contains a cycle C through e such that

$$\begin{aligned} |C| &\geq (n-1)^r + 3 \\ &= ((n-1)^r + 1) + 2 \\ &\geq ((n-1) + 1)^r + 2 \quad \text{(by 3.1)} \\ &> \left(\frac{n}{2t}\right)^r + 2. \end{aligned}$$

Therefore, we may assume that $G - z$ is not 3-connected. By Theorem 2.2, we can decompose $G - z$ into 3-connected components. Let $\mathcal{H} = H_1 \dots H_h$ be a block chain in $G - z$ such that

(i) $\{x, y\} \subseteq V(H_1)$, and $\{x, y\} \neq V(H_1) \cap V(H_2)$ when $k \neq 1$,
(ii) H_h contains an extreme 3-block of $G - z$, and
(iii) subject to (i) and (ii), $|\mathcal{H}|$ is maximum.

Note that $H_1 \dots H_k$ can be found in $O(|G|)$ time.

We claim that $|\mathcal{H}| \geq \frac{n-1}{t}$. Since G is 3-connected, each extreme 3-block of $G - z$ distinct from H_1 contains a neighbor of z that is not incident with xy. Therefore, there are at most t extreme 3-blocks of $G - z$ different from H_1. Thus there are at most t different block chains in $G-z$ starting with a 3-block or cycle chain containing $\{x, y\}$ and ending with an extreme 3-block of $G-z$ or a cycle chain in $G-z$ containing an extreme 3-block. Since all such chains cover the whole graph $G-z$, it follows from (iii) that $|\mathcal{H}| - 2 \geq \frac{n-3}{t}$, and thus $|\mathcal{H}| \geq \frac{n-1}{t}$.

Let $V(H_i) \cap V(H_{i+1}) = \{x_i, y_i\}$, $1 \leq i \leq h - 1$, and assume that the notation is chosen so that H_i contains disjoint paths from x_{i-1}, y_{i-1} to x_i, y_i, respectively, where $x_0 = x$ and $y_0 = y$. See Figure 6. Next, we show how to find the desired cycle in $G - z$.

Case 1. There exists some $1 \leq i \leq h$ such that $|H_i| \geq \frac{n}{2t}$. We choose H_i so that $|H_i| \geq |H_j|$ for all $1 \leq j \leq h$. Then $|H_i| \geq \frac{n}{2t}$.

First, assume that $|H_i| = 3$. Then $3 \geq \frac{n}{2t}$. By the choice of H_i, $|H_j| = 3$ for all $1 \leq j \leq h$. Since $\mathcal{H}$ does not contain two consecutive cycle chains, we have $h = 1$. Hence $G - z$ is a union of triangles which share the edge xy. Therefore, there are exactly t triangles. Because $n \geq 6$, we have $t \geq 3$, and thus $n = t + 3 \leq 2t$. Hence $C := H_i$ is a cycle through xy in $G - z$, and $|C| = 3 \geq (\frac{n}{2t})^r + 2$.

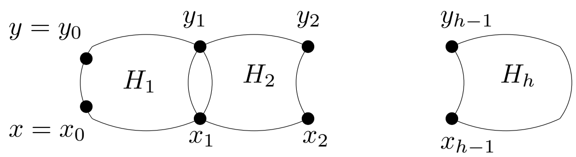

FIG. 6. *Block chain* $\mathcal{H} = H_1 \dots H_h$.

We may assume that $|H_i| \geq 4$. If $H_i = K_4$ or H_i is a cycle chain in $G - z$, then, by Proposition 2.4, let C_i denote a Hamilton cycle through $x_{i-1}y_{i-1}, x_iy_i$ in H_i. By Lemma 3.2, $|C_i| = |H_i| \geq (|H_i|)^r + 2 \geq (\frac{n}{2t})^r + 2$.

Thus assume that H_i is 3-connected and $|H_i| \geq 5$. Since $|H_i| < |G|$ and the maximum degree of H_i is at most d, Theorem 2.1 holds for H_i. By Theorem 2.1 (c), there is a cycle C_i through $e_i := x_{i-1}y_{i-1}$ in H_i such that $|C_i| \geq |H_i|^r + 3 \geq (\frac{n}{2t})^r + 2$.

We can obtain a cycle C in G by replacing virtual edges contained in C_i with disjoint paths in G (in particular, replacing $x_{i-1}y_{i-1}$ by a path through xy in G). Therefore, C is a cycle through xy in G, and $|C| \geq |C_i| \geq (\frac{n}{2t})^r + 2$.

Case 2. For each $1 \leq i \leq h$, $|H_i| < \frac{n}{2t}$. Since $|\mathcal{H}| \geq \frac{n-1}{t} > \frac{n}{2t}$, we have $h \geq 2$. We will find a cycle C_i through $x_{i-1}y_{i-1}$ and x_iy_i in each H_i, where x_hy_h is an arbitrary edge of H_h.

If $H_i = K_4$ or H_i is a cycle chain in $G - z$, then, by Proposition 2.4, let C_i be a Hamilton cycle through both $x_{i-1}y_{i-1}$ and x_iy_i in H_i. By Lemma 3.2, $|C_i| = |H_i| \geq (\frac{|H_i|}{2(d-1)})^r + 2$.

Assume that $H_i \neq K_4$ and H_i is not a cycle chain in $G - z$. Then $|H_i| \geq 5$ and H_i is a 3-connected graph with maximum degree d. Since $|H_i| < n$, Theorem 2.1 holds for H_i. By Theorem 2.1 (b), there is a cycle C_i through $x_{i-1}y_{i-1}$ and x_iy_i in H_i such that $|C_i| \geq (\frac{|H_i|}{2(d-1)})^r + 3$.

Note that $C_1 - x_1y_1$, $C_h - x_{h-1}y_{h-1}$, and $C_i - \{x_{i-1}y_{i-1}, x_iy_i\}$, $2 \leq i \leq h-1$, are disjoint edges, and their union is a cycle C' through xy in $\mathcal{H}$. By replacing the virtual edges in C' with disjoint paths in G, we can produce a cycle C through e in $G - z$ such that $|C| \geq |C'|$. Hence $|C| \geq (\frac{|H_1|}{2(d-1)})^r + \cdots + (\frac{|H_h|}{2(d-1)})^r + 2$.

Note that $h \geq 2$ and the vertices in $V(H_1) \cap V(H_2)$ are counted twice in $|H_1| + \ldots + |H_h|$. Hence $|H_1| + \cdots + |H_h| > \frac{n-1}{t} + 1 \geq \frac{n}{t}$. Consider the function $f(x_1, \cdots, x_h) = x_1^r + \ldots + x_h^r + 2$, with $x_1 + \cdots + x_h \geq \frac{n}{2(d-1)t}$ and $0 \leq x_i \leq \frac{n}{4(d-1)t}$. By the convexity of $f(x_1, \ldots, x_h)$, the minimum of $f(x_1, \cdots, x_h)$ is achieved on the boundary of its domain. In particular, the minimum is achieved when $x_1 = x_2 = \frac{n}{4(d-1)t}$ and $x_3 = \cdots = x_h = 0$. Hence

$$
\begin{aligned}
f(x_1, \ldots, x_h) &\geq f\left(\frac{n}{4(d-1)t}, \frac{n}{4(d-1)t}, 0, \ldots, 0\right) \\
&= 2\left(\frac{n}{4(d-1)t}\right)^r + 2 \\
&> \left(\frac{n}{2t}\right)^r + 2.
\end{aligned}
$$

The final inequality follows from the fact that $r = \log_{2(d-1)^2+1} 2$ and $2 = (2(d-1)^2 + 1)^r$. Therefore, $|C| \geq (\frac{n}{2t})^r + 2$. ◻

As we can see from the above proof, the desired cycle through xy in $G - z$ can be found either (1) directly, (2) by finding a long cycle through e_i in some H_i with

$|H_i| \geq \frac{n}{2t}$, or (3) by finding long cycles through $x_{j-1}y_{j-1}$ and x_jy_j in H_j, $1 \leq j \leq h$. Next we show that the proof of Lemma 4.1 implies that this process can be done in $O(|G|)$ time.

Algorithm Avoidvertex. Let G be a 3-connected graph, $e = xy \in E(G)$, and $z \in V(G) - \{x, y\}$, satisfying the conditions of Lemma 4.1. The algorithm performs the following steps.

1. *Preprocessing.* Replace G by a 3-connected spanning graph of G with $O(|G|)$ edges. (This can be done in $O(|E(G)|)$ time using Lemma 3.4.)
2. Decompose $G - z$ into 3-connected components. (This can be done in $O(|G|)$ time using Lemma 2.2.)
3. If there is only one 3-block of $G-z$, then $G-z$ is 3-connected and we proceed to find a cycle D through $e = xy$ in $G-z$ such that $|D| \geq (|G|-1)^r+3$. That is, we reduce (a) for G, xy, z to (c) for $G-z, xy$. (Clearly, this reduction can be done in constant time.)
4. If there are at least two 3-blocks of $G-z$, then $G-z$ is not 3-connected. We find a block chain $\mathcal{H} = H_1 \ldots H_k$ in $G-z$ such that $\{x, y\} \subseteq V(H_1)$, $\{x, y\} \neq V(H_1) \cap V(H_2)$, and $|\mathcal{H}| \geq \frac{n-1}{t}$. Let $V(H_i) \cap V(H_{i+1}) = \{x_i, y_i\}$ for $1 \leq i \leq h-1$. (Note that $\mathcal{H}$ can be found in $O(|G|)$ time by a simple search.)
5. Either find some H_i with $|H_i| \geq \frac{n}{2t}$, or certify that $|H_i| < \frac{n}{2t}$ for all $1 \leq i \leq h$. (This can be done in $O(|G|)$ time by a simple search.)
6. Suppose there exists some $1 \leq i \leq h$ for which $|H_i| \geq \frac{n}{2t}$.
   - If $H_i = K_4$ or H_i is a cycle chain, then let C_i denote a Hamilton cycle in H_i through the edge $x_{i-1}y_{i-1}$. Let C be a cycle in G obtained from C_i by replacing virtual edges with paths in G, and make sure $e \in E(C)$. (Note that C_i can be found in $O(|G|)$ time using Proposition 2.4, and so C can be found in $O(|G|)$ time.)
   - If H_i is 3-connected and $H_i \neq K_4$, then to find the desired cycle in $G-z$ through e it suffices to find a cycle D in H_i through $x_{i-1}y_{i-1}$ such that $|D| \geq |H_i|^r + 3$. Hence, we reduce (a) for G, e, z to (c) for $H_i, x_{i-1}y_{i-1}$. (This can be done in constant time.)
7. Now assume that, for all $1 \leq j \leq h$, $|H_j| < \frac{n}{2t}$. Then $h \geq 2$. For each $1 \leq j \leq h$, we perform the following:
   - If $H_j = K_4$ or H_j is a cycle chain in $G-z$, let C_j denote a Hamilton cycle through both $x_{j-1}y_{j-1}$ and x_jy_j in H_j. (Note that C_j can be found in $O(|H_j|)$ time using Proposition 2.4.)
   - If H_j is 3-connected and $H_j \neq K_4$, then it suffices to find a cycle D in H_j through $x_{j-1}y_{j-1}$ and x_jy_j such that $|D| \geq (\frac{|H_j|}{2(d-1)})^r$. Hence, we reduce (a) for G, xy, z to (b) for $H_j, x_{j-1}y_{j-1}, x_jy_j$, for all H_j which are not cycle chains and are not isomorphic to K_4. (Clearly, this can done in $O(|G|)$ time. Moreover, any such H_j contains a vertex that does not belong to any other H_k—this is why we want $H_j \neq K_4$, and it will be used in the final complexity analysis.)

The correctness of the algorithm follows from the proof of Lemma 4.1. To summarize, we have the following result.

Proposition 4.2. *Let $G, e = xy, z, t, d, r$ be as in Theorem* 2.1 (a). *Then, in $O(|E(G)|)$ time, we can either*

(1) *find a cycle C through e in $G-z$ with $|C| \geq (\frac{|G|}{2t})^r + 2$,*
(2) *reduce* (a) *of Theorem* 2.1 *for G, xy, z to* (c) *of Theorem* 2.1 *for some* 3*-block H_i of $G-z$ that is* 3*-connected and $|H_i| \geq \max\{5, \frac{|G|}{2t}\}$, or*

(3) *reduce* (a) *of Theorem* 2.1 *for* G, xy, z *to* (b) *of Theorem* 2.1 *for* $H_j, x_{j-1}y_{j-1}$, x_jy_j *for some* 3*-connected* 3*-blocks* $H_j \neq K_4$.

Moreover, in (3)*, each* H_j *contains a vertex that does not belong to any other* H_k, $k \neq j$.

5. Cycles through two edges. In this section, we show how to reduce (b) of Theorem 2.1 to (a) or (b) of the same theorem for smaller graphs. We will show that such a reduction can be performed in linear time.

LEMMA 5.1. *Let* $n \geq 6$ *and* $d \geq 3$ *be integers, let* $r = \log_{2(d-1)^2+1} 2$*, and assume that Theorem* 2.1 *holds for graphs with at most* $n-1$ *vertices. Suppose* G *is a* 3*-connected graph on* n *vertices and that the maximum degree of* G *is at most* d*. Then for any* $\{e, f\} \subseteq E(G)$ *there is a cycle* C *through* e, f *in* G *such that* $|C| \geq (\frac{n}{2(d-1)})^r + 3$.

Proof. First, assume that e is incident with f. Let $e = xz$ and $f = yz$, and let $G' := G + xy$. Then G' is a 3-connected graph with maximum degree at most $d+1$, and the possible vertices of degree $d+1$ in G' are x and y, which are incident with the edge zx or zy. By applying Lemma 4.1 to G', xy, z, there is a cycle C' through xy in $G' - z$ such that $|C'| \geq (\frac{n}{2t})^r + 2$, where t is the number of neighbors of z in G' distinct from x and y. Since $zx, zy \in E(G)$, $t \leq d-1$. Now let $C := C' - xy + \{e, f\}$. Then $|C| \geq (\frac{n}{2t})^r + 3 \geq (\frac{n}{2(d-1)})^r + 3$, and C gives the desired cycle.

Therefore, we may assume that e and f are not incident. Let $e = xy$, and consider $G - y$. Since y is not incident with f, $f \in E(G-y)$. Since G is 3-connected, $G - y$ is 2-connected.

Suppose that $G - y$ is 3-connected. Let $y' \neq x$ be a neighbor of y. Then $G' := (G-y)+xy'$ is a 3-connected graph with maximum degree at most d, and $5 \leq |G'| < n$. Hence Theorem 2.1 holds for G'. By Theorem 2.1 (b), there is a cycle C' through xy' and f in G' such that $|C'| \geq (\frac{n-1}{2(d-1)})^r + 3$. Let $C := (C' - xy') + \{y, xy, yy'\}$. Then

$$\begin{aligned} |C| &= |C'| + 1 \\ &\geq \left(\frac{n-1}{2(d-1)}\right)^r + 3 + 1 \\ &\geq \left(\frac{n-1}{2(d-1)} + 1\right)^r + 3 \quad \text{(by Lemma 3.1)} \\ &> \left(\frac{n}{2(d-1)}\right)^r + 3. \end{aligned}$$

Assume that $G - y$ is not 3-connected. By Theorem 2.2, we can decompose $G - y$ into 3-connected components. Let $\mathcal{H} := H_1 \dots H_h$ be a block chain in $G - y$ such that $x \in V(H_h) - V(H_{h-1})$, $f \in E(H_1)$, and f is not incident with both vertices in $V(H_1) \cap V(H_2)$. See Figure 7. Define $V(H_s) \cap V(H_{s+1}) = \{a_s, b_s\}$ for $1 \leq s \leq h-1$.

For each $1 \leq s \leq h$ we define A_s, which consists of vertices of H_s to be counted when applying induction. If H_s is 3-connected, then let $A_s := V(H_s)$. If $h = 1$ and $H_s = C_1 \dots C_k$ is a cycle chain in $G - y$, then let A_s consist of the vertices incident with f and the vertices in $\bigcup_{i=1}^{k-1} V(C_i \cap C_{i+1})$. If $h > 1$ and $H_1 = C_1 \dots C_k$ is a cycle chain in $G - y$, then let A_1 consist of those vertices of $H_1 - \{a_1, b_1\}$ which are incident with f or contained in $\bigcup_{i=1}^{k-1} V(C_i \cap C_{i+1})$. If $1 < s < h$ and $H_s = C_1 \dots C_k$ is a cycle chain, then let $A_s := (\bigcup_{i=1}^{k-1} V(C_i \cap C_{i+1})) - (\{a_{s-1}, b_{s-1}\} \cup \{a_s, b_s\})$. If $1 < s = h$ and $H_s = C_1 \dots C_k$ is a cycle chain, then let $A_s := (\bigcup_{i=1}^{k-1} V(C_i \cap C_{i+1})) - \{a_{s-1}, b_{s-1}\}$.

Define $\sigma(\mathcal{H}) := |\bigcup_{s=1}^{h} A_s|$. Intuitively, $\sigma(\mathcal{H})$ consists of the vertices incident with f, and those vertices which are of degree at least three in $\mathcal{H}$ (when viewed as a graph).

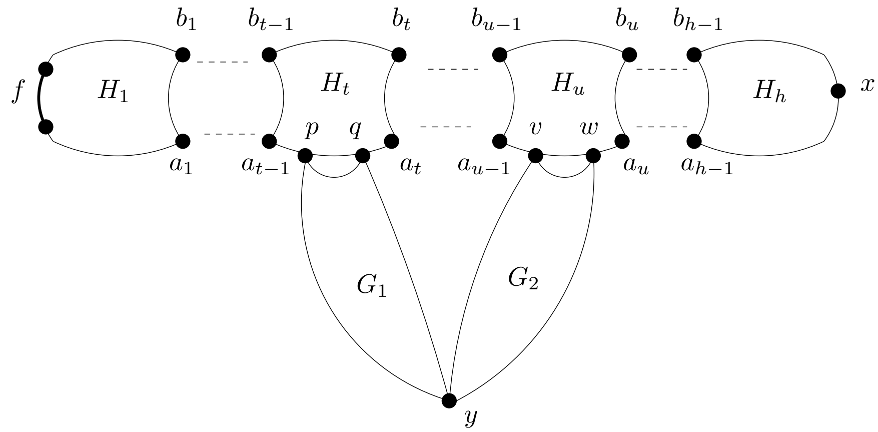

FIG. 7. *Block chains $\mathcal{H}$, $\mathcal{I}$, and $\mathcal{J}$.*

We wish route our cycle through two large "parts" of $G - y$. For this purpose, we consider chains $\mathcal{I}$ and $\mathcal{J}$ in $G - y$ defined below.

Let $\mathcal{I} := I_1 \dots I_i$ be a block chain in $G - y$ such that (i) $|V(I_1) \cap V(\mathcal{H})| = 2$, (ii) $V(\mathcal{I}) \cap V(\mathcal{H}) = V(I_1) \cap V(\mathcal{H})$, (iii) I_i is an extreme 3-block of $G - y$, and (iv) subject to (i), (ii), and (iii), $|V(\mathcal{I})|$ is maximum. When $\mathcal{I}$ is nonempty, let $V(I_1) \cap V(\mathcal{H}) = \{p, q\}$. In this case, $\{p, q, y\}$ is a 3-cut of G, and we let G_1 denote the subgraph of G by deleting those components of $G - \{p, q, y\}$ which contain an element of $V(\mathcal{H})$. (Note that G_1 can be defined in a more direct way; however, defining it from $\mathcal{I}$ is more natural for our algorithm because we have all 3-blocks.) See Figure 7.

Since all degree two vertices in $\mathcal{H}$ are contained in some other 3-blocks of $G - y$ or are neighbors of y, $G - y$ can be covered by at most $d - 1$ block chains starting from a 3-block containing f and ending with an extreme 3-block (or a cycle chain containing an extreme 3-block). Hence, we have the following.

Observation 1. If $G_1 \neq \emptyset$ then $|G_1| \geq \frac{n - \sigma(\mathcal{H})}{d-1}$.

Let $\mathcal{J} := J_1 \dots J_j$ be a block chain in $G - y$ such that (i) $|V(J_1) \cap (V(\mathcal{H}) \cup V(G_1))| = 2$, (ii) $V(\mathcal{J}) \cap (V(\mathcal{H}) \cup V(G_1)) = V(J_1) \cap (V(\mathcal{H}) \cup V(G_1))$, (iii) J_j is an extreme 3-block of $G - y$, and (iv) subject to (i), (ii), and (iii), $|V(\mathcal{J})|$ is maximum. When $\mathcal{J}$ is nonempty, let $V(\mathcal{J}) \cap (V(\mathcal{H}) \cup V(G_1)) = \{v, w\}$. By the choice of G_1, $\{v, w\} \neq \{p, q\}$ and $\{v, w\} \subseteq V(\mathcal{H})$. In this case, $\{v, w, y\}$ is a 3-cut of G, and we let G_2 denote the subgraph of G by deleting those components of $G - \{v, w, y\}$ that contain an element of $V(G_1) \cup V(\mathcal{H})$. Note that $V(G_1) \cap V(G_2) \subseteq \{p, q, y\} \cap \{v, w, y\}$ and $|V(G_1) \cap V(G_2)| \leq 2$ (because $\{v, w\} \neq \{p, q\}$). See Figure 7.

By the same reasoning as for Observation 1, we have following two observations.

Observation 2. If $G_2 \neq \emptyset$, then $|G_2| \geq \frac{n - \sigma(\mathcal{H}) - (|G_1| - 1)}{d-2}$.

Observation 3. If $\sigma(\mathcal{H}) \geq |G_2|$, then $\sigma(\mathcal{H}) \geq \frac{|G|}{d-1}$.

Next we distinguish two cases by comparing $\sigma(\mathcal{H})$ and $|G_2|$.

Case 1. $\sigma(\mathcal{H}) \geq |G_2|$. In this case, it suffices to consider $\mathcal{H}$ and G_1. Clearly, there is some $1 \leq t \leq h$ such that $\{p, q\} \subseteq V(H_t)$, and $\{p, q\} \neq \{a_{t-1}, b_{t-1}\}$ when $t \neq 1$. Let a_0, b_0 be the vertices incident with f. We will find paths in H_s, $1 \leq s \leq h$, and a path in G_1 to form the desired cycle.

(1) If $s = 1 < t$, then there is a path P_1 from a_1 to b_1 in H_1 such that $f \in E(P_1)$ and $|E(P_1)| \geq (\frac{|A_s|}{2(d-1)})^r + 1$. If $1 < s < t$, then there exists $P_s \subseteq H_s$, consisting of

disjoint paths from $\{a_{s-1}, b_{s-1}\}$ to $\{a_s, b_s\}$, such that $|E(P_s)| \geq (\frac{|A_s|}{2(d-1)})^r + 1$.

Suppose $H_s = K_4$ or H_s is a cycle chain. By Proposition 2.4, let C_s denote a Hamilton cycle through $a_{s-1}b_{s-1}$ and a_sb_s in H_s. Since $|H_s| \geq 3$ and by Lemma 3.2,

$$|C_s| = |H_s| \geq \left(\frac{|H_s|}{2(d-1)}\right)^r + 2 \geq \left(\frac{|A_s|}{2(d-1)}\right)^r + 2.$$

If $s = 1$, then $P_1 := C_1 - a_1b_1$ gives the desired path for (1). Now assume $1 < s < t$. If $|H_s| \geq 4$, then $|C_s| = |H_s| \geq (\frac{|H_s|}{2(d-1)})^r + 3$, and if $|H_s| = 3$, then $|A_s| = 0$, and hence $|C_s| = |H_s| = (\frac{|A_s|}{2(d-1)})^r + 3$. Hence $P_s := C_s - \{a_{s-1}b_{s-1}, a_sb_s\}$ gives the desired path for (1).

Now suppose H_s is 3-connected and $H_s \neq K_4$. Then $5 \leq |H_s| < n$, and hence Theorem 2.1 holds for H_s. By Theorem 2.1 (b), there is a cycle C_s through $a_{s-1}b_{s-1}$ and a_sb_s in H_s such that $|C_s| = (\frac{|H_s|}{2(d-1)})^r + 3 \geq (\frac{|A_s|}{2(d-1)})^r + 3$.

When $s \neq 1$, then $P_s := C_s - \{a_{s-1}b_{s-1}, a_sb_s\}$ is as desired, and when $s = 1$, then $P_s := C_s - a_sb_s$ gives the desired path for (1).

(2) Next we find $P_t \subseteq H_t$, and to do so, we consider three subcases.

(2a) First, assume that $1 = t = h$. We will find a path P_t from x to $\{p, q\}$ such that $f \in E(P_t)$, $pq \notin E(P_t)$ unless $pq = f$, and $|E(P_t)| \geq (\frac{\sigma(\mathcal{H})}{2(d-1)})^r + 1$.

If H_t is a cycle chain, then by Proposition 2.8, let P_t denote a path from x to $\{p, q\}$ in H_t such that $f \in E(P_t)$, $pq \notin E(P_t)$ unless $pq = f$, and $A_t \subseteq V(P_t)$. When H_t consists of only one 3-block of $G - y$, then $|E(P_t)| \geq 2 = |A_t| > (\frac{|A_t|}{2(d-1)})^r + 1 = (\frac{\sigma(\mathcal{H})}{2(d-1)})^r + 1$ (by Lemma 3.2). When H_t has at least two 3-blocks of $G - y$, then $|A_t| \geq 3$ and $|E(P_t)| \geq |A_t| - 1 \geq (\frac{|A_t|}{2(d-1)})^r + 1 = (\frac{\sigma(\mathcal{H})}{2(d-1)})^r + 1$ (by Lemma 3.2). So P_t gives the desired path for (2a).

If $H_t = K_4$, then $\sigma(\mathcal{H}) = 4$. Let P_t denote a Hamilton path from x to $\{p, q\}$ in H_t such that $f \in E(P_t)$, and $pq \notin E(P_t)$ unless $pq = f$. Then $|E(P_t)| = 3 \geq (\frac{\sigma(\mathcal{H})}{2(d-1)})^r + 2$. Hence, P_t gives the desired path for (2a).

Now assume that H_t is not a cycle chain and $H_t \neq K_4$.

If $x \in \{p, q\}$, then $f \neq pq$ since x is not incident with f. Since $5 \leq |H_t| < n$, Theorem 2.1 holds for H_t. By Theorem 2.1 (b), there exists a cycle C_t in H_t such that $pq, f \in E(P_t)$ and $|C_t| \geq (\frac{|H_t|}{2(d-1)})^r + 3 = (\frac{\sigma(\mathcal{H})}{2(d-1)})^r + 3$. Hence $P_t := C_t - pq$ gives the desired path.

Assume $x \notin \{p, q\}$.

Suppose $f \neq pq$. Let H_t' be obtained from H_t by a T-transform at $\{x, pq\}$, and let x' denote the new vertex. By Lemma 3.3 and since x has degree at most $d - 1$ in H_t, H_t' is a 3-connected graph with maximum degree at most d. Since $G - y$ is not 3-connected, $|H_t| < n - 1$. Hence $5 \leq |H_t'| < n$, and Theorem 2.1 holds for H_t'. By Theorem 2.1 (b), there exists a cycle C_t in H_t' such that $f, xx' \in E(C_t)$ and $|C_t| \geq (\frac{|H_t|}{2(d-1)})^r + 3 = (\frac{\sigma(\mathcal{H})}{2(d-1)})^r + 3$. Hence $P_t := C_t - x'$ gives the desired path for (2a).

Finally, assume that $f = pq$. Let $H_t' := H_t + \{px, qx\}$. Then H_t' is a 3-connected graph with maximum degree at most $d + 1$, and all vertices of degree $d + 1$ must be incident with px or pq. By Theorem 2.1, we can find a cycle C_t in $H_t' - p$ through xq such that $|C_t| \geq (\frac{|H_t|}{2t})^r + 2 = (\frac{\sigma(\mathcal{H})}{2t})^r + 2$, where $t \leq d - 1$ is the number of neighbors of p distinct from x and q. Hence $P_t := (C_t - qx) + \{p, pq\}$ gives the desired path for (2a).

(2b) Now assume that $1 \le t < h$. If $t = 1$, we will find a path P_t from $\{a_t, b_t\}$ to $\{p, q\}$ in H_t such that $f \in E(P_t)$, $pq \notin E(P_t)$ unless $pq = f$, and $|E(P_t)| \ge (\frac{|A_t|}{2(d-1)})^r$. If $t \ne 1$, we will find $P_t \subseteq H_t$, consisting of disjoint paths from $\{p, q\}$ and $\{a_t, b_t\}$ to $\{a_{t-1}, b_{t-1}\}$ such that $|E(P_t)| \ge (\frac{|A_t|+2}{2(d-1)})^r - 1$.

Suppose H_t is a cycle chain. If $\{a_t, b_t\} = \{p, q\}$, then by Proposition 2.4 let C_t denote a Hamilton path in H_t from a_t to b_t through f. If $\{a_t, b_t\} \ne \{p, q\}$, then by Proposition 2.7, let C_t denote a path in $H_t - a_t b_t$ from $\{a_t, b_t\}$ to $\{p, q\}$ such that $a_{t-1}b_{t-1} \in E(C_t)$, $pq \notin E(C_t)$, and $A_t \subseteq V(C_t)$. From the definition of A_t and since $t < h$, $a_t \notin A_t$ and $b_t \notin A_t$. Also note that if $t \ne 1$, then $a_{t-1} \notin A_t$ or $b_{t-1} \notin A_t$. So if $t = 1$, then $|E(C_t)| \ge |A_t|$, and if $t \ne 1$, then $|E(C_t)| \ge |A_t| + 1$. Let $P_t := C_t$ if $t = 1$, and let $P_t := C_t - a_{t-1}b_{t-1}$ if $t \ne 1$. Then P_t is as desired for (2b).

If $H_t = K_4$, then let C_t denote a Hamilton path in $H_t - \{pq, a_t b_t\}$ from $\{a_t, b_t\}$ to $\{p, q\}$ through $a_{t-1}b_{t-1}$. Then $|E(C_t)| = 3 > (\frac{|A_t|}{2(d-1)})^r + 1$, and so $P_t := C_t - a_{t-1}b_{t-1}$ is as desired for (2b).

Now assume that H_t is not a cycle and $H_t \ne K_4$.

Suppose $\{a_t, b_t\} = \{p, q\}$. Since $5 \le |H_t| < n$, Theorem 2.1 holds for H_t. By Theorem 2.1 (b), there is a cycle C_t through $a_{t-1}b_{t-1}$ and $a_t b_t$ in H_t such that $|C_t| \ge (\frac{|H_t|}{2(d-1)})^r + 3 = (\frac{|A_t|}{2(d-1)})^r + 3$. If $t = 1$ then $P_t := C_t - a_t b_t$ gives the desired path, and if $t \ne 1$ then $P_t := C_t - \{a_{t-1}b_{t-1}, a_t b_t\}$ is as desired for (2b).

Now assume that $\{a_t, b_t\} \ne \{p, q\}$. Let H_t' be obtained from H_t by an H-transform at $\{a_t b_t, pq\}$, and let a', b' denote the new vertices. By Lemma 3.3, H_t' is a 3-connected graph with maximum degree at most d. Since $G - y$ is not 3-connected and since $\mathcal{I}$ is nonempty (when $\{p, q\}$ is defined), we have $|H_t| \le n - 3$. Hence $5 \le |H_t'| < n$. Hence Theorem 2.1 holds for H_t'. By Theorem 2.1 (b), there is a cycle C_t through $a'b'$ and $a_{t-1}b_{t-1}$ in H_t' such that $|C_t| \ge (\frac{|H_t'|}{2(d-1)})^r + 3 = (\frac{|A_t|+2}{2(d-1)})^r + 3$. If $t = 1$ then $P_t := C_t - \{a, a'\}$ gives the desired path, and if $t \ne 1$ then $P_t := C_t - \{a, a', a_{t-1}b_{t-1}\}$ is as desired for (2b).

(2c) Finally, assume $1 < t = h$. We will find $P_t \subseteq H_t$, consisting of disjoint paths from x and $\{p, q\}$ to $\{a_{t-1}, b_{t-1}\}$, such that $|E(P_t)| \ge (\frac{|A_t|}{2(d-1)})^r$.

If H_t is a cycle chain, then by Proposition 2.8, let C_t denote a path in H_t from x to $\{p, q\}$ such that $a_{t-1}b_{t-1} \in E(C_t)$, $pq \notin E(P_t)$, and $A_t \subseteq V(P_t)$. Since $x, a_{t-1}, b_{t-1} \notin A_t$, $|E(C_t)| \ge |A_t| + 1 > (\frac{|A_t|}{2(d-1)})^r + 1$. Hence $P_t := C_t - a_{t-1}b_{t-1}$ is as desired for (2c).

If $H_t = K_4$, then let C_t denote a Hamilton path in $H_t - pq$ from x to $\{p, q\}$ through $a_{t-1}b_{t-1}$. Then $|E(C_t)| = 3 \ge (\frac{|H_t|}{2(d-1)})^r + 2 = (\frac{|A_t|}{2(d-1)})^r + 2$. Hence $P_t := C_t - a_{t-1}b_{t-1}$ is as desired for (2c).

Now assume that H_t is not a cycle chain and $H_t \ne K_4$.

Suppose $x \in \{p, q\}$. Since $5 \le |H_t| < n$, Theorem 2.1 holds for H_t. By Theorem 2.1 (b), there is a cycle C_t through $a_{t-1}b_{t-1}$ and pq in H_t such that $|C_t| \ge (\frac{|H_t|}{2(d-1)})^r + 3 = (\frac{|A_t|}{2(d-1)})^r + 3$. Then $P_t := C_t - \{pq, a_{t-1}b_{t-1}\}$ is as desired.

Now assume that $x \notin \{p, q\}$. Recall that $\{p, q\} \ne \{a_{t-1}, b_{t-1}\}$. Let H_t' be obtained from H_t by a T-transform $\{x, pq\}$ and let c' denote the new vertex. By Lemma 3.3, H_t' is a 3-connected graph with maximum degree at most d (because the degree of x in H_t is at most $d - 1$). Since $G - y$ is not 3-connected, $|H_t| \le n - 2$, and so, $5 \le |H_t'| < n$. Hence Theorem 2.1 holds for H_t'. By Theorem 2.1 (b), there is a cycle C_t through xc' and $a_{t-1}b_{t-1}$ in H_t' such that $|C_t| \ge (\frac{|H_t'|}{2(d-1)})^r + 3 > (\frac{|A_t|}{2(d-1)})^r + 3$.

Hence $P_t := C_t - \{c', a_{t-1}b_{t-1}\}$ is as desired for (2c).

(3) For each $t+1 \le s \le h$, we will find a path $P_s \subseteq H_s$ such that $|E(P_s)| \ge (\frac{|A_s|}{2(d-1)})^r$ when $s \ne h$, $|E(P_s)| \ge (\frac{|A_s|}{2(d-1)})^r + 1$ when $s = h$, and $\bigcup_{s=t+1}^h P_s$ is a path from x to the end of P_t contained in $\{a_t, b_t\}$ and is otherwise disjoint from P_t.

We find P_s in the order $s = t+1, \ldots, h$.

Suppose P_{s-1} is found, and the notation of $\{a_{s-1}, b_{s-1}\}$ is chosen so that a_{s-1} is an end of P_{s-1}, and assume that the notation of $\{a_s, b_s\}$ is chosen so that $a_s \notin \{a_{s-1}, b_{s-1}\}$.

First, assume that H_s is a cycle chain. If $s \ne h$, then by Proposition 2.5, let P_s denote a path in $H_s - \{a_{s-1}b_{s-1}, a_sb_s\}$ from a_{s-1} to $\{a_s, b_s\}$ such that $A_s \subseteq V(P_s)$. Since $a_{s-1}, b_{s-1}, a_s, b_s \notin A_s$, we have that $|E(P_s)| \ge |A_s|+1 \ge (\frac{|A_s|}{2(d-1)})^r+1$. If $s = h$, then by Proposition 2.6 let P_s be a path from x to a_{s-1} in $H_s - a_{s-1}b_{s-1}$ such that $A_s \subseteq V(P_s)$. Since $x, a_{s-1}, b_{s-1} \notin A_s$, we have that $|E(P_s)| \ge |A_s|+1 > (\frac{|A_s|}{2(d-1)})^r+1$.

Now assume $H_s = K_4$. If $s \ne h$, then let P_s denote a path in $H_s - \{a_{s-1}b_{s-1}, a_sb_s\}$ from a_{s-1} to a_s with $|E(P_s)| \ge 1 \ge (\frac{|A_s|}{2(d-1)})^r$. If $s = h$, then let P_s be a path in $H_s - a_{s-1}b_{s-1}$ from a_{s-1} to x with $|E(P_s)| \ge 2 > (\frac{|A_s|}{2(d-1)})^r + 1$.

Assume that H_s is not a cycle chain and $H_s \ne K_4$.

Suppose $s \ne h$. If $b_{s-1} = b_s$, then let $H_s' := H_s + a_{s-1}a_s$. Clearly, H_s' is 3-connected with maximum degree at most $d+1$, and the vertices of degree $d+1$ must be incident with $a_{s-1}b_{s-1}$ or a_sb_{s-1}. Thus by Theorem 2.1 (a), there is a cycle C_s in $H_s' - b_{s-1}$ such that $a_{s-1}a_s \in E(C_s)$ and $|C_s| \ge (\frac{|H_s'|}{2(d-1)})^r + 2$. Let $P_s := C_s - a_{s-1}a_s$. Then $|E(P_s)| \ge (\frac{|A_s|}{2(d-1)})^r + 1$. So assume $b_{s-1} \ne b_s$. Let H_s'' be obtained from H_s by a T-transform at $\{b_{s-1}, a_sb_s\}$, and let a' denote the new vertex. Let $H_s' := H_s'' + a_{s-1}a'$. Since $G - y$ is not 3-connected, $|H_s''| \le n-2$, and so $5 \le |H_s'| < n$. By Lemma 3.3, H_s'' is a 3-connected graph with maximum degree at most $d+1$, and the vertices of degree $d+1$ must be incident with $b_{s-1}a'$ or $b_{s-1}a_{s-1}$. Thus $H_s', a_{s-1}a', b_{s-1}$ satisfy the conditions Theorem 2.1 (a). By Theorem 2.1 (a), there is a cycle C_s through $a_{s-1}a'$ in $H_s' - b_{s-1}$ such that $|C_s| \ge (\frac{|H_s'|}{2(d-1)})^r + 2$. Let $P_s := C_s - a'$. Then $|E(P_s)| \ge (\frac{|A_s|}{2(d-1)})^r$.

Now assume $s = h$. Let $H_s' := H_s + \{xb_{s-1}, xa_{s-1}\}$. Then H_s' is a 3-connected graph, the vertices x, a_{s-1}, b_{s-1} have degree at most $d+1$, and all other vertices of H_s' have degree at most d. Thus $H_s', a_{s-1}x, b_{s-1}$ satisfy the conditions of Theorem 2.1 (a). By Theorem 2.1 (a), there is a cycle C_s' through $a_{s-1}x$ in $H_s' - b_{s-1}$ such that $|C_s'| \ge (\frac{|H_s'|}{2(d-1)})^r + 2$. Let $P_s := C_s' - a_{s-1}x$. Then P_s is a path from a_{s-1} to x and $|E(P_s)| \ge (\frac{|H_s'|}{2(d-1)})^r + 1 \ge (\frac{|A_s|}{2(d-1)})^r + 1$.

It is easy to see that $\bigcup_{s=t+1}^h P_s$ is a path from x to the end of P_t in $\{a_t, b_t\}$ and is otherwise disjoint from P_t.

(4) Let $P := \bigcup_{s=1}^h P_s$. We claim that P is a path from x to $\{p, q\}$, $f \in E(P)$, $pq \notin E(P)$ unless $pq = f$, and $|E(P)| \ge (\frac{\sigma(\mathcal{H})}{2(d-1)})^r + 1$.

This is obvious if $h = 1$ (by (2a)). So assume that $h \ge 2$.

Suppose $t \ne 1$. Then $|E(P_s)| \ge (\frac{|A_s|}{2(d-1)})^r + 1$ for $1 \le s \le t-1$ (by (1)), $|E(P_t)| \ge (\frac{|A_t|+2}{2(d-1)})^r - 1$ when $t \ne h$ (by (2b)), $|E(P_t)| \ge (\frac{|A_t|}{2(d-1)})^r$ when $t = h$ (by (2c)), $|E(P_s)| \ge (\frac{|A_s|}{2(d-1)})^r$ when $t+1 \le s < h$ (by (3)), and $|E(P_h)| \ge (\frac{|A_h|}{2(d-1)})^r) + 1$

when $t < h$ (by (3)). Hence we have

$$\begin{aligned}|E(P)| &= \sum_{s=1}^{h} |E(P_s)| \\ &\geq \left(\sum_{s=1}^{h} \left(\frac{|A_s|}{2(d-1)} \right)^r \right) + 1 \\ &\geq \left(\frac{\sum_{s=1}^{h} |A_s|}{2(d-1)} \right)^r + 1 \quad \text{(by Lemma 3.1)} \\ &\geq \left(\frac{\sigma(\mathcal{H})}{2(d-1)} \right)^r + 1.\end{aligned}$$

Now suppose $t = 1$. Then $|E(P_t)| \geq (\frac{|A_t|}{2(d-1)})^r$ (by (2b)), $|E(P_s)| \geq (\frac{|A_s|}{2(d-1)})^r$ for $2 \leq s \leq h-1$ (by (3)), and $|E(P_h)| \geq (\frac{|A_h|}{2(d-1)})^r + 1$ (by (3)). Hence by the same argument as in the above paragraph, we have $|E(P)| \geq (\frac{\sigma(\mathcal{H})}{2(d-1)})^r + 1$. Thus, we have (4).

By (4), we may assume that the notation of $\{p, q\}$ is chosen so that P is from x to p.

(5) We claim that there is a path Q in $G_1 - q$ from p to y such that $|E(Q)| \geq (\frac{|G_1|}{2(d-1)})^r + 1$.

Note that $|G_1| \geq 4$ and that G_1 is not a cycle. Thus $G_1' := G_1 + \{yp, yq, pq\}$ is a 3-connected graph.

If $G_1' = K_4$, then we can find a path Q in $G_1' - q$ from p to y such that $|E(Q)| = 2 \geq (\frac{|G_1|}{2(d-1)})^r + 1$.

Now assume that $G_1' \neq K_4$. Then Theorem 2.1 holds for G_1'. Note that all vertices of G_1' have degree at most d, except possibly y, p, q, which have degree at most $d+1$. By Theorem 2.1 (a), there is a cycle C_1 through py in $G_1' - q$ such that $|C_1| \geq (\frac{|G_1|}{2(d-1)})^r + 2$. Let $Q := C_1 - py$. Then Q gives the desired path.

(6) Finally, let $C := (P \cup Q) + xy$. Then C is a cycle through e and f in G and, by (4) and (5), $|C| \geq ((\frac{\sigma(\mathcal{H})}{2(d-1)})^r + 1) + ((\frac{|G_1|}{2(d-1)})^r + 1) + 1 = (\frac{\sigma(\mathcal{H})}{2(d-1)})^r + (\frac{|G_1|}{2(d-1)})^r + 3$. Recall that $\sigma(\mathcal{H}) \geq |G_2|$ and $|G_1| \geq |G_2|$.

If $\sigma(\mathcal{H}) < |G_1|$, then

$$\begin{aligned}|C| &\geq \left(\frac{\sigma(\mathcal{H})}{2(d-1)} \right)^r + \left(\frac{|G_1|}{2(d-1)} \right)^r + 3 \\ &\geq \left((d-1)\sigma(\mathcal{H}) + \frac{|G_1|}{2(d-1)} \right)^r + 3 \quad \text{(by Lemma 3.1 and since } \sigma(\mathcal{H}) < |G_1|) \\ &\geq \left(\frac{n}{2(d-1)} \right)^r + 3 \quad \text{(by Observation 3).}\end{aligned}$$

Otherwise, $\sigma(\mathcal{H}) \geq |G_1|$. Hence,

$$\begin{aligned}|C| &\geq \left(\frac{\sigma(\mathcal{H})}{2(d-1)} \right)^r + \left(\frac{|G_1|}{2(d-1)} \right)^r + 3 \\ &\geq \left(\frac{\sigma(\mathcal{H})}{2(d-1)} + (d-1)|G_1| \right)^r + 3 \quad \text{(by Lemma 3.1 and since } \sigma(\mathcal{H}) \geq |G_1|)\end{aligned}$$

$$> \left(\frac{n}{2(d-1)}\right)^r + 3 \quad \text{(by Observation 1).}$$

Case 2. $\sigma(\mathcal{H}) \le |G_2|$. In this case G_2 is nonempty. We will use G_1 and G_2 to find the desired cycle. Let $V(G_2) \cap V(\mathcal{H}) = \{v, w\}$. Then there exists some $1 \le u \le h$ such that $\{v, w\} \subseteq V(H_u)$, and $\{v, w\} \ne \{a_{u-1}, b_{u-1}\}$ when $u \ne 1$. Also there exists some $1 \le t \le h$ such that $\{p, q\} \subseteq V(H_t)$, and $\{p, q\} \ne \{a_{t-1}, b_{t-1}\}$ when $t \ne 1$.

(1) We claim that we can find, in $O(|\mathcal{H}|)$ time, a path P from x to some $z \in \{p, q\} \cup \{v, w\}$ in $\bigcup_{s=1}^{h} H_s$ for which

(i) $f \in E(P)$,

(ii) $pq \in E(P)$ or $vw \in E(P)$, and

(iii) if $pq \in E(P)$, then $z \in \{v, w\}$, and $vw \notin E(P)$ unless $vw = f$, and if $vw \in E(P)$, then $z \in \{p, q\}$, and $pq \notin E(P)$ unless $pq = f$.

To prove (1), let us assume that $t \le u$; the case $t \ge u$ can be taken care of in exactly the same way.

When $t \ne 1$, we use Lemma 3.5 to find a cycle Q' in $\bigcup_{s=1}^{t-1} H_s$ through $a_{t-1}b_{t-1}$ and f. Let $Q := Q' - a_{t-1}b_{t-1}$, which is a path from a_{t-1} to b_{t-1} through f. Let $Q = \emptyset$ when $t = 1$. We distinguish two cases.

Subcase (1a). $t < u$. By choosing the notation of $\{a_t, b_t\}$, we may assume that $(\bigcup_{s=t+1}^{h} H_s) - b_t$ contains a path X from a_t to x through vw.

If $b_t \in \{p, q\}$, then we use Lemma 3.5 to find a cycle C_t through $a_{t-1}b_{t-1}$ and $a_t b_t$ in H_t. If $pq \notin E(C_t)$ or $b_t \in \{a_{t-1}, b_{t-1}\}$, let $P_t := C_t - \{a_{t-1}b_{t-1}, a_t b_t\}$ when $t \ne 1$, and let $P_t := C_t - a_t b_t$ when $t = 1$. Then $P := Q \cup X \cup P_t$ gives the desired path for (1) (with $z = b_t$). So assume $pq \in E(C_t)$ and $b_t \notin \{a_{t-1}, b_{t-1}\}$. Then let $P_t := C_t - \{a_{t-1}b_{t-1}, b_t\}$ when $t \ne 1$, and let $P_t := C_t - b_t$ when $t = 1$. Then $P := Q \cup X \cup P_t$ gives the desired path for (1) (with $z = p$).

We may therefore assume that $b_t \notin \{p, q\}$.

Suppose H_t is not a cycle chain. Then H_t is 3-connected. Let H'_t be obtained from H_t by a T-transform at $\{a_t, pq\}$ and let a' denote the new vertex. Then $H'_t - b_t$ is 2-connected. By Lemma 3.5, we find a cycle C'_t through $a_{t-1}b_{t-1}$ and $a' a_t$ in $H'_t - b_t$. If $t = 1$, then let $P_t := C'_t - a'$, and if $t \ne 1$, then let $P_t := C'_t - \{a', a_{t-1}b_{t-1}\}$. Then $P := Q \cup P_t \cup X$ gives the desired path for (1).

Now assume that H_t is a cycle chain. By the structure of a cycle chain, we can, in $O(|H_t|)$ time, either find a path C_t in $H_t - a_t b_t$ from a_t to $\{p, q\}$ through $a_{t-1}b_{t-1}$, or find a path C'_t in H_t from a_t to b_t through $a_{t-1}b_{t-1}$ and pq.

If we find C_t, then let $P_t := C_t - a_{t-1}b_{t-1}$ when $t \ne 1$ and $P_t := C_t$ when $t = 1$. In this case, $P := Q \cup P_t \cup X$ gives the desired path for (1).

Assume that we find C'_t. In this case, we cannot use X. Let $P_t := C'_t$ if $t = 1$, and otherwise let $P_t := C'_t - a_{t-1}b_{t-1}$. Let $H := \bigcup_{s=t+1}^{h} H_s$. If $x \in \{v, w\}$, then find a cycle C' in H through $a_t b_t$ and vw, and so $P := Q \cup P_t \cup (C' - \{a_t b_t, vw\})$ gives the desired path for (1). So assume that $x \notin \{v, w\}$. Let H' be obtained from H by a T-transform at $\{x, vw\}$, and let x' denote the new vertex. Then H' is a 2-connected graph. By Lemma 3.5, we find a cycle C' through $a_t b_t$ and xx'. Now $P := Q \cup P_t \cup (C' - \{x', a_t b_t\})$ gives the desired path for (1).

Subcase (1b). $t = u$. Recall that $\{p, q\} \ne \{v, w\}$.

First, assume that $t \ne h$. We claim that there is a path Q_t in H_t from $\{a_t, b_t\}$ to some $z \in \{p, q\} \cup \{v, w\}$ such that (i) $a_{t-1}b_{t-1} \in E(Q_t)$, (ii) $pq \in E(Q_t)$ or $vw \in E(Q_t)$, and (iii) if $pq \in E(Q_t)$, then $z \in \{v, w\}$, and $vw \notin E(Q_t)$ unless $vw = a_{t-1}b_{t-1}$, and if $vw \in E(Q_t)$, then $z \in \{p, q\}$, and $pq \notin E(Q_t)$ unless $pq = a_{t-1}b_{t-1}$.

This is easy to see if H_t is a cycle chain, and otherwise, it follows from Lemma 3.6. Assume without loss of generality that $a_t \in V(Q_t)$. In $(\bigcup_{s=t+1}^{h} H_s) - b_t$, we find a path R from a_t to x. Now $P := Q \cup Q_t \cup R$ gives the desired path for (1).

Now assume that $t = h$. We note that there is a path Q_t in H_t from x to $z \in \{p,q\} \cup \{v,w\}$ such that (i) $a_{t-1}b_{t-1} \in E(Q_t)$, (ii) $pq \in E(Q_t)$ or $vw \in E(Q_t)$, and (iii) if $pq \in E(Q_t)$, then $z \in \{v,w\}$, and $vw \notin E(Q_t)$ unless $vw = a_{t-1}b_{t-1}$, and if $vw \in E(Q_t)$, then $z \in \{p,q\}$, and $pq \notin E(Q_t)$ unless $pq = a_{t-1}b_{t-1}$. This is easy to see if H_t is a cycle chain, and otherwise, it follows from Lemma 3.7. Now $P := Q \cup Q_t$ gives the desired path for (1).

Assume that $vw \in E(P)$ in (1) (the case $pq \in E(P)$ is similar), and assume p is an end of P.

(2) Note that $G_1' := G_1 + \{yp, yq, pq\}$ is a 3-connected graph, vertices y, p, q have degree at most $d+1$ in G_1', and all other vertices of G_1' have degree at most d. If $G_1' = K_4$, then we can find a path P_1 from p to y in $G_1' - q$ such that $|E(P_1)| = 2 \geq (\frac{|G_1|}{2(d-1)})^r + 1$. If $G_1' \neq K_4$, then Theorem 2.1 holds for G_1'. By Theorem 2.1 (a), there is a cycle C_1 through py in $G_1' - q$ such that $|C_1| \geq (\frac{|G_1|}{2t_1})^r + 2$, where t_1 is the number of neighbors of q in G_1' distinct from p and y. Let $P_1 := C_1 - py$. Then P_1 is a path from p to y in $G_1' - q$. Since $t_1 \leq d-1$, we have $|E(P_1)| \geq (\frac{|G_1|}{2(d-1)})^r + 1$.

(3) Note that $G_2' := G_2 + \{yv, yw, vw\}$ is a 3-connected graph, vertices y, v, w have degree at most $d+1$ in G_2', and all other vertices of G_2' have degree at most d. If $G_2' = K_4$, then we can find a path P_2 from v to w in $G_2' - y$ such that $|E(P_2)| = 2 \geq (\frac{|G_2|}{2(d-1)})^r + 1$. If $G_2' \neq K_4$, then Theorem 2.1 holds for G_2'. By Theorem 2.1 (a), there is a cycle C_2 through vw in $G_2' - y$ such that $|C_2| \geq (\frac{|G_2'|}{2t_2})^r + 2$, where t_2 is the number of neighbors of y in G_2' distinct from v and w. Let $P_2 := C_2 - vw$. Then P_2 is a path from v to w in $G_2' - y$. Since $t_2 \leq d-1$, we have $|E(P_2)| \geq (\frac{|G_2|}{2(d-1)})^r + 1$.

Let $C := ((P - vw) \cup P_1 \cup P_2) + e$. Then C is a cycle through e and f in G and

$$
\begin{aligned}
|C| &= |E(P)| + |E(P_1)| + |E(P_2)| + 1 \\
&\geq \left(\frac{|G_1|}{2(d-1)}\right)^r + \left(\frac{|G_2|}{2(d-1)}\right)^r + 3 \quad \text{(by (2) and (3))} \\
&\geq \left(\frac{|G_1|}{2(d-1)} + (d-1)|G_2|\right)^r + 3 \quad \text{(by Lemma 3.1 and since } |G_1| \geq |G_2|) \\
&\geq \left(\frac{n}{2(d-1)}\right)^r + 3,
\end{aligned}
$$

where the final inequality holds because of Observation 2 and since $|G_2| \geq \sigma(\mathcal{H})$. $\square$

Next we show that the above proof gives an $O(|G|)$ algorithm which reduces Theorem 2.1 (b) to (a) and (b) of the same theorem (for smaller graphs).

Algorithm Twoedge. Let n, d, r, G, e, f be as in Lemma 5.1.

1. *Preprocessing* Replace G with a 3-connected spanning subgraph of G with $O(|G|)$ edges. (This can be done in $O(|E(G)|)$ time by 3.4.)
2. If e is adjacent to f, then let $e = xz$ and $f = yz$. It suffices to find a cycle C' through xy in $G' := (G + xy) - z$ such that $|C'| \geq (\frac{|G'|}{2t})^r + 2$, where t is the number of neighbors of z in G' distinct from x and y. That is, we reduce Theorem 2.1 (b) for G, e, f to Theorem 2.1 (a) for G', xy, z. We apply Algorithm Avoidvertex to G', xy, z. (By Proposition 4.2, we can, in $O(|G|)$ time, either find the desired cycle C' or reduce it to (a) or (c) for smaller

graphs. Moreover, each smaller graph contains a vertex that does not belong to any other smaller graph.)

3. Now assume that e is not adjacent to f, and let $e = xy$. Decompose $G-y$ into 3-connected components. (This can be done in $O(|G|)$ time using Theorem 2.2.)
4. Suppose there is only one 3-connected component of $G - y$. Then $G - y$ is 3-connected, and let y' denote a neighbor of y distinct from x. Let $G' := (G - y) + xy'$ and $e' = xy'$. To find the desired cycle through e and f in G, it suffices to find a cycle C' through e' and f in G' such that $|C'| \geq (\frac{|G'|}{2(d-1)})^r + 3$. Thus we reduce Theorem 2.1 (b) for G, e, f to Theorem 2.1 (b) for G', e', f, with $|G'| < |G|$. (This reduction can be done in constant time.)
5. Now assume that $G - y$ has at least two 3-connected components. Find the block chain $\mathcal{H} = H_1 \dots H_h$ such that $f \in E(H_1)$, $x \in V(H_h) - V(H_{h-1})$, and f is not incident with both vertices in $V(H_1) \cap V(H_2)$. Find G_1 and G_2 as in the proof Lemma 5.1. (This can be done in $O(|G|)$ time.)
6. Suppose $\sigma(\mathcal{H}) > |G_2|$.
   - Assume $1 \leq s \leq t-1$. We need to find P_s as in (1) of Case 1 in the proof of 5.1. If $H_s = K_4$ or H_s is a cycle chain then we find P_s, and otherwise, we need to find a cycle C_s through $a_{s-1}b_{s-1}$ and $a_s b_s$ in H_s such that $|C_s| \geq (\frac{|H_s|}{2(d-1)})^r + 3$. So either we find P_s in $O(|H_s|)$ time or we reduce the problem of finding P_s to Theorem 2.1 (b) for $H_s, a_{s-1}b_{s-1}, a_s b_s$ in constant time.
   - We need to find $P_t \subseteq H_t$ as in (2) of Case 1 in the proof of Lemma 5.1. If H_t is a cycle chain or $H_t = K_4$ then we find $P_t \subseteq H_t$ as in (2) of Case 1 in the proof of Lemma 5.1. (This can be done in $O(|H_t|)$ time.) If H_t is not a cycle chain and $H_t \neq K_4$, then we reduce the problem of finding P_t to the following:Theorem 2.1 (b) for H_t, f, pq or H_t', f, xx'; Theorem 2.1 (a) for H_t', px, q (as in (2a) of Case 1); Theorem 2.1 (b) for $H_t, a_{t-1}b_{t-1}, a_t b_t$ or $H_t', a_{t-1}b_{t-1}, a'b'$ (as in (2b) of Case 1); Theorem 2.1 (b) for $H_t, a_{t-1}b_{t-1}, pq$ or $H_t', a_{t-1}b_{t-1}, xc'$ (as in (2c) of Case 1). (This reduction can be done in constant time.)
   - Suppose $t + 1 \leq s \leq h$. We need to find P_s as in (3) of Case 1 in the proof of Lemma 5.1. If $H_s = K_4$ or H_s is a cycle chain, we find a path P_s. (This can be done in $O(|H_s|)$ time.) If $H_s \neq K_4$ and H_s is not a cycle chain, then we reduce the problem of finding P_s to the following: Theorem 2.1 (a) for $H_s', a_{s-1}a_s, b_{s-1}$, or $H_s', a_{s-1}a', b_{s-1}$, or $H_s', a_{s-1}x, b_{s-1}$. (This reduction can be done in constant time.)
   - Let $G_1' := G_1 + \{yp, yq, pq\}$. We need to find a path Q in G_1' as in (5) of Case 1 in the proof of Lemma 5.1. If $G_1' = K_4$, we find a path Q in $O(|G_1|)$ time, and otherwise, we reduce the problem of finding Q to Theorem 2.1 (a) for G_1', py, q, in constant time.

   (The operations in step 6 can be done in $O(|G|)$ time. Also each 3-connected graph reduced to from H_s's or G_1 contains a vertex which does not belong to any other 3-connected graphs reduced to from H_s's or G_1.)
7. Now assume $\sigma(\mathcal{H}) \leq |G_2|$.
   - First, we find H_t and H_u such that $\{p, q\} \subseteq V(H_t)$, $\{p, q\} \neq \{a_{t-1}, b_{t-1}\}$ when $t \neq 1$, $\{v, w\} \subseteq V(H_u)$, and $\{v, w\} \neq \{a_{u-1}, b_{u-1}\}$ when $u \neq 1$. (This can be done in $O(|G|)$ time by searching the 3-connected components of $G - y$.)

- Assume that $t \le u$ ($u \ge t$ can be treated similarly). Find a path P in $\bigcup_{s=1}^{t} H_s$ from x to $\{p, q\}$ (or $\{v, w\}$) to $\{v, w\}$ (or $\{p, q\}$) through f and vw (or pq). (This can be done in $O(|G|)$ time as in (1) of Case 2 in the proof of Lemma 5.1.)
- Assume P is from x to p and through f and vw. If $G_1' = K_4$, then we find a path P_1 in $G_1' - q$ from p to y of length 2. If $G_1' \neq K_4$, then we need to apply Theorem 2.1 (a) to G_1', yp, q. (This reduction can be done in constant time, as in (2) of Case 2 in the proof of Lemma 5.1.)
- If $G_2' = K_4$, then find a path P_2 in $G_2' - y$ from v to w of length 2. If $G_2' \neq K_4$, then we need to apply Theorem 2.1 (a) to G_2', vw, y. (Again, this can be done in constant time, as in (3) of Case 2 in the proof of Lemma 5.1.)

To summarize, we have the following.

Proposition 5.2. *Given G, e, f, n, d, r as in Lemma 5.1, we can, in $O(|E(G)|)$ time, either*

(1) *find a cycle C through e and f in G such that $|C| \ge (\frac{|G|}{2(d-1)})^r + 3$, or*
(2) *reduce Theorem 2.1 (b) for G, e, f to Theorem 2.1 (a) or (b) for smaller 3-connected graphs.*

Moreover, any smaller graph in (2) comes from a 3-connected 3-block of $G - y$ that is not K_4. Hence, any smaller graph in (2) contains a vertex that does not belong to any other smaller graph in (2).

6. Cycles through one edge. In this section, we show how to reduce Theorem 2.1 (c), in linear time, to Theorem 2.1 (a), (b), or (c) of 2.1 for smaller graphs. As in the previous two sections, we state the reduction as a lemma.

Lemma 6.1. *Let $n \ge 6$ and $d \ge 3$ be integers, let $r = \log_{2(d-1)^2+1} 2$, and assume that Theorem 2.1 holds for graphs with at most $n-1$ vertices. Let G be a 3-connected graph on n vertices, and assume that the maximum degree of G is at most d. Then for any $e \in E(G)$ there is a cycle C through e in G such that $|C| \ge n^r + 3$.*

Proof. Let $e = xy \in E(G)$, and consider $G - y$.

If $G - y$ is 3-connected, then let y' be a neighbor of y other than x. Clearly, $G' := (G - y) + xy'$ is a 3-connected graph with maximum degree at most d. Since $5 \le |G'| < n$, Theorem 2.1 holds for G'. By Theorem 2.1 (c), there is a cycle C' through xy' in G' such that $|C'| \ge (n-1)^r + 3$. Now let $C := (C' - xy') + \{y, xy, yy'\}$. Then C is a cycle through xy in G and

$$\begin{aligned} |C| &= |C'| + 1 \\ &\ge (n-1)^r + 1 + 3 \\ &\ge n^r + 3 \quad \text{(by Lemma 3.1)}. \end{aligned}$$

Therefore, we may assume that $G - y$ is not 3-connected. Since G is 3-connected, $G - y$ is 2-connected. By Theorem 2.2, we can decompose $G - y$ into 3-connected components.

First, let us consider the case where all 3-blocks of $G - y$ are cycles. Let $\mathcal{I} = I_1 \dots I_i$ be a block chain in $G - y$ such that (i) $x \in V(I_1) - V(I_2)$, (ii) I_i is an extreme 3-block of $G - y$, and (iii) subject to (i) and (ii), $|V(\mathcal{I})|$ is maximum. For convenience, let $B := I_1$. Then $|V(\mathcal{I})| \ge \frac{(n-1)-|B|}{t-1} + |B| = \frac{n+(t-2)|B|-1}{t-1}$, where t is the number of extreme 3-blocks of $G - y$ distinct from I_1. So $n \ge t + 4$ (since $|B| \ge 3$) and $t \le d - 1$. It is easy to see that there is some $y' \in V(\mathcal{I}) - \{x\}$ such that $\bigcup_{s=1}^{i} I_s$ contains a Hamilton path P from x to y' and G has a path Q from y' to y

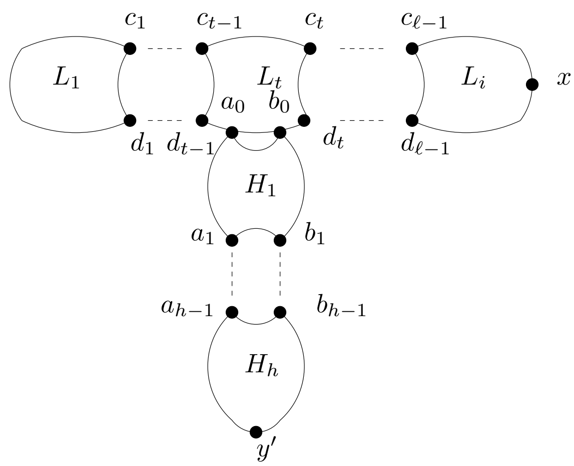

FIG. 8. *Block chains $\mathcal{L}$ and $\mathcal{H}$.*

disjoint from $V(\mathcal{I}) - \{y\}$. (Note that P and y' can be found in $O(|G|)$ time.) Let $C := (P \cup Q) + \{y, xy, yy'\}$. Then $|C| \geq |V(\mathcal{I})| + 1 \geq \frac{n+(t-2)|B|-1}{t-1} + 1$. Next we show that $|C| - 3 \geq n^r$. Note that $|C| - 3 \geq \frac{n+(t-2)|B|-1}{t-1} - 2 \geq \frac{n+t-5}{t-1}$ (since $|B| \geq 3$). One can prove that $\frac{x+t-5}{t-1} - x^r$ is an increasing function when $x \geq 2(d-1)^2 + 1$. Hence when $n \geq 2(d-1)^2 + 1$, $\frac{n+t-5}{t-1} \geq n^r$. Now if $t + 4 \leq n \leq 2(d-1)^2$, then $\frac{n+t-5}{t-1} > 2 > n^r$. Therefore, $|C| - 3 \geq n^r$, and so $|C| \geq n^r + 3$.

Hence, we may assume that some 3-block of $G - y$ is 3-connected. Let L denote a 3-connected 3-block of $G - y$ with $|L|$ maximum. Then $|L| \geq 4$. Let $\mathcal{L} := L_1 \dots L_\ell$ denote a block chain in $G - y$ such that $L_1 = L$ and $x \in V(L_\ell) - V(L_{\ell-1})$, where $\ell \geq 1$. See Figure 8.

Let $V(L_i) \cap V(L_{i+1}) = \{c_i, d_i\}$ for $1 \leq i \leq \ell - 1$. For each $1 \leq i \leq \ell$, we define B_i as follows: if L_i is 3-connected, then $B_i := V(L_i)$; if $i < \ell$ and $L_i = C_1 \dots C_k$ is a cycle chain, then $B_i = (\bigcup_{i=1}^{k-1} V(C_i \cap C_{i+1})) - (\{c_{i-1}, d_{i-1}\} \cup \{c_i, d_i\})$; if $L_\ell = C_1 \dots C_k$ is a cycle chain, then B_ℓ consists of x and the vertices in $(\bigcup_{i=1}^{k-1} V(C_i \cap C_{i+1})) - \{c_{\ell-1}, d_{\ell-1}\}$. Define $\sigma(\mathcal{L}) := |\bigcup_{s=1}^{\ell} B_s|$.

If $V(\mathcal{L}) = V(G) - \{y\}$, then let $\mathcal{H} = \emptyset$. Otherwise, let $\mathcal{H} := H_1 \dots H_h$ denote a block chain in $G - y$ such that (i) $|V(\mathcal{H}) \cap V(\mathcal{L})| = 2$, (ii) $V(\mathcal{H}) \cap V(\mathcal{L}) = V(H_1) \cap V(\mathcal{L}) \neq V(H_1) \cap V(H_2)$, (iii) H_h is an extreme 3-block of $G - y$. See Figure 8. Let $V(H_i) \cap V(H_{i+1}) = \{a_i, b_i\}$, $1 \leq i \leq h - 1$. Let a_0, b_0 denote the vertices in $V(\mathcal{H}) \cap V(\mathcal{L})$. If $\mathcal{H} = \emptyset$, then let y' denote a neighbor of y distinct from x and let $a_0 = b_0 = y'$. If $\mathcal{H} \neq \emptyset$, then let y' be a neighbor of y in $V(H_h) - \{a_{h-1}, b_{h-1}\}$. For each $1 \leq s \leq h$, we define A_s as follows: if H_s is 3-connected, then $A_s := V(H_s)$; if $h = 1$ and $H_1 = C_1 \dots C_k$ is a cycle chain, then $A_1 := (\bigcup_{i=1}^{k-1} V(C_i \cap C_{i+1})) \cup \{a_0, b_0\}$; if $h > 1$ and $H_1 = C_1 \dots C_k$ is a cycle chain, then A_1 consists of a_0, b_0 and those vertices in $(\bigcup_{i=1}^{k-1} V(C_i \cap C_{i+1})) - \{a_1, b_1\}$; if $1 < s < h$ and $H_s = C_1 \dots C_k$ is a cycle chain, then $A_s = (\bigcup_{i=1}^{k-1} V(C_i \cap C_{i+1})) - \{a_{s-1}, b_{s-1}, a_s, b_s\}$; if $h > 1$ and $H_h = C_1 \dots C_k$ is a cycle chain, then A_h consists of y' and those vertices in $(\bigcup_{i=1}^{k-1} V(C_i \cap C_{i+1})) - \{a_{h-1}, b_{h-1}\}$. Define $\sigma(\mathcal{H}) := |\bigcup_{s=1}^{h} A_s|$.

We choose $\mathcal{H} = H_1 \dots H_h$ so that, subject to (i)–(iii), $\sigma(\mathcal{H})$ is maximum. Without loss of generality, we may assume that, for some $1 \leq t \leq \ell$, $a_0, b_0 \in V(L_t)$ and

$\{a_0, b_0\} \neq \{c_{t-1}, d_{t-1}\}$ (or $y' \notin \{c_{t-1}, d_{t-1}\}$ if $a_0 = b_0 = y'$) when $t \geq 2$.

Note that each vertex in $(\bigcup_{i=1}^{\ell}(V(L_i) - B_i)) \cup (\bigcup_{s=1}^{h}(V(H_s) - A_s))$ either appears in some other block chain in $G - y$ or is adjacent to y. By the choice of $\mathcal{H}$ and since there are at most $d - 1$ extreme blocks of $G - y$ not containing x, we have $\sigma(\mathcal{H}) \geq \frac{(n-1)-\sigma(\mathcal{L})}{d-1}$. Hence we have the following.

Observation. $\sigma(\mathcal{L}) + (d-1)\sigma(\mathcal{H}) \geq n - 1$.

We consider three cases.

Case 1. $\ell = 1$. In this case, $\sigma(\mathcal{L}) = |L_1|$ (because L_1 is not a cycle chain). Since $G - y$ is not 3-connected, $\mathcal{H} \neq \emptyset$.

(1) First, we find a path P in $\mathcal{L} - a_0b_0$ from x to $\{a_0, b_0\}$ such that $|E(P)| \geq (\sigma(\mathcal{L}) + 1)^r + 1$.

If $L_1 = K_4$, then we can find a Hamilton path P from x to $\{a_0, b_0\}$ in $L_1 - a_0b_0$. Hence by Lemma 3.2, $|E(P)| = 3 \geq |L_1|^r + 2 = (\sigma(\mathcal{L}) + 1)^r + 1$. So assume that $L_1 \neq K_4$.

If $x \in \{a_0, b_0\}$, then by Theorem 2.1 (c), there is a cycle C_1 through a_0b_0 in L_1 such that $|C_1| \geq |L_1|^r + 3$. Now $P := C_1 - a_0b_0$ is a path from x to $\{a_0, b_0\}$ in $L_1 - a_0b_0$ and $|E(P)| \geq (\sigma(\mathcal{L})^r + 2(\sigma(\mathcal{L}) + 1)^r + 1$ (by Lemma 3.1).

Assume $x \notin \{a_0, b_0\}$. Let L_1' denote the graph obtained from L_1 by a T-transform at $\{x, a_0b_0\}$, and let x' denote the new vertex. By Lemma 3.3 and because x has degree at most $d-1$ in L_1, L_1' is a 3-connected graph with maximum degree at most d. Note that $5 \leq |L_1'| < n$. Thus Theorem 2.1 holds for L_1'. By Theorem 2.1 (c), there is a cycle C_1 through xx' in L_1' such that $|C_1'| \geq |L_1'|^r + 3 = (|L_1| + 1)^r + 3$. Now $P := C_1 - x'$ gives the desired path.

Without loss of generality, we may assume that the path P found in (1) is from x to a_0.

(2) For $i = 1, \ldots, h-1$, we find paths Q_i from $\{a_{i-1}, b_{i-1}\}$ to $\{a_i, b_i\}$ in H_i such that

(i) $\bigcup_{i=1}^{h-1} Q_i$ is a path from a_0 to $\{a_{h-1}, b_{h-1}\}$,

(ii) if H_i is 3-connected, then $|E(Q_i)| \geq (\frac{|A_i|}{2(d-1)})^r$,

(iii) if H_1 is a cycle chain, then $|A_1| \geq 2$ and $|E(Q_1)| \geq |A_1|$, and

(iv) if $1 < i < h$ and H_i is a cycle chain, then $|E(Q_i)| \geq |A_i| + 1$.

Suppose that, for some $1 \leq i \leq h-1$, we have found paths Q_j, $1 \leq j \leq i-1$, from $\{a_{j-1}, b_{j-1}\}$ to $\{a_j, b_j\}$ in H_j such that $\bigcup_{j=1}^{i-1} Q_j$ is a path from a_0 to $\{a_{i-1}, b_{i-1}\}$ and (ii)–(iv) above are satisfied for H_j, Q_j, A_j. For ease of presentation, we may assume that a_{i-1} is an end of $\bigcup_{j=1}^{i-1} Q_j$.

If $H_i = K_4$, then we find a path Q_i in $H_i - b_{i-1}$ from a_{i-1} to $\{a_i, b_i\}$ such that $|E(Q_i)| \geq 2$. Clearly, $|E(Q_i)| \geq 2 \geq (\frac{|A_i|}{2(d-1)})^r$.

If H_i is a cycle chain, then by Proposition 2.5 let Q_i be a path in $H_i - \{a_{i-1}b_{i-1}, a_ib_i\}$ from a_{i-1} to $\{a_i, b_i\}$ such that $A_i \subseteq V(Q_i)$. If $i = 1$, then since $a_0, b_0 \in A_1$ and $(\{a_1, b_1\} - \{a_0, b_0\}) \cap A_1 = \emptyset$, we have $|A_1| \geq 2$ and $|E(Q_i)| \geq |A_i|$. If $i \neq 1$, then since $(\{a_{i-1}, b_{i-1}\} \cup \{a_i, b_i\}) \cap A_i = \emptyset$, we have $|E(Q_i)| \geq |A_i| + 1$.

Now assume that H_i is not a cycle chain and $H_i \neq K_4$.

If $b_{i-1} \in \{a_i, b_i\}$, then assume $b_{i-1} = b_i$ (by choosing the notation of $\{a_i, b_i\}$), and let $H_i' := H_i + a_{i-1}a_i$. Clearly, H_i' is 3-connected with maximum degree at most $d + 1$, and the possible vertices of degree $d + 1$ are incident with $a_{i-1}b_{i-1}$ or a_ib_{i-1}. By Theorem 2.1 (a), there is a cycle D_i through $a_{i-1}a_i$ in $H_i' - b_{i-1}$ such that $|E(D_i)| \geq (\frac{|A_i|}{2t_i})^r + 2$, where t_i is the number of neighbors of b_{i-1} in H_i' distinct from a_{i-1} and a_i. So $t_i \leq d - 1$. Let $Q_i := D_i - a_{i-1}a_i$. Then $|E(Q_i)| \geq (\frac{|A_i|}{2(d-1)})^r + 1$.

Hence, assume $b_{i-1} \notin \{a_i, b_i\}$. Let H_i'' be obtained from H_i by a T-transform at $\{b_{i-1}, a_i b_i\}$, and let a denote the new vertex. By Lemma 3.3, H_i'' is a 3-connected graph. Let $H_i' := H_i'' + a_{i-1}a$. Then $5 \leq |H_i'| < n$. Also a_{i-1}, b_{i-1}, a have degrees at most $d+1$ in H_i', and all other vertices have degree at most d in H_i'. Thus by Theorem 2.1 (a), there is a cycle D_i through $a_{i-1}a$ in $H_i' - b_{i-1}$ such that $|D_i| \geq (\frac{|H_i'|}{2t_i})^r + 2$, where t_i is the number of neighbors of b_{i-1} in H_i' distinct from a and a_{i-1}. Thus $t_i \leq d-1$. Let $Q_i := D_i - a$. Then Q_i is a path in $H_i - b_{i-1}$ between a_{i-1} and $\{a_i, b_i\}$ and $|E(Q_i)| \geq (\frac{|H_i|}{2(d-1)})^r$.

(3) We find a path Q_h in H_h from $\{a_{h-1}, b_{h-1}\}$ to y' such that

(i) $\bigcup_{i=1}^h Q_i$ is a path from a_0 to y',

(ii) if H_h is 3-connected, then $|E(Q_h)| \geq (\frac{|A_h|}{2(d-1)})^r + 1$,

(iii) if $h = 1$ and H_h is a cycle chain, then $|A_h| \geq 2$ and $|E(Q_h)| \geq |A_h|$, and

(iv) if $h \neq 1$ and H_h is a cycle chain, then $|E(Q_h)| \geq |A_h| + 1$.

For ease of presentation, let us assume that $\bigcup_{i=1}^{h-1} Q_i$ is from a_0 to a_{h-1}.

If H_h is a cycle chain, then by Proposition 2.6, let Q_h denote a path from a_{h-1} to y' in $H_h - b_{h-1}$ such that $A_h \subseteq V(Q_h)$. If $h = 1$, then since $a_0, b_0 \in A_h$ and $y' \notin A_h$, we have (iii). If $h \neq 1$, then $a_{h-1}, b_{h-1}, y' \notin A_h$, and so we have (iv).

If $H_h = K_4$, then let Q_h be a Hamilton path from a_{h-1} to y' in $H_h - b_{h-1}$. Then $|E(Q_h)| = 2 \geq (\frac{|H_h|}{2(d-1)})^r + 1 = (\frac{|A_h|}{2(d-1)})^r + 1$, and (ii) holds.

Now assume that H_h is not a cycle chain and $H_h \neq K_4$. Let $H_h' := H_h + \{a_{h-1}y', b_{h-1}y'\}$. Then H_h' is 3-connected, the vertices a_{h-1}, b_{h-1}, y' have degree at most $d+1$ in H_h', and all other vertices have degree at most d in H_h'. By Theorem 2.1 (a), there is a cycle D_h through $a_{h-1}y'$ in $H_h' - b_{h-1}$ such that $|D_h| \geq (\frac{|H_h|}{2(d-1)})^r + 2$. Let $Q_h := D_h - a_{h-1}y'$. Then $|E(Q_h)| \geq (\frac{|A_h|}{2(d-1)})^r + 1$, and we have (ii).

(4) Let $C := (P \cup (\bigcup_{i=1}^h Q_i)) + \{y, xy, yy'\}$. Now C is a cycle in G through xy and, by (1)–(3), we have

$$|C| \geq (\sigma(\mathcal{L}) + 1)^r + 1 + \left(\sum |A_i|\right) + \left(\sum \left(\frac{|A_i|}{2(d-1)}\right)^r\right) + 2,$$

where the first summation is over all cycle chains H_i and the second summation is over all 3-connected H_i. Note that each $|A_i|$ in the first summation can be written as $1^r + \cdots + 1^r$ ($|A_i|$ times), and this allows us to apply Lemma 3.1 in the following inequalities. Hence

$$\begin{aligned} |C| &\geq \left(\sigma(\mathcal{L}) + 1 + (d-1)\sum_{i=1}^{h} |A_i|\right)^r + 3 \quad \text{(by Lemma 3.1)} \\ &= (\sigma(\mathcal{L}) + 1 + (d-1)\sigma(\mathcal{H}))^r + 3 \\ &\geq n^r + 3 \quad \text{(by Observation).} \end{aligned}$$

Case 2. $1 = t < \ell$. Recall that if $\mathcal{H} = \emptyset$, then $a_0 = b_0 = y'$, and $\{a_0, b_0\} = \{y'\}$.

(1) We find a path P_1 from $\{a_0, b_0\}$ to $\{c_1, d_1\}$ in $L_1 = L$ such that $|E(P_1)| \geq (|B_1| + 2)^r$. (Note that $|B_1| = |L_1|$ by definition.)

If $L_1 = K_4$, then we can find a Hamilton path P_1 from $\{a_0, b_0\}$ to $\{c_1, d_1\}$ in $L_1 - \{a_0 b_0, c_1 d_1\}$ (or in $L_1 - c_1 d_1$ when $\mathcal{H} = \emptyset$). Thus, $|E(P_1)| = 3 \geq |L_1|^r + 1 = |B_1|^r + 1 > (|B_1| + 2)^r$ (by Lemma 3.1).

Now assume that $L_1 \neq K_4$. If $\{a_0, b_0\} \subseteq \{c_1, d_1\}$, then by Theorem 2.1 (c) there is a cycle C_1 through c_1d_1 in L_1 such that $|C_1| \geq |L_1|^r + 3$. Let $P_1 := C_1 - c_1d_1$; then P_1 is a path between c_1 and $d_1 \in \{a_0, b_0\}$, and $|E(P_1)| \geq |L_1|^r + 2 = |B_1|^r + 2 > (|B_1| + 2)^r$ (by Lemma 3.1).

Assume that $\{a_0, b_0\} \not\subseteq \{c_1, d_1\}$. If $\mathcal{H} = \emptyset$, then let L_1' be obtained from L_1 by a T-transform at $\{y', c_1d_1\}$, and let c denote the new vertex. Of $\mathcal{H} \neq \emptyset$ then let L_1' be obtained from L_1 by an H-transform at $\{a_0b_0, c_1d_1\}$, and let a, c denote the new vertices with a adjacent to a_0 and b_0 and c adjacent to c_1 and d_1. By Lemma 3.3, L_1' is a 3-connected graph with maximum degree at most d. Since $5 \leq |L_1'| < n$, Theorem 2.1 holds for L_1'. By Theorem 2.1 (c), there is a cycle C_1 in L_1' through ac such that $|C_1| \geq |L_1'|^r + 3$. If $\mathcal{H} = \emptyset$, let $P_1 := C_1 - c$; then P_1 is a path from $\{a_0, b_0\}$ to $\{c_1, d_1\}$ in L_1 and $|E(P_1)| \geq |L_1'|^r + 1 = (|L_1| + 1)^r + 1 \geq (|L_1| + 2)^r = (|B_1| + 2)^r$ (by Lemma 3.1). If $\mathcal{H} \neq \emptyset$ let $P_1 := C_1 - \{a, c\}$; then P_1 is a path from $\{a_0, b_0\}$ to $\{c_1, d_1\}$ in L_1 and $|E(P_1)| \geq |L_1'|^r = (|L_1| + 2)^r = (|B_1| + 2)^r$.

Without loss of generality, assume that the notation of $\{a_0, b_0\}$ and $\{c_1, d_1\}$ is chosen so that P_1 is between c_1 and a_0.

(2) For $i = 2, \ldots, \ell - 1$, we find paths P_i from $\{c_{i-1}, d_{i-1}\}$ to $\{c_i, d_i\}$ in L_i such that

(i) $\bigcup_{i=2}^{\ell-1} P_i$ is a path from c_1 to $\{c_{\ell-1}, d_{\ell-1}\}$,

(ii) if L_i is 3-connected, then $|E(P_i)| \geq (\frac{|B_i|}{2(d-1)})^r$, and

(iii) if L_i is a cycle chain, then $|E(P_i)| \geq |B_i| + 1$.

Assume that, for some $2 \leq i \leq \ell - 1$, we have found paths P_j, $1 \leq j \leq i - 1$, from $\{c_{j-1}, d_{j-1}\}$ to $\{c_j, d_j\}$ in L_j such that $\bigcup_{j=2}^{i-1} P_j$ is a path from c_1 to $\{c_{i-1}, d_{i-1}\}$ and (ii) and (iii) are satisfied for L_j, B_j, P_j. Without loss of generality, assume that c_{i-1} is an end of $\bigcup_{j=2}^{i-1} P_j$ other than c_1.

If L_i is a cycle chain, then by Proposition 2.5 let P_i be a path from c_{i-1} to $\{c_i, d_i\}$ in $L_i - \{c_{i-1}d_{i-1}, c_id_i\}$ such that $B_i \subseteq V(P_i)$. Then, since $c_{i-1}, d_{i-1}, c_i, d_i \notin B_i$, we see that $|E(P_i)| \geq |B_i| + 1$, and we have (iii).

Now assume that L_i is not a cycle chain.

If $c_{i-1} \in \{c_i, d_i\}$, then, by choosing the notation of $\{c_i, d_i\}$, we may assume $c_i \notin \{c_{i-1}, d_{i-1}\}$. If $L_i = K_4$, then let P_i be a path from c_{i-1} to c_i in $L_i - d_{i-1}$ such that $|E(P_i)| = 2 \geq (\frac{|B_i|}{2(d-1)})^r$. If $L_i \neq K_4$, then let $L_i' := L_i + c_id_{i-1}$. By Theorem 2.1 (a), there is a cycle C_i through $c_{i-1}c_i$ in $L_i' - d_{i-1}$ such that $|C_i| \geq (\frac{|L_i'|}{2(d-1)})^r + 2 = (\frac{|B_i|}{2(d-1)})^r + 2$. Let $P_i := C_i - c_ic_{i-1}$; then P_i is a path from c_{i-1} to c_i in $L_i - d_{i-1}$, $|E(P_i)| \geq (\frac{|B_i|}{2(d-1)})^r + 1$, and we have (iii).

Assume that $c_{i-1} \notin \{c_i, d_i\}$. Let L_i'' be obtained from L_i by a T-transform at $\{c_{i-1}, c_id_i\}$, and let c denote the new vertex. Let $L_i' := L_i'' + d_{i-1}c$. By Lemma 3.3, L_i' is 3-connected. Note that c, c_i, d_{i-1} have degree at most $d+1$ in L_i', and all other vertices have degree at most d in L_i'. By Theorem 2.1 (a), there is a cycle C_i through $c_{i-1}c$ in $L_i' - d_{i-1}$ such that $|C_i| \geq (\frac{|L_i'|}{2(d-1)})^r + 2$. Let $P_i := C_i - c_{i-1}'$. Then P_i is a path from c_{i-1} to $\{c_i, d_i\}$ in $L_i - d_{i-1}$ and $|E(P_i)| \geq (\frac{|B_i|}{2(d-1)})^r$, and we have (iii).

(3) We find a path P_ℓ from $\{c_{\ell-1}, d_{\ell-1}\}$ to x in L_ℓ such that

(i) $\bigcup_{i=2}^{\ell} P_i$ is a path from c_1 to x,

(ii) if L_ℓ is 3-connected, then $|E(P_\ell)| \geq (\frac{|B_\ell|}{2(d-1)})^r + 1$, and

(iii) if L_ℓ is a cycle chain, then $|B_\ell| \geq 1$ and $|E(P_\ell)| \geq |B_\ell|$.

By choosing the notation of $\{c_{\ell-1}, d_{\ell-1}\}$, we may assume that $\bigcup_{i=2}^{\ell-1} P_i$ is between

c_1 and $c_{\ell-1}$.

If L_ℓ is a cycle chain, then by Proposition 2.6 let P_ℓ be a path from x to $c_{\ell-1}$ in L_ℓ such that $B_\ell \subseteq V(P_\ell)$. Since $c_{\ell-1}, d_{\ell-1} \notin B_\ell$, we have $|E(P_\ell)| \geq |B_\ell|$. Note that $|B_\ell| = 1$ only if L_ℓ is a cycle and $B_\ell = \{x\}$.

If $L_\ell = K_4$, then we can find a path P_ℓ from x to $c_{\ell-1}$ in $L_\ell - d_{\ell-1}$ with $|E(P_\ell)| = 2 \geq (\frac{|L_\ell|}{2(d-1)})^r + 1 = (\frac{|B_\ell|}{2(d-1)})^r + 1$.

Now assume that L_ℓ is not a cycle chain and that $L_\ell \neq K_4$. Let $L'_\ell := L_\ell + \{xc_{\ell-1}, xd_{\ell-1}\}$. Then L'_ℓ is a 3-connected graph, the vertices $x, c_{\ell-1}, d_{\ell-1}$ have degree at most $d+1$ in L'_ℓ, and all other vertices of L'_ℓ have degree at most d. So by Theorem 2.1 (a), there is a cycle C_ℓ through $xc_{\ell-1}$ in $L'_\ell - d_{\ell-1}$ such that $|C_\ell| \geq (\frac{|L_\ell|}{2(d-1)})^r + 2$. Now $P_\ell := C_\ell - xc_{\ell-1}$ gives the desired path for (ii).

(4) Let $P := \bigcup_{i=1}^{\ell} P_i$. Clearly P is a path in $\bigcup_{i=1}^{\ell} L_i$ from x to a_0. We claim that $|E(P)| \geq (\sigma(\mathcal{L}) + 1)^r + 1$.

By (2), we have

$$|E(P)| \geq |E(P_1)| + \left(\sum |B_i|\right) + \left(\sum \left(\frac{|B_i|}{2(d-1)}\right)^r\right) + |E(P_\ell)|,$$

where the first summation is over all cycle chains L_i and the second is over all 3-connected L_i. Note that each B_i in the first summation can be written as $1^r + \cdots + 1^r$ ($|B_i|$ times), and this allows the application of Lemma 3.1 in the following argument.

If L_ℓ is 3-connected, then by (1) and (3),

$$\begin{aligned}
|E(P)| &\geq (|B_1| + 2)^r + \left(\sum |B_i|\right) + \left(\sum \left(\frac{|B_i|}{2(d-1)}\right)^r\right) + \left(\left(\frac{|B_\ell|}{2(d-1)}\right)^r + 1\right) \\
&\geq \left(2 + \sum_{i=1}^{\ell} |B_i|\right)^r + 1 \quad \text{(by Lemma 3.1)} \\
&> (\sigma(\mathcal{L}))^r + 1.
\end{aligned}$$

If L_ℓ is a cycle chain, then by (1) and (3),

$$\begin{aligned}
|E(P)| &\geq (|B_1| + 2)^r + \left(\sum |B_i|\right) + \left(\sum \left(\frac{|B_i|}{2(d-1)}\right)^r\right) + (|B_\ell| - 1) + 1 \\
&\geq \left(1 + \sum_{i=1}^{\ell} |B_i|\right)^r + 1 \quad \text{(by Lemma 3.1)} \\
&= (\sigma(\mathcal{L}) + 1)^r + 1.
\end{aligned}$$

(5) For $i = 1, \ldots, h-1$, we find paths Q_i from $\{a_{i-1}, b_{i-1}\}$ to $\{a_i, b_i\}$ in H_i, as in (2) of Case 1.

(6) We find a path Q_h in H_h as in (3) of Case 1.

(7) Let $C := (P \cup (\bigcup_{i=1}^{h} Q_i)) + \{y, xy, yy'\}$. Now C is a cycle in G through xy and

$$|C| \geq ((\sigma(\mathcal{L}) + 1)^r + 1) + \left(\sum |A_i|\right) + \left(\sum \left(\frac{|A_i|}{2(d-1)}\right)^r\right) + 2,$$

where the first summation is over all cycle chains H_i and the second is over all 3-connected H_i. Again, when we apply Lemma 3.1 in the following argument, each $|A_i|$

in the first summation may be written as $1^r + \cdots + 1^r$. Since $\sigma(\mathcal{L}) \geq |L_1| \geq |A_i|$ for all 3-connected H_i, we have

$$\begin{aligned}|C| &\geq \left(\sigma(\mathcal{L}) + 1 + (d-1)\sum_{i=1}^{h} |A_i|\right) r + 3 \quad \text{(by Lemma 3.1)}\\ &= (\sigma(\mathcal{L}) + 1 + (d-1)\sigma(\mathcal{H}))^r + 3\\ &\geq n^r + 3 \quad \text{(by Observation).}\end{aligned}$$

Case 3. $1 < t \leq \ell$. Note that in this case there exist $y'' \in V(L_1) - \{c_1, d_1\}$ and a path Y in G from y to y'' disjoint from $(V(\mathcal{L}) - \{y''\}) \cup V(\mathcal{H})$. For convenience, let $S := (\bigcup_{i=2}^{t} B_i) - (B_1 \cup B_{t+1})$. We consider two subcases by comparing $\sigma(\mathcal{H})$ and $|S|$.

Subcase 3.1. $|S| < \sigma(\mathcal{H})$. Then $\mathcal{H} \neq \emptyset$ and $a_0 \neq b_0$. Since $\sum_{i=1}^{h} |A_i| = \sigma(\mathcal{H})$ and there are at most $d-2$ extreme 3-blocks of $G - y$ containing neither x nor y'', we have the following inequality:

$$\sigma(\mathcal{L}) - |S| + (d-1)\sum_{i=1}^{h} |A_i| \geq n - 1.$$

(1) First, we find a path P_1 from c_1 to d_1 in L_1 such that $|E(P_1)| \geq (|B_1|+1)^r + 1$.

If $L_1 = K_4$, then let P_1 be a Hamilton path from c_1 to d_1 in L_1. Hence $|E(P_1)| = 3 \geq (|L_1| + 1)^r + 1 = (|B_1| + 1)^r + 1$. If $L_1 \neq K_4$ then $5 \leq |L_1| < n$. By Theorem 2.1 (c), there is a cycle C_1 through c_1d_1 in L_1 such that $|C_1| \geq |L_1|^r + 3 = |B_1|^r + 3 > (|B_1| + 1)^r + 1$. Then $P_1 := C_1 - c_1d_1$ gives the desired path.

(2) Next, we find $Q \subseteq (\bigcup_{i=2}^{t} L_i) - \{a_0b_0, c_td_t\}$ such that (1) $P_1 \cup Q$ is a path from $\{a_0, b_0\}$ to $\{c_t, d_t\}$ when $t \neq \ell$, and (2) $P_1 \cup Q$ is a path from $\{a_0, b_0\}$ to x when $t = \ell$.

Let $K := \bigcup_{i=2}^{t} L_i$. First, assume that $t \neq \ell$. If $\{a_0, b_0\} = \{c_t, d_t\}$, then by Lemma 3.5 we find a cycle D through a_0b_0 and c_1d_1 in K, and $Q := D - \{a_0b_0, c_1d_1\}$ is as desired. If $\{a_0, b_0\} \neq \{c_t, d_t\}$, then let K' be obtained from K by an H-transform at $\{a_0b_0, c_td_t\}$, and let a, c denote the new vertices. By Lemma 3.5, we find a cycle D' through ac and c_1d_1, and $Q := D' - \{a, c, c_1d_1\}$ is as desired.

Now assume that $t = \ell$. If $x \in \{a_0, b_0\}$, then by Lemma 3.5 there is a cycle D in K through a_0b_0 and c_1d_1, and $Q := D - \{a_0b_0, c_1d_1\}$ is as desired. So assume that $x \notin \{a_0, b_0\}$. Let K' be obtained from K by a T-transform at $\{x, a_0b_0\}$, and let a denote the new vertex. By Lemma 3.5, there is a cycle D in K' through xa and c_1d_1. Now $Q := D - \{a, c_1d_1\}$ is as desired.

We choose the notation of $\{a_0, b_0\}$ and $\{c_t, d_t\}$ so that $P_1 \cup Q$ is from a_0 to c_t.

(3) For each $t+1 \leq i \leq \ell - 1$, we find a path P_i in L_i from $\{c_{i-1}, d_{i-1}\}$ to $\{c_i, d_i\}$ exactly as in (2) of Case 2 such that (i) $(\bigcup_{i=t+1}^{\ell-1} P_i) \cup P_1 \cup Q$ is a path from $\{c_{\ell-1}, d_{\ell-1}\}$ to $\{a_0, b_0\}$, (ii) $|E(P_i)| \geq (\frac{|B_i|}{2(d-1)})^r$ when L_i is 3-connected, and (iii) $|E(P_i)| \geq |B_i| + 1$ when L_i is a cycle chain.

(4) If $t \neq \ell$, we find a path P_ℓ between $\{c_{\ell-1}, d_{\ell-1}\}$ and x exactly as in (3) of Case 2 such that (i) $(\bigcup_{i=t+1}^{\ell} P_i) \cup P_1 \cup Q$ is a path from $\{a_0, b_0\}$ to x, (iii) $|E(P_\ell)| \geq (\frac{|B_\ell|}{2(d-1)})^r + 1$ when L_ℓ is 3-connected, and (iii) $|E(P_\ell)| \geq |B_\ell|$ when L_ℓ is a cycle chain.

(5) For $i = 1, \ldots, h-1$, we find paths Q_i from $\{a_{i-1}, b_{i-1}\}$ to $\{a_i, b_i\}$ in $H_i - b_{i-1}$, as in (2) of Case 1, such that (i) $\bigcup_{i=1}^{h-1} Q_i$ is a path from a_0 to $\{a_{h-1}, b_{h-1}\}$, (ii) $|E(Q_i)| \geq (\frac{|A_i|}{2(d-1)})^r$ when H_i is 3-connected, (iii) $|E(Q_1)| \geq |A_1| \geq 2$ when H_1 is a cycle chain, and (iv) $|E(Q_i)| \geq |A_i| + 1$ when $1 < i < h$ and H_i is a cycle chain.

(6) We find a path Q_h exactly as in (3) of Case 1 such that (i) $\bigcup_{i=1}^{h} Q_i$ is a path from a_0 to y', (ii) $|E(Q_h)| \geq (\frac{|A_h|}{2(d-1)})^r + 1$ when H_h is 3-connected, (iii) $|E(Q_h)| \geq |A_h| \geq 2$ when $h = 1$ and H_h is a cycle chain, and (iv) $|E(Q_h)| \geq |A_h| + 1$ when $h > 1$ and H_h is a cycle chain.

(7) Let $C := (P_1 \cup Q \cup (\bigcup_{i=t+1}^{\ell} P_i) \cup (\bigcup_{i=1}^{h} Q_i)) + \{y, xy, yy'\}$. Then C is a cycle in G through xy and, by (1)–(6), we have

$$|C| \geq (|L|+1)^r + \Big(\sum |B_i|\Big) + \Big(\sum \Big(\frac{|B_i|}{2(d-1)}\Big)^r\Big) + \Big(\sum |A_i|\Big) + \Big(\sum \Big(\frac{|A_i|}{2(d-1)}\Big)^r\Big) + 3,$$

where the first sum is taken over all cycle chains L_i for $t+1 \leq i \leq \ell$, the second is over all 3-connected L_i for $t+1 \leq i \leq \ell$, the third is over all cycle chains H_i, and the fourth is over all 3-connected H_i. Because $\sigma(\mathcal{L}) \geq |A_i|$ for all 3-connected H_i, and $\sigma(\mathcal{L}) \geq |B_j|$ for all 3-connected L_j, we have

$$\begin{aligned} |C| &\geq \left(\sigma(\mathcal{L}) + 1 - |S| + (d-1)\sum_{i=1}^{h} |A_i|\right)^r + 3 \quad \text{(by Lemma 3.1)} \\ &\geq n^r + 3. \end{aligned}$$

The second inequality follows from the inequality in the first paragraph of this subcase.

Subcase 3.2. $|S| \geq \sigma(\mathcal{H})$. As in the previous subcase, we deduce the following inequality:

$$|B_1| + (d-1)\sum_{i=2}^{\ell} |B_i| \geq n - 1.$$

(1) First, we find a path P_1 from y'' to $\{c_1, d_1\}$ in $L_1 - c_1d_1$ such that $|E(P_1)| \geq (|B_1| + 1)^r + 1$.

Let L_1' denote the graph obtained from L_1 by a T-transform at $\{y'', c_1d_1\}$, and let y^* denote the new vertex. By Lemma 3.3 and since y'' has degree at most $d-1$ in L_1, L_1' is a 3-connected graph with maximum degree at most d. Since $5 \leq |L_1'| < n$, Theorem 2.1 holds for L_1'. By Theorem 2.1 (c), L_1' has a cycle C_1 through y^*y'' such that $|C_1| \geq |L_1'|^r + 3 = (|B_1| + 1)^r + 3$. Then $P_1 := C_1 - y^*$ gives the desired path for (1).

We may choose the notation of $\{c_1, d_1\}$ so that P_1 is between y'' and c_1.

(2) For each $2 \leq i \leq \ell - 1$, we find a path P_i in L_i as in (2) of Case 2 such that (i) $\bigcup_{i=2}^{\ell-1} P_i$ is a path from c_1 to $\{a_{\ell-1}, b_{\ell-1}\}$, (ii) $|E(P_i)| \geq (\frac{|B_i|}{2(d-1)})^r$ when L_i is 3-connected, and (iii) $|E(P_i)| \geq |B_i| + 1$ when L_i is a cycle chain.

(3) We find a path P_ℓ as in (3) of Case 2 such that (i) $\bigcup_{i=2}^{\ell} P_i$ is a path from c_1 to x, (ii) $|E(P_\ell)| \geq (\frac{|B_\ell|}{2(d-1)})^r + 1$ when L_ℓ is 3-connected, and (iii) $|E(P_\ell)| \geq |B_\ell|$ when L_ℓ is a cycle chain.

(4) Let $C := (Y \cup (\bigcup_{i=1}^{\ell} P_i)) + xy$. Then C is a cycle in G through xy and

$$|C| \geq (|B_1|^r + 1) + \Big(\sum |B_i|\Big) + \left(\sum \Big(\frac{|B_i|}{2(d-1)}\Big)^r\right) + 2,$$

where the first sum is over all L_i which are cycle chains and the second is over all 3-connected L_i. Again, we may view $|B_i|$ in the first summation as $1^r + \cdots 1^r$ ($|B_i|$

times). Since $|B_1| \geq |B_i|$ for all 3-connected L_i, we have

$$\begin{aligned} |C| &\geq \left(|B_1| + 1 + (d-1)\sum_{i=2}^{\ell} |B_i|\right)^r + 3 \quad \text{(by Lemma 3.1)} \\ &\geq n^r + 3. \end{aligned}$$

The second inequality follows from the inequality in the first paragraph of this case. □

Next we show that the above proof gives rise to an $O(|G|)$ algorithm which reduces Theorem 2.1 (c) to Theorem 2.1 (a), (b), and (c) for smaller graphs.

Algorithm Oneedge. Let n, d, r, G, e, be as in Lemma 6.1.

1. *Preprocessing*. Replace G by a 3-connected spanning subgraph of G with $O(|G|)$ edges. (This can be done in $O(|E(G)|)$ time using Lemma 3.4.)
2. Let $e = xy$. Decompose $G - y$ into 3-connected components. (This can be done in $O(|G|)$ time using Theorem 2.2.)
3. If there is only one 3-connected component of $G-y$, then $G-y$ is 3-connected. Let y' denote a neighbor of y other than x, and let $G' := (G - y) + xy'$ and $e' = xy'$. It suffices to to find a cycle C' through e' in G' such that $|C'| \geq |G'|^r + 3$. So reduce Theorem 2.1 (c) for G, e to Theorem 2.1 (c) for G', e', with $|G'| < |G|$. (Note that this reduction takes constant time.)
4. Now assume that $G - y$ has at least two 3-connected components. If all 3-blocks of $G - y$ are cycles, find a cycle chain $\mathcal{I} = I_1 \ldots I_i$ such that (i) $x \in V(I_1) - V(I_2)$, (ii) I_i is an extreme 3-block of $G - y$, and (iii) subject to (i) and (ii), $|V(\mathcal{I})|$ is maximum. (This can be done in $O(|G|)$ time by a simple search.) Then, find a neighbor $y' \in V(\mathcal{I}) - \{x\}$ of y, a Hamilton path P in $\mathcal{I}$ from x to y', and a path Q from y to y' disjoint from $V(\mathcal{I}) - \{y'\}$, so that $(P \cup Q) + \{y, xy, yy'\}$ gives the desired cycle. (These paths can be computed in $O(|G|)$ time as in the proof of Lemma 6.1.)
5. Now assume that $G - y$ has at least two 3-blocks, and at least one is 3-connected. We choose a 3-connected 3-block L of $G - y$ such that $|L|$ is maximum. Find the block chain $\mathcal{L} = L_1 \ldots L_\ell$ such that $L_1 = L$ and $x \in V(L_\ell) - V(L_{\ell-1})$. Find a block chain $\mathcal{H} = H_1 \ldots H_h$ in $G - y$ with $y' \in H_h$, and define a_i, b_i, A_i, c_j, d_j, B_j for $i = 1, \ldots h$ and $j = 1, \ldots, \ell$ as in the proof of Lemma 6.1. (All these can be done in $O(|G|)$ time by searching the 3-blocks of $G - y$.)
6. Suppose $\ell = 1$.
   - First, we need to find a path P in $\mathcal{L}$ from x to $\{a_0, b_0\}$ as in (1) of Case 1. We either find the desired P or reduce the problem of finding P to Theorem 2.1 (c) for L_1, a_0b_0 or L_1', xx', both are smaller graphs. (From (1) of Case 1 in the proof of 6.1, this can be done in constant time.)
   - For each $1 \leq i \leq h-1$, we want to find a path Q_i from $\{a_{i-1}, b_{i-1}\}$ to $\{a_i, b_i\}$ in H_i as in (2) of Case 1 in the proof of Lemma 6.1. We either find the desired Q_i or reduce the problem of finding Q_i to Theorem 2.1 (a) for $H_i', a_{i-1}a_i, b_{i-1}$ or $H_i', a_{i-1}a, b_{i-1}$. (From (2) of Case 1 in the proof of Lemma 6.1, this can be done in $O(|H_i|)$ time.)
   - We need to find a path Q_h in H_h from a_{h-1} to y' as in (3) of Case 1. We either find the desired Q_h or reduce the problem of finding Q_h to Theorem 2.1 (a) for $H_h', a_{h-1}y', b_{h-1}$. (From (3) of Case 1 in the proof of Lemma 6.1, this can be done in $O(|H_h|)$ time.)

- Since $\sum_{i=1}^{h}(|H_i|-2) = |V(\mathcal{H})|-2$, we see that this step takes $O(|G|)$ time.

7. Suppose $1 = t < \ell$.
   - First, we need to find a path P_1 from $\{a_0, b_0\}$ to $\{c_1, d_1\}$ in L_1, as in (1) of Case 2 in the proof of Lemma 6.1. We either find the desired P_1 or reduce the problem of finding P_1 to Theorem 2.1 (c) for either L_1, c_1d_1 or L_1', ac. (From (1) of Case 2 in the proof of Lemma 6.1, this can be done in constant time.)
     Assume that the notation is chosen so that P_1 is a path from a_0 to c_1.
   - For each $2 \le i \le \ell - 1$, we need to find a path P_i from $\{c_{i-1}, d_{i-1}\}$ to $\{c_i, d_i\}$ in L_i as in (2) of Case 2 in the proof of Lemma 6.1. We either find the desired P_i or reduce the problem of finding P_i to Theorem 2.1 (a) for either $L_i', c_{i-1}c_i, d_{i-1}$ or $L_i', c_{i-1}c, d_{i-1}$. (From (2) of Case 2 in the proof of Lemma 6.1, this can be done in $O(|L_i|)$ time.)
   - We need to find a path P_ℓ from $\{c_{\ell-1}, d_{\ell-1}\}$ to x in L_ℓ, as in (3) of Case 2 in the proof of Lemma 6.1. We either find the desired path P_ℓ or reduce the problem of finding P_ℓ to Theorem 2.1 (a) for $L_\ell', xc_{\ell-1}, d_{\ell-1}$. (From (3) of Case 2 in the proof of Lemma 6.1, this can be done in $O(|L_\ell|)$ time.)
   - Next we need to find paths Q_i, $1 \le i \le h$. This is taken care of exactly as in step 6 above.
     (Since $\sum_{i=1}^{h}(|H_i|-2) = |V(\mathcal{H})|-2$ and $\sum_{i=1}^{\ell}(|L_i|-2) = |V(\mathcal{L})|-2$, we see that all operations in this step can be done in $O(|G|)$ time.)
8. Suppose $1 < t \le \ell$. First, find a path Y from y to $y'' \in V(L_1) - V(L_2)$ such that $Y - y''$ is disjoint from $V(\mathcal{L}) \cup V(\mathcal{H})$. Define $S := (\bigcup_{i=2}^{t} B_i) - (B_1 \cup B_{t+1})$. (Note that Y and all B_i's can be found in $O(|G|)$ time, as in Case 3 in the proof of Lemma 6.1.)
9. Suppose $|S| < \sigma(\mathcal{H})$.
   - We need to find a path P_1 from c_1 to d_1 in L_1, as in (1) of Subcase 3.1 in the proof of Lemma 6.1. We either find the desired P_1 or reduce the problem of finding P_1 to Theorem 2.1 (c) for L_1, c_1d_1. (From (1) of Subcase 3.1 in the proof of Lemma 6.1, this can be done in $O(|L_1|)$ time.)
   - Find $Q \subseteq (\bigcup_{i=2}^{t} L_i) - \{a_0b_0, c_td_t\}$ as in (2) of Subcase 3.1 in the proof of Lemma 6.1. (From (2) of Subcase 3.1 in the proof of Lemma 6.1, this can be done in $O(|G|)$ time.)
   - For each $t+1 \le i \le \ell - 1$, we need to find a path P_i from $\{c_{i-1}, d_{i-1}\}$ to $\{c_i, d_i\}$, as in (3) of Subcase 3.1 in the proof of Lemma 6.1. (This can be done as in step 7 above, and hence in $O(|G|)$ time.)
   - Next, we need to find a path P_ℓ from $\{c_{\ell-1}, d_{\ell-1}\}$ to x in L_ℓ as in (4) of Subcase 3.1 in the proof of Lemma 6.1. (This can be done as in step 7 above, and hence in $O(|L_\ell|)$ time.)
   - For each $1 \le i \le h-1$, we want to find a path Q_i as in (5) of Subcase 3.1 in the proof of Lemma 6.1. (This can be done as in step 6 above, and hence in $O(|H_i|)$ time.)
   - Finally, we find a Q_h in H_h from $\{a_{h-1}, b_{h-1}\}$ to y' as in (6) of Subcase 3.1 in the proof of Lemma 6.1. (This can be done as in step 6 above, and hence in $O(|H_h|)$ time.)
     (Since $\sum_{i=1}^{h}(|H_i|-2) = |V(\mathcal{H})|-2$ and $\sum_{i=1}^{\ell}(|L_i|-2) = |V(\mathcal{L})|-2$,

we see that all operations in this step can be done in $O(|G|)$ time.)

10. Suppose $|S| \geq \sigma(\mathcal{H})$.
    - First, we need to find a path P_1 from y'' to $\{c_1, d_1\}$ in L_1, as in (1) of Subcase 3.2 in the proof of Lemma 6.1. We either find the desired P_1 or reduce the problem of finding P_1 to Theorem 2.1 (c) for L_1', y^*y''. (From (1) of Subcase 3.2 in the proof of Lemma 6.1, this can be done in $O(|L_1|)$ time.)
    - For each $2 \leq i \leq \ell - 1$, we need to find a P_i as in (2) of Subcase 3.2 in the proof of Lemma 6.1. (This can be done as in step 7 above for each i, and hence, in $O(|L_i|)$ time.)
    - Next, we want to find a path P_ℓ from $\{c_{\ell-1}, d_{\ell-1}\}$ to x in L_ℓ as in (3) of Subcase 3.2 in the proof Lemma 6.1. (This can be done as in step 7 above, and hence, in $O(|L_\ell|)$ time.)

    (All operations in this step can be done in $O(|G|)$ time since $\sum_{i=1}^{\ell}(|L_i| - 2) = |V(\mathcal{L})| - 2$.)

We summarize the above procedure as follows.

PROPOSITION 6.2. *Given G, e, n, d, r as in Lemma 6.1, we can, in $O(|E(G)|)$ time, either*

(1) *find a cycle C through e in G such that $|C| \geq |G|^r + 3$, or*

(2) *reduce Theorem 2.1* (c) *of for G, e to Theorem 2.1* (a) *or* (b) *or* (c) *for smaller 3-connected graphs.*

Moreover, any smaller graph in (2) *results from a 3-connected 3-block of $G - y$ which is not K_4. Hence, any smaller graph in* (2) *contains a vertex that does not belong to any other smaller graph in* (2).

7. Conclusions. We now complete the proof of Theorem 2.1. Let n, d, r, G be given as in the theorem. We will prove te conclusions by applying induction on n. When $n = 5$, then G is isomorphic to one the following three graphs: K_5, K_5 minus an edge, or the wheel on five vertices. In each case, we can verify that Theorem 2.1 holds. So assume that $n \geq 6$ and that Theorem 2.1 holds for all 3-connected graphs with at most $n - 1$ vertices. Then (a) holds by Lemma 4.1, (b) holds by Lemma 5.1, and (c) holds by Lemma 6.1. This completes the proof of Theorem 2.1. □

ALGORITHM CYCLE. Let G be a 3-connected graph with maximum degree at most d, let $e = xy \in E(G)$, and assume $|G| \geq 5$. The following procedure finds a cycle C through e in G with $|C| \geq |G|^r + 3$.

1. *Preprocessing* Replace G with a 3-connected spanning subgraph of G with $O(|G|)$ edges.
2. Apply Algorithm Oneedge to G, e. We either find the desired cycle C or we reduce the problem to Theorem 2.1 (a), (b), or (c) of 2.1 for some 3-connected graphs G_i, for which $|G_i| < |G|$ and each G_i contains a vertex which does not belong to any other G_i.
3. Replace each G_i with a 3-connected spanning subgraph of G_i with $O(|G_i|)$ edges.
4. Apply Algorithm Avoidvertex to those G_i for which Theorem 2.1 (a) needs to be applied. Apply Algorithm Twoedge to those G_i for which (Theorem 2.1 (b) needs to be applied. Apply Algorithm Oneedge to those G_i for which Theorem 2.1 (c) needs to be applied.
5. Repeat steps 3 and 4 for new 3-connected graphs.
6. In the final output, replace all virtual edges by paths in G to complete the desired cycle C.

Note that step 1 takes $O(|E(G)|)$ time by Lemma 3.4 and step 2 takes $O(|E(G)|)$ time by Proposition 4.2.

By Lemma 3.4, step 3 spends $O(|E(G_i)|)$ time for each G_i from step 2. Note that each G_i in step 2 contains a vertex which does not belong to any other G_i. By Theorem 2.2 and since each G_i contributes at most three additional edges due to T-transform or H-transform, the total number of edges in step 2 is at most $3|E(G)| - 6 + 3|V(G)|$. Hence step 3 takes $O(|E(G)|)$ time.

From Propositions 4.2, 5.2, and 6.2, we see that step 4 spends $O(|E(G_i)|)$ time for each G_i from step 2. By Theorem 2.2 and since each G_i contributes at most three additional edges due to T-transform or H-transform, the total number of edges in step 4 is at most $\sum_i (3|E(G_i)| - 6 + 3|V(G_i)|)$. Since each G_i in step 2 contains a vertex which does not belong to any other G_i, Step 4 takes $O(|G|^2)$ time.

Since there are at most $|G|$ iterations, we see that Algorithm Cycle takes $O(|G|^3)$ time.

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