

Two new families of q -positive integers

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Abstract. Let n, p and k be three non negative integers. We prove that the apparently rational fractions of q :

$$\begin{bmatrix} n \\ k \end{bmatrix}_q {}_3\phi_2 \left[\begin{matrix} q^{1-k}, q^{-p}, q^{p-n} \\ q, q^{1-n} \end{matrix} \middle| q; q^{k+1} \right] \quad \text{and} \quad q^{(n-p)p} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_3\phi_2 \left[\begin{matrix} q^{1-k}, q^{-p}, q^{p-n} \\ q, q^{1-n} \end{matrix} \middle| q; q \right]$$

are actually polynomials of q with positive integer coefficients. This generalizes a recent result of Lassalle (Ann. Comb. 6(2002), no. 3-4, 399-405), in the same way as the classical q -binomial coefficients refine the ordinary binomial coefficients.

Keywords: q -binomial coefficients, q -integers, Basic hypergeometric functions,

MR Subject Classifications: Primary 05A30; Secondary 33D15;

1 Introduction

In [3] Lassalle introduced a new family of positive integers generalizing the classical binomial coefficients. We refer the readers to [4] for some motivations from the study of Jack polynomials and to [2] for some further results and extensions of these coefficients. In this paper we will present a different generalization of Lassalle's coefficients. More precisely, using basic hypergeometric functions we will show that all the results of Lassalle [3] have *natural* q -analogues.

In order to present its q -analogues we need first to introduce some notations. Throughout this paper q is a complex variable such that $|q| < 1$. For any integer $k \geq 0$, the q -raising factorial is defined by

$$(a; q)_k = (1 - a)(1 - aq) \dots (1 - aq^{k-1}) \quad \text{for } k \geq 1,$$

and $(a; q)_0 = 1$. We shall also use the concise notation $(a_1, \dots, a_s; q)_k = (a_1; q)_k \dots (a_s; q)_k$ for $s \geq 1$. For any integer n (not necessary positive) the q -binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^{n-k+1}; q)_k}{(q; q)_k}.$$

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So $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k > n$. Furthermore we will use the standard notation for basic hypergeometric series [1]:

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} \middle| q; z \right] = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k} (-1)^{(1+s-r)k} q^{(1+s-r)\binom{k}{2}} \frac{z^k}{(q; q)_k}.$$

The q -binomial coefficients have two less obvious formulas as follows:

$$\begin{bmatrix} n \\ p \end{bmatrix}_q = {}_2\phi_1 \left[\begin{matrix} q^{-p}, q^{p-n} \\ q \end{matrix} \middle| q; q^{n+1} \right] = q^{p(p-n)} {}_2\phi_1 \left[\begin{matrix} q^{-p}, q^{p-n} \\ q \end{matrix} \middle| q; q \right]. \quad (1)$$

Actually these expressions are obtained by specializing $a = q^{p-n}$, $c = q$ in the celebrated q -Chu-Vandermonde formula [1, p. 236]:

$$\frac{(c/a; q)_p}{(c; q)_p} = {}_2\phi_1 \left[\begin{matrix} q^{-p}, a \\ c \end{matrix} \middle| q; cq^p/a \right] = a^{-p} {}_2\phi_1 \left[\begin{matrix} q^{-p}, a \\ c \end{matrix} \middle| q; q \right].$$

Of course using this relation as a definition of q -binomial coefficients would be rather tautological. However, quite surprisingly, it is possible to define two new families of q -positive integers, i.e., a polynomial of q with *non negative integer coefficients*, by slightly modifying the q -Chu-Vandermonde formula. In fact there are two such q -analogues of Lassalle's generalized binomial coefficients.

In the next two sections we present the first q -analogue and its *raison d'être* in the context of linearization problem of q -binomial coefficients. In Section 4 we outline the second q -analogue. At the end this paper we give the first values of our generalized q -binomial coefficients, which seems to suggest some *unimodal* properties of the coefficients of these polynomials.

2 The first q -analogue

For any non negative integers n, p and k , define the coefficient

$$\binom{n}{p, k}_q = q^{(n-p)p} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_3\phi_2 \left[\begin{matrix} q^{1-k}, q^{-p}, q^{p-n} \\ q, q^{1-n} \end{matrix} \middle| q; q \right]. \quad (2)$$

Applying Sears' transformation [1, p. 61, (3.2.5)]

$${}_3\phi_2 \left[\begin{matrix} q^{-p}, a, b \\ c, d \end{matrix} \middle| q; q \right] = \frac{(c/a; q)_p}{(c; q)_p} a^p {}_3\phi_2 \left[\begin{matrix} q^{-p}, a, d/b \\ d, q^{1-p}a/c \end{matrix} \middle| q; bq/c \right] \quad (3)$$

with $a = q^{p-n}$, $b = q^{1-k}$, $c = q$ and $d = q^{1-n}$, we get

$$\binom{n}{p, k}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ p \end{bmatrix}_q {}_3\phi_2 \left[\begin{matrix} q^{-p}, q^{p-n}, q^{k-n} \\ q^{1-n}, q^{-n} \end{matrix} \middle| q; q^{1-k} \right]. \quad (4)$$

It follows that

$$\binom{n}{p, k}_q = 0 \quad \text{for } k > n \quad \text{or } p > n,$$

and for $p, k \leq n$,

$$\binom{n}{p, k}_q = \binom{n}{n-p, k}_q, \quad \binom{n}{p, n}_q = \begin{bmatrix} n \\ p \end{bmatrix}_q,$$

the last equations following directly from the q -Chu-Vandermonde formula.

Set $[n]_q = 1 + q + \dots + q^{n-1} = (1 - q^n)/(1 - q)$ for $n \geq 0$. We can rewrite (2) as follows:

$$\binom{n}{p, k}_q = \frac{[n]_q}{[k]_q} \sum_{r \geq 0} \begin{bmatrix} p \\ r \end{bmatrix}_q \begin{bmatrix} n-p \\ r \end{bmatrix}_q \begin{bmatrix} n-r-1 \\ k-r-1 \end{bmatrix}_q q^{(n-p)p+r(r-k)}. \quad (5)$$

Therefore

$$\binom{n}{0, k}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q, \quad \binom{n}{1, k}_q = [k]_q q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

and

$$\binom{n}{2, k}_q = q^{2n-3-k} [k]_q \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{2(n-k)} \frac{[n]_q [n-3]_q}{[2]_q} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q.$$

For $p > 0$, we have also

$$\begin{aligned} \binom{n}{p, 0}_q &= 0, & \binom{n}{p, 1}_q &= q^{(n-p)p} [n]_q, \\ \binom{n}{p, 2}_q &= q^{(n-p)p} \frac{[n]_q}{[2]_q} ([n-1]_q + [p]_q [n-p]_q). \end{aligned}$$

These results suggest that $\binom{n}{p, k}_q$ are polynomials of q with *non negative integer coefficients*. Indeed, it is not hard to see that they are polynomials with integer coefficients.

We can rewrite (4) as follows

$$\binom{n}{p, k}_q = \sum_{i \geq 0} (-1)^i \begin{bmatrix} n-i \\ k \end{bmatrix}_q \frac{[n]_q}{[n-i]_q} \begin{bmatrix} n-i \\ i \end{bmatrix}_q \begin{bmatrix} n-2i \\ p-i \end{bmatrix}_q q^{i(i-1)/2}. \quad (6)$$

Since

$$\frac{[n]_q}{[n-i]_q} \begin{bmatrix} n-i \\ i \end{bmatrix}_q = \frac{1 - q^{n-i} + q^{n-i} - q^n}{1 - q^{n-i}} \begin{bmatrix} n-i \\ i \end{bmatrix}_q = \begin{bmatrix} n-i \\ i \end{bmatrix}_q + q^{n-i} \begin{bmatrix} n-i-1 \\ i-1 \end{bmatrix}_q,$$

we see that $\binom{n}{p, k}_q$ is a q -integer, i.e., a polynomial of q with integer coefficients.

Two generating functions for $\binom{n}{p, k}_1$ were published in [3]. We consider their q -analogues below.

Theorem 1. *There holds*

$$\begin{aligned} & \sum_{0 \leq k, p \leq n} \binom{n}{p, k}_q x^p q^{\binom{p}{2}} y^k q^{\binom{k}{2}} \\ &= (-y; q)_n (-x; q)_n \sum_{i \geq 0} \frac{1 - q^n}{1 - q^{n-i}} \begin{bmatrix} n-i \\ i \end{bmatrix}_q \frac{(-x)^i q^{i^2-i}}{(-yq^{n-i}; q)_i (-x; q)_i (-xq^{n-i}; q)_i}. \end{aligned} \quad (7)$$

Proof. Using (6) one has

$$\begin{aligned}
& \sum_{0 \leq k, p \leq n} \binom{n}{p, k}_q x^p q^{\binom{p}{2}} y^k q^{\binom{k}{2}} \\
&= \sum_{0 \leq k, p \leq n} \sum_{i \geq 0} (-1)^i \frac{1 - q^n}{1 - q^{n-i}} \begin{bmatrix} n-i \\ k \end{bmatrix}_q \begin{bmatrix} n-i \\ i \end{bmatrix}_q \begin{bmatrix} n-2i \\ p-i \end{bmatrix}_q q^{\binom{i}{2}} y^k q^{\binom{k}{2}} x^p q^{\binom{p}{2}} \\
&= \sum_{i \geq 0} (-1)^i \frac{1 - q^n}{1 - q^{n-i}} x^i \sum_{k \geq 0} \begin{bmatrix} n-i \\ k \end{bmatrix}_q y^k q^{\binom{k}{2}} \sum_{p \geq 0} \begin{bmatrix} n-2i \\ p-i \end{bmatrix}_q (xq^i)^{p-i} q^{\binom{p-i}{2}} q^{i^2-i} \\
&= \sum_{i \geq 0} (-1)^i \frac{1 - q^n}{1 - q^{n-i}} x^i (-y; q)_{n-i} (-xq^i; q)_{n-2i} q^{i^2-i},
\end{aligned}$$

which yields (7). \square

Remark. When $q = 1$, since

$$\sum_{k < n} \binom{n-k}{k} \frac{n}{n-k} z^k = \left(\frac{1 + \sqrt{1+4z}}{2} \right)^n + \left(\frac{1 - \sqrt{1+4z}}{2} \right)^n,$$

setting $z = \frac{-x}{(1+x)^2(1+y)}$, the above generating functions can be written as follows:

$$\sum_{0 \leq k, p \leq n} \binom{n}{p, k}_1 x^p y^k = 2^{-n} [(1+y)(1+x)]^n \left((1 + \sqrt{1+4z})^n + (1 - \sqrt{1+4z})^n \right).$$

The following is the q -analogue of Lassalle's recurrence relation in [3, Lemma 3.2].

Proposition 1. For $k \neq 0$ and $0 \leq p \leq n$, we have

$$(1 - q^{n-p+1}) \binom{n}{p-1, k}_q - (1 - q^p) \binom{n}{p, k}_q = \frac{[n]_q}{[n-1]_q} q^{p-1} (1 - q^{n-2p+1}) \binom{n-1}{p-1, k}_q. \quad (8)$$

Proof. Indeed, up to the factor $[n]_q/[k]_q$ and using (5), the left-hand side can be written as

$$\begin{aligned}
& \sum_{r \geq 0} q^A \begin{bmatrix} n-r-1 \\ k-r-1 \end{bmatrix}_q \left((1 - q^{n-p+1}) q^{p-1} \begin{bmatrix} p-1 \\ r \end{bmatrix}_q \begin{bmatrix} n-p+1 \\ r \end{bmatrix}_q - (1 - q^p) q^{n-p} \begin{bmatrix} p \\ r \end{bmatrix}_q \begin{bmatrix} n-p \\ r \end{bmatrix}_q \right) \\
&= (1 - q^{n-2p+1}) \sum_{r \geq 0} q^A \begin{bmatrix} n-r-1 \\ k-r-1 \end{bmatrix}_q \left(q^{p-1} \begin{bmatrix} p-1 \\ r \end{bmatrix}_q \begin{bmatrix} n-p \\ r \end{bmatrix}_q - q^{n-2r+1} \begin{bmatrix} p-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} n-p \\ r-1 \end{bmatrix}_q \right) \\
&= (1 - q^{n-2p+1}) \sum_{r \geq 0} q^A \begin{bmatrix} p-1 \\ r \end{bmatrix}_q \begin{bmatrix} n-p \\ r \end{bmatrix}_q q^{p-1} \left(\begin{bmatrix} n-r-1 \\ k-r-1 \end{bmatrix}_q - q^{n-k} \begin{bmatrix} n-r-2 \\ k-r-2 \end{bmatrix}_q \right) \\
&= (1 - q^{n-2p+1}) \sum_{r \geq 0} q^A \begin{bmatrix} p-1 \\ r \end{bmatrix}_q \begin{bmatrix} n-p \\ r \end{bmatrix}_q \begin{bmatrix} n-r-2 \\ k-r-1 \end{bmatrix}_q q^{p-1},
\end{aligned}$$

where $A = r(r-k) + (n-p)(p-1)$. The result follows then from Equation (5). \square

For the second generating function, we will further need the q -binomial formula [1, 236]:

$$\sum_{n \geq 0} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad (9)$$

Jackson's q -Pfaff-Kummer transformation [1, p.241]:

$${}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix} \middle| q; z \right] = \frac{(az; q)_\infty}{(z; q)_\infty} {}_2\phi_2 \left[\begin{matrix} a, c/b \\ c, az \end{matrix} \middle| q; bz \right], \quad (10)$$

and another transformation of Sears [1, p. 61, (3.2.2)]:

$${}_3\phi_2 \left[\begin{matrix} q^{-p}, a, b \\ c, d \end{matrix} \middle| q; cdq^p/ab \right] = \frac{(c/a; q)_p}{(c; q)_p} {}_3\phi_2 \left[\begin{matrix} q^{-p}, a, d/b \\ d, q^{1-p}a/c \end{matrix} \middle| q; q \right]. \quad (11)$$

Theorem 2. For $0 \leq p \leq n$, we have

$$\sum_{k \geq 1} \binom{n}{p, k}_q y^k q^{\binom{k}{2}} = yq^{p(n-p)} (-y; q)_p [n]_q {}_2\phi_1 \left[\begin{matrix} q^{p+1}, q^{p-n+1} \\ q^2 \end{matrix} \middle| q; -yq^{n-p} \right]. \quad (12)$$

Proof. By applying (3) to (2) and then applying (11) we get:

$$\begin{aligned} \binom{n}{p, k}_q &= q^{(n-p)p} \binom{n}{k} \frac{(q^{1-p}; q)_{k-1}}{(q^{1-n}; q)_{k-1}} q^{(k-1)(p-n)} {}_3\phi_2 \left[\begin{matrix} q^{1-k}, q^{p-n}, q^{1+p} \\ q^{p-k+1}, q \end{matrix} \middle| q; q^{n-p} \right] \\ &= q^{(n-p)p} \binom{n}{k} \frac{(q^{1-p}; q)_{k-1}}{(q^{1-n}; q)_{k-1}} q^{(p-n)(k-1)} \frac{(q^{-k}; q)_{k-1}}{(q^{p-k+1}; q)_{k-1}} {}_3\phi_2 \left[\begin{matrix} q^{1-k}, q^{1+p}, q^{n-p+1} \\ q, q^2 \end{matrix} \middle| q; q \right] \\ &= (-1)^{k-1} q^{(n-p)p - \frac{(k-1)(k+2)}{2}} [n]_q {}_3\phi_2 \left[\begin{matrix} q^{1-k}, q^{1+p}, q^{n-p+1} \\ q, q^2 \end{matrix} \middle| q; q \right]. \end{aligned} \quad (13)$$

Hence

$$\sum_{k \geq 1} \binom{n}{p, k}_q y^k q^{\binom{k}{2}} = yq^{p(n-p)} [n]_q \sum_{l \geq 0} \frac{(q^{p+1}; q)_l (q^{n-p+1}; q)_l}{(q^2; q)_l (q; q)_l (q; q)_l} q^l \sum_{k \geq l} (q^{-k}; q)_l (-yq^{-1})^k.$$

But the q -binomial formula (9) implies that

$$\sum_{k \geq l} (q^{-k}; q)_l (-yq^{-1})^k = (-1)^l q^{\binom{l}{2}} \sum_{k \geq l} (q^{k-l+1}; q)_l (-yq^{-l-1})^k = \frac{(q; q)_l}{(1 + yq^{-1})(-y^{-1}q^2; q)_l}.$$

Hence

$$\sum_{k \geq 1} \binom{n}{p, k}_q y^k q^{\binom{k}{2}} = \frac{yq^{p(n-p)}}{1 + yq^{-1}} [n]_q \sum_{l \geq 0} \frac{(q^{p+1}; q)_l (q^{n-p+1}; q)_l q^l}{(-y^{-1}q^2; q)_l (q^2; q)_l (q; q)_l}. \quad (14)$$

Using the formula

$$(a; q)_n = (a^{-1}; q^{-1})_n (-a)^n q^{\binom{n}{2}} \quad (15)$$

and Jackson's transformation, we can rewrite the above sum as follows:

$$\begin{aligned} \sum_{k \geq 1} \binom{n}{p, k}_q y^k q^{\binom{k}{2}} &= \frac{yq^{p(n-p)}}{1+yq^{-1}} [n]_{q^2} \phi_2 \left[\begin{matrix} q^{-1-p}, q^{p-n-1} \\ q^{-2}, -yq^{-2} \end{matrix} \middle| q^{-1}; -yq^{n-2} \right] \\ &= yq^{p(n-p)} (-y; q)_p [n]_{q^2} \phi_1 \left[\begin{matrix} q^{-1-p}, q^{p-n-1} \\ q^{-2} \end{matrix} \middle| q^{-1}; -yq^{p-1} \right]. \end{aligned}$$

We then recover (12) by applying (15) again. \square

Remark. When $q = 1$, the above theorem reduces to Lassalle's generating function [3], which was proved by using induction and contiguous relation. One could give another proof of Theorem 2 by using Proposition 1.

The following result is crucial to prove that $\binom{n}{p, k}_q$ is a q -positive integer.

Corollary 1. *We have*

$$\binom{n}{p, k}_q = \frac{[n]_q}{[p]_q} \sum_{i=0}^{k-1} \begin{bmatrix} n-p \\ k-1-i \end{bmatrix}_q \begin{bmatrix} n-p+i \\ i \end{bmatrix}_q \begin{bmatrix} p \\ i+1 \end{bmatrix}_q q^{(i+1)(i+1-k)+(n-p)p}.$$

Proof. Using the q -binomial formula we have

$$(-y; q)_p = \sum_{j=0}^p \begin{bmatrix} p \\ j \end{bmatrix} y^j q^{\binom{j}{2}}.$$

Extracting the coefficient of y^k in (12) we obtain

$$\binom{n}{p, k}_q = \frac{[n]_q}{[n-p]_q} \sum_{i=0}^{k-1} \begin{bmatrix} p \\ k-1-i \end{bmatrix}_q \begin{bmatrix} p+i \\ i \end{bmatrix}_q \begin{bmatrix} n-p \\ i+1 \end{bmatrix}_q q^{(n-p)p+(i+1)(i+1-k)}.$$

As $\binom{n}{p, k}_q = \binom{n}{n-p, k}_q$, substituting p by $n-p$ yields the desired identity. \square

By applying

$$\binom{n}{p, k}_q = \frac{[p]_q}{[n]_q} \binom{n}{p, k}_q + q^p \frac{[n-p]_q}{[n]_q} \binom{n}{n-p, k}_q,$$

we can write

$$\begin{aligned} \binom{n}{p, k}_q &= \sum_{i=0}^{k-1} \begin{bmatrix} n-p \\ k-1-i \end{bmatrix}_q \begin{bmatrix} n-p+i \\ i \end{bmatrix}_q \begin{bmatrix} p \\ i+1 \end{bmatrix}_q q^{(n-p)p+(i+1)(i+1-k)} \\ &\quad + \sum_{i=0}^{k-1} \begin{bmatrix} p \\ k-1-i \end{bmatrix}_q \begin{bmatrix} p+i \\ i \end{bmatrix}_q \begin{bmatrix} n-p \\ i+1 \end{bmatrix}_q q^{(n-p)(p+1)+(i+1)(i+1-k)}, \end{aligned}$$

which implies the following

Theorem 3. *The polynomials $\binom{n}{p, k}_q$ is a q -positive integer.*

Since q -binomial coefficients have various nice combinatorial interpretations, it would be possible to derive a combinatorial interpretation for $\binom{n}{p, k}_q$ from the above expression.

3 Further extensions

It is surprising that the general numbers $c_k^{(\mathbf{r})}$ of Lassalle [3] have also a q -analogue, which are also q -positive integers. We shall explain such a q -analog in this section. Note that the q -Chu-Vandermonde formula:

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_q = \sum_{i \geq 0} \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} m \\ k-i \end{bmatrix}_q q^{(n-i)(k-i)} \quad (16)$$

implies that

$$\begin{aligned} \begin{bmatrix} x \\ r_1 \end{bmatrix}_q \begin{bmatrix} x \\ r_2 \end{bmatrix}_q &= \begin{bmatrix} x \\ r_1 \end{bmatrix}_q \sum_{k \geq 0} \begin{bmatrix} r_1 \\ k \end{bmatrix}_q \begin{bmatrix} x-r_1 \\ r_2-k \end{bmatrix}_q q^{(r_1-k)(r_2-k)} \\ &= \sum_{k \geq 0} q^{(r_1-k)(r_2-k)} \begin{bmatrix} r_1+r_2-k \\ k, r_1-k, r_2-k \end{bmatrix}_q \begin{bmatrix} x \\ r_1+r_2-k \end{bmatrix}_q. \end{aligned} \quad (17)$$

Set $\mathbf{r} = (r_1, \dots, r_m)$ and $|\mathbf{r}| = r_1 + \dots + r_m$. Iterating (17) yields:

$$\begin{bmatrix} x \\ r_1 \end{bmatrix}_q \cdots \begin{bmatrix} x \\ r_m \end{bmatrix}_q = \sum_{l \geq 0} d_l(\mathbf{r}; q) \begin{bmatrix} x \\ l \end{bmatrix}_q, \quad (18)$$

where $d_l(\mathbf{r}; q)$ are q -positive integers given by

$$\begin{aligned} d_l(\mathbf{r}; q) &= \sum_{k_1, \dots, k_{m-2} \geq 0} \begin{bmatrix} r_1+r_2-k_1 \\ k_1, r_1-k_1, r_2-k_1 \end{bmatrix}_q \begin{bmatrix} r_1+r_2+r_3-k_1-k_2 \\ k_2, r_1+r_2-k_1-k_2, r_3-k_2 \end{bmatrix}_q \\ &\quad \times \cdots \times \begin{bmatrix} r_1+\dots+r_{m-2}-k_1-\dots-k_{m-3} \\ k_{m-3}, r_1+\dots+r_{m-3}-k_1-\dots-k_{m-3}, r_{m-2}-k_{m-3} \end{bmatrix}_q \\ &\quad \times \begin{bmatrix} l \\ r_m \end{bmatrix}_q \begin{bmatrix} r_m \\ |\mathbf{r}|-k_1-\dots-k_{m-2}-l \end{bmatrix}_q q^B, \end{aligned} \quad (19)$$

where

$$\begin{aligned} B &= (r_1-k_1)(r_2-k_1) + (r_1+r_2-k_1-k_2)(r_3-k_2) + \cdots \\ &\quad + (r_1+r_2+\dots+r_{m-2}-k_1-\dots-k_{m-2})(r_{m-1}-k_{m-2}) \\ &\quad + (l-r_m)(l-r_1-\dots-r_{m-1}+k_1+\dots+k_{m-2}). \end{aligned}$$

In particular, for $m = 2$ we have

$$d_l(r_1, r_2; q) = q^{(l-r_1)(l-r_2)} \begin{bmatrix} l \\ r_1 \end{bmatrix}_q \begin{bmatrix} r_1 \\ l-r_2 \end{bmatrix}_q. \quad (20)$$

On the other hand, identity (16) implies also

$$\begin{bmatrix} x+r_1-1 \\ r_1 \end{bmatrix}_q = \sum_{k \geq 0} \begin{bmatrix} r_1-1 \\ r_1-k \end{bmatrix}_q \begin{bmatrix} x \\ k \end{bmatrix}_q q^{k(k-1)}. \quad (21)$$

From (18) and (21) we derive the following result.

$$\begin{bmatrix} x + r_1 - 1 \\ r_1 \end{bmatrix}_q \cdots \begin{bmatrix} x + r_m - 1 \\ r_m \end{bmatrix}_q = \sum_{l \geq 0} \tilde{c}_l(\mathbf{r}; q) \begin{bmatrix} x \\ l \end{bmatrix}_q.$$

where

$$\tilde{c}_l(\mathbf{r}; q) = \sum_{\mathbf{k}} d_l(\mathbf{k}; q) \prod_{i=1}^m \begin{bmatrix} r_i - 1 \\ k_i - 1 \end{bmatrix}_q q^{k_i(k_i-1)}. \quad (22)$$

Theorem 4. *The polynomial*

$$c_l(\mathbf{r}; q) = \frac{[r_1 + \cdots + r_m]_q}{[l]_q} \tilde{c}_l(\mathbf{r}; q)$$

is a q -positive integer.

Proof. By (19) there is a polynomial $P_m(\mathbf{k}; q) \in \mathbb{N}[q]$ such that

$$d_l(\mathbf{k}; q) = \frac{[l]_q}{[k_m]_q} P_m(\mathbf{k}; q).$$

Since $d_l(\mathbf{k}; q)$ is symmetric with respect to $\mathbf{k} = (k_1, \dots, k_m)$, the above formula infers that there is a polynomial $P_i(\mathbf{k}; q) \in \mathbb{N}[q]$ for each $j \in \{1, \dots, m\}$ such that

$$d_l(\mathbf{k}; q) = \frac{[l]_q}{[k_j]_q} P_j(\mathbf{k}; q).$$

Therefore, using (3) we have

$$\begin{aligned} c_l(\mathbf{r}; q) &= \sum_{j=1}^m \frac{[r_j]_q q^{r_1 + \cdots + r_{j-1}}}{[r_1 + \cdots + r_m]_q} c_l(\mathbf{r}; q) \\ &= \sum_{j=1}^m q^{r_1 + \cdots + r_{j-1}} \sum_{\mathbf{k}} P_j(\mathbf{k}; q) \begin{bmatrix} r_j \\ k_j \end{bmatrix}_q q^{k_j(k_j-1)} \prod_{i \neq j} \begin{bmatrix} r_i - 1 \\ k_i - 1 \end{bmatrix}_q q^{k_i(k_i-1)}, \end{aligned}$$

which is clearly a q -positive integer. \square

We can also derive a simpler formula for $\tilde{c}_j(r_1, \dots, r_m; q)$ using the q -difference operator. Set $[x; q] = (q^x - 1)/(q - 1)$ and

$$[x; q]_n = [x; q][x - 1; q] \cdots [x - n + 1; q] = \frac{(q^{x-n+1}; q)_n}{(1 - q)^n}.$$

We define the q -difference operator Δ_q by

$$\Delta_q^0 f(x) = f(x), \quad \Delta_q^{n+1} f(x) = \Delta_q^n (E - q^n I) f(x),$$

where $I f(x) = f(x)$ and $E f(x) = f(x + 1)$. Note that

$$\Delta_q^n f(x) = (E - q^{n-1} I)(E - q^{n-2} I) \cdots (E - I) f(x).$$

By the q -Chu-Vandermonde formula, we have

$$\Delta_q^n f(x) = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(x+n-k). \quad (23)$$

It's easy to see that

$$\Delta_q^n [x; q]_m = [m; q]_n [x; q]_{m-n} q^{n(x+n-m)}.$$

We have also

$$p(x) = \sum_{n \geq 0} \frac{\Delta_q^n p(0)}{[n]!} [x; q]_n. \quad (24)$$

It follows from (24), (23) and (3) that

$$\tilde{c}_k(r_1, \dots, r_m; q) = \sum_{j=1}^k (-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{k-j}{2}} \prod_{l=1}^m \begin{bmatrix} j+r_l-1 \\ r_l \end{bmatrix}_q. \quad (25)$$

Set $\mathbf{r} = (r_1, \dots, r_m)$ and

$$\begin{aligned} c_k(\mathbf{r}; q) &= \frac{[r_1 + \dots + r_m]_q}{[k]_q} \tilde{c}_k(r_1, \dots, r_m; q) \\ &= \sum_{i=1}^m q^{r_1 + \dots + r_{i-1}} \frac{[r_i]_q}{[k]_q} \sum_{j=1}^k (-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{k-j}{2}} \prod_{l=1}^m \begin{bmatrix} j+r_l-1 \\ r_l \end{bmatrix}_q \\ &= \sum_{i=1}^m \sum_{j=1}^k (-1)^{k-j} q^{r_1 + \dots + r_{i-1} + \binom{k-j}{2}} \\ &\quad \times \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q \begin{bmatrix} j+r_i-1 \\ r_i \end{bmatrix}_q \prod_{l=1, l \neq j}^m \begin{bmatrix} r_l+i-1 \\ r_l \end{bmatrix}_q. \end{aligned} \quad (26)$$

Thus we have obtained another proof of the q -integrality of $c_k(\mathbf{r}; q)$.

Finally, when $\mathbf{r} = (r_1, r_2)$, we have the following result.

Theorem 5. *The coefficients $\binom{r_1+r_2}{r_1, k}_q$ satisfy*

$$\begin{bmatrix} x+r_1-1 \\ r_1 \end{bmatrix}_q \begin{bmatrix} x+r_2-1 \\ r_2 \end{bmatrix}_q = \sum_{k \geq 1} \frac{[k]_q q^{k(k-1)-r_1 r_2}}{[r_1+r_2]_q} \binom{r_1+r_2}{r_1, k}_q \begin{bmatrix} x \\ k \end{bmatrix}_q. \quad (27)$$

Proof. By (25) we have

$$\begin{aligned} c_k(r_1, r_2; q) &= \frac{[r_1+r_2]_q}{[k]_q} \sum_{j=0}^k (-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{k-j}{2}} \begin{bmatrix} j+r_1-1 \\ r_1 \end{bmatrix}_q \begin{bmatrix} j+r_2-1 \\ r_2 \end{bmatrix}_q \\ &= (-1)^{k-1} [r_1+r_2]_q q^{\binom{k-1}{2}} {}_3\phi_2 \left[\begin{matrix} q^{1-k}, q^{1+r_1}, q^{1+r_2} \\ q, q^2 \end{matrix} \middle| q; q \right]. \end{aligned}$$

Comparing with (13) we see that $c_k(r_1, r_2; q) = q^{k(k-1)-r_1 r_2} \binom{r_1+r_2}{r_1, k}_q$. \square

4 The Second q -analogue

In this section, we give another new family of q -positive integers $\begin{bmatrix} n \\ p, k \end{bmatrix}_q$, which have the similar properties as $\binom{n}{p, k}_q$. Since the proofs are similar we omit the details.

Definition 1. For any positive integers n, p, k , define

$$\begin{bmatrix} n \\ p, k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q {}_3\phi_2 \left[\begin{matrix} q^{1-k}, q^{-p}, q^{p-n} \\ q, q^{1-n} \end{matrix} \middle| q; q^{k+1} \right]. \quad (28)$$

Obviously we have

$$\begin{bmatrix} n \\ p, k \end{bmatrix}_q = \begin{bmatrix} n \\ n-p, k \end{bmatrix}_q, \quad \begin{bmatrix} n \\ p, n \end{bmatrix}_q = \begin{bmatrix} n \\ p \end{bmatrix}_q.$$

The definition (28) could be written as follows:

$$\begin{bmatrix} n \\ p, k \end{bmatrix}_q = \frac{[n]_q}{[k]_q} \sum_{r \geq 0} \begin{bmatrix} p \\ r \end{bmatrix}_q \begin{bmatrix} n-p \\ r \end{bmatrix}_q \begin{bmatrix} n-r-1 \\ k-r-1 \end{bmatrix}_q q^{r^2}.$$

Therefore

$$\begin{bmatrix} n \\ 0, k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q, \quad \begin{bmatrix} n \\ 1, k \end{bmatrix}_q = [k]_q \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

and

$$\begin{bmatrix} n \\ 2, k \end{bmatrix}_q = [k]_q \begin{bmatrix} n \\ k \end{bmatrix}_q + q^2 \frac{[n]_q [n-3]_q}{[2]_q} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q.$$

For $p \geq 0$, we have also

$$\begin{bmatrix} n \\ p, 0 \end{bmatrix}_q = 0, \quad \begin{bmatrix} n \\ p, 1 \end{bmatrix}_q = [n]_q, \quad \begin{bmatrix} n \\ p, 2 \end{bmatrix}_q = \frac{[n]_q}{[2]_q} ([n-1]_q + [p]_q [n-p]_q).$$

Applying (11) to the definition (28), we get

$$\begin{bmatrix} n \\ p, k \end{bmatrix}_q = \sum_{r \geq 0} (-1)^i \begin{bmatrix} n-i \\ k \end{bmatrix}_q \frac{[n]_q}{[n-i]_q} \begin{bmatrix} n-i \\ i \end{bmatrix}_q \begin{bmatrix} n-2i \\ p-i \end{bmatrix}_q q^{i(i-1)/2+ki}.$$

The following two generating functions are obtained in the same way as for $\binom{n}{p, k}_q$.

Theorem 6. *There holds*

$$\begin{aligned} & \sum_{0 \leq k, p \leq n} \begin{bmatrix} n \\ p, k \end{bmatrix}_q x^p q^{\binom{p}{2}} y^k q^{\binom{k}{2}} \\ &= (-y; q)_n (-x; q)_n \sum_{i \geq 0} \frac{1-q^n}{1-q^{n-i}} \begin{bmatrix} n-i \\ i \end{bmatrix}_q \frac{(-x)^i q^{i^2-i}}{(-y; q)_i (-x; q)_i (-xq^{n-i}; q)_i}. \end{aligned} \quad (29)$$

Theorem 7. For $0 \leq p \leq n$, there holds

$$\sum_{k \geq 1} \begin{bmatrix} n \\ p, k \end{bmatrix}_q y^k q^{\binom{k}{2}} = y(-yq^{n-p}; q)_p [n]_q {}_2\phi_1 \left[\begin{matrix} q^{p+1}, q^{p-n+1} \\ q^2 \end{matrix} \middle| q; -yq^{n-p} \right]. \quad (30)$$

As (8) we get the recurrence relation

Proposition 2. For $k \neq 0$ and $0 \leq p \leq n$, there holds

$$(1 - q^{n-p+1}) \begin{bmatrix} n \\ p-1, k \end{bmatrix}_q - (1 - q^p) \begin{bmatrix} n \\ p, k \end{bmatrix}_q = \frac{[n]_q}{[n-1]_q} (1 - q^{n-2p+1}) q^{k+p-1} \begin{bmatrix} n-1 \\ p-1, k \end{bmatrix}_q. \quad (31)$$

Corollary 2. We have

$$\begin{bmatrix} n \\ p, k \end{bmatrix}_q = \frac{[n]_q}{[p]_q} \sum_{i=0}^{k-1} \begin{bmatrix} n-p \\ k-1-i \end{bmatrix}_q \begin{bmatrix} n-p+i \\ i \end{bmatrix}_q \begin{bmatrix} p \\ i+1 \end{bmatrix}_q q^{(i+1+p-n)(i+1-k)}.$$

As in the proof of Theorem 3, we can prove that $\begin{bmatrix} n \\ p, k \end{bmatrix}_q$ is also a q -positive integer by using the above Corollary.

5 Tables of the generalized q -binomial coefficients

- $\begin{bmatrix} n \\ 0, k \end{bmatrix}_q = \binom{n}{0, k} = \begin{bmatrix} n \\ k \end{bmatrix}_q$ for $k \geq 0$,
- $\begin{bmatrix} n \\ p, 0 \end{bmatrix}_q = \binom{n}{p, 0} = 0$, for $p \geq 0$.
- $n = 1$, $\begin{bmatrix} n \\ 1, 1 \end{bmatrix}_q = \binom{n}{1, 1} = 1$.

Tables of $\binom{n}{p, k}_q$

$n = 2$

$p \backslash k$	1	2
1	$q[2]_q$	$[2]_q$
2	$[2]_q$	1

$n = 3$,

$p \backslash k$	1	2	3
1	$q^2[3]_q$	$q[2]_q[3]_q$	$[3]_q$
2	$q^2[3]_q$	$q[2]_q[3]_q$	$[3]_q$
3	$[3]_q$	$[3]_q$	1

$n=4$,

$p \backslash k$	1	2	3	4
1	$q^3[4]_q$	$q^2[3]_q[4]_q$	$q[2]_q[3]_q(1+q^2)$	$[4]_q$
2	$q^4[4]_q$	$(1+q^2)(1+3q+2q^2+q^3)$	$[2]_q(1+q^2)(2+q+q^2)$	$[3]_q(1+q^2)$
3	$q^3[4]_q$	$q^2[3]_q[4]_q$	$q[2]_q[3]_q(1+q^2)$	$[4]_q$
4	$[4]_q$	$[2]_q[3]_q$	$[4]_q$	1

$n=5$,

$p \backslash k$	1	2	3	4	5
1	$[5]_q$	$[2]_q[5]_q(1+q^2)$	$[3]_q[5]_q(1+q^2)$	$[2]_q[5]_q(1+q^2)$	$[5]_q$
2	$[5]_q$	$[5]_q(1+q+2q^2+q^3)$	$[3]_q[5]_q(1+2q^2)$	$[5]_q(1+q+2q^2+2q^3+q^4)$	$[5]_q(1+q^2)$
3	$[5]_q$	$[5]_q(1+q+2q^2+q^3)$	$[3]_q[5]_q(1+2q^2)$	$[5]_q(1+q+2q^2+2q^3+q^4)$	$[5]_q(1+q^2)$
4	$[5]_q$	$[2]_q[5]_q(1+q^2)$	$[3]_q[5]_q(1+q^2)$	$[2]_q[5]_q(1+q^2)$	$[5]_q$
5	$[5]_q$	$[5]_q(1+q^2)$	$[5]_q(1+q^2)$	$[5]_q$	1

Tables of $\begin{bmatrix} n \\ p, k \end{bmatrix}_q$

$$n = 2,$$

$p \backslash k$	1	2
1	$[2]_q$	$[2]_q$
2	$[2]_q$	1

$$n = 3,$$

$p \backslash k$	1	2	3
1	$[3]_q$	$[2]_q[3]_q$	$[3]_q$
2	$[3]_q$	$[2]_q[3]_q$	$[3]_q$
3	$[3]_q$	$[3]_q$	1

$$n = 4,$$

$p \backslash k$	1	2	3	4
1	$[4]_q$	$[3]_q[4]_q$	$[2]_q[3]_q(1+q^2)$	$[4]_q$
2	$[4]_q$	$(1+q^2)(1+2q+3q^2+q^3)$	$[2]_q(1+q^2)(1+q+2q^2)$	$(1+q^2)[3]_q$
3	$[4]_q$	$[3]_q[4]_q$	$[2]_q[3]_q(1+q^2)$	$[4]_q$
4	$[4]_q$	$[3]_q(1+q^2)$	$[4]_q$	1

$$n = 5,$$

$p \backslash k$	1	2	3	4	5
1	$q^4[5]_q$	$[2]_q[5]_q(1+q^2)q^3$	$[3]_q[5]_q(1+q^2)q^2$	$[2]_q[5]_q(1+q^2)q$	$[5]_q$
2	$q^6[5]_q$	$[5]_q(1+2q+q^2+q^3)q^5$	$[3]_q[5]_q(2+q^2)q^4$	$[5]_q(1+2q+2q^2+q^3+q^4)q^2$	$[5]_q(1+q^2)$
3	$q^6[5]_q$	$[5]_q(1+2q+q^2+q^3)q^5$	$[3]_q[5]_q(2+q^2)q^4$	$[5]_q(1+2q+2q^2+q^3+q^4)q^2$	$[5]_q(1+q^2)$
4	$q^4[5]_q$	$[2]_q[5]_q(1+q^2)q^3$	$[3]_q[5]_q(1+q^2)q^2$	$[2]_q[5]_q(1+q^2)q$	$[5]_q$
5	$[5]_q$	$[5]_q(1+q^2)$	$[5]_q(1+q^2)$	$[5]_q$	1

The above tables seem to suggest that the sequences of the coefficients in $\binom{n}{p, k}_q$ and $\begin{bmatrix} n \\ p, k \end{bmatrix}_q$ are *unimodal* for each fixed pair (n, p) or (n, k) .

Remark H. Rosengren noticed that the algebraic expressions of our coefficients $\binom{n}{p, k}_q$ and $\begin{bmatrix} n \\ p, k \end{bmatrix}_q$ are limit cases of his coefficients R_k^l (see "An elementary approach to $6j$ -symbols" available at <http://www.arxiv.org/abs/math.CA/0312310>). It would be interesting to see whether it is possible to obtain further (interesting) families of q -positive integers as degenerate cases of his coefficients.

Acknowledgements

This work was done under the auspices of the National Science Foundation of China.

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