Numerical Triangles and Several Classical Sequences

Yidong Sun
Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P.R. China
and
Department of Applied Mathematics
Dalian University of Technology, Dalian 116024, P. R. China
Email:sydcom@eyou.com

September 3, 2004

Abstract: In 1991 Ferri, Faccio and D'Amico introduced and investigated two numerical triangles, called the DFF and DFFz triangles. Later Trzaska also considered the DFF triangle. And in 1994 Jeannin generalized the two triangles. In this paper, we focus our attention on the generalized Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas polynomials, and several numerical triangles deduced by them.

1. The Generalized Fibonacci and Lucas Polynomials

Let us define a sequence of polynomials $\{F_n(y)\}$ by the recurrence relation

$$F_{n+1}(y) = F_n(y) + yF_{n-1}(y), \qquad n \geq 1, \tag{1.1}$$

where $F_0(y) = a,\ F_1(y) = b$. Notice that (1.1) yield the Fibonacci and Lucas sequences F_n and L_n when $y = 1$ with the initial values $a = b = 1,\ a = 2,\ b = 1$, respectively.

Define

$$F_n(y) = \sum_{k=0}^{n} f_{n,\ k}\, y^k, \quad F(x, y) = \sum_{n\geq 0} F_n(y)x^n. \tag{1.2}$$

By (1.1) and (1.2) it is easy to derive

$$F(x, y) = \frac{a + (b - a)x}{1 - x - x^2 y}, \tag{1.3}$$

and

$$\begin{aligned} f_{n,\ k} &= [x^n y^k]F(x, y) = [x^n y^k](a + (b - a)x)\sum_{k\geq 0} \frac{x^{2k}y^k}{(1 - x)^{k+1}} \\ &= a\binom{n-k}{n-2k} + (b-a)\binom{n-k-1}{n-2k-1} \\ &= a\binom{n-k-1}{k-1} + b\binom{n-k-1}{k}, \end{aligned}$$

which satisfies the recurrence $f_{n+1,\ k} = f_{n,\ k} + f_{n-1,\ k-1}$, with the initial conditions $f_{0,\ 0} = a,\ f_{1,\ 0} = b$.

Let $a_{n+1,\ k} = f_{2n+2,\ n-k+1}$ and $b_{n,\ k} = f_{2n+1,\ n-k}$, then we have

$$a_{n+1,\ k} = a\binom{n+k}{2k} + b\binom{n+k}{2k-1}, \qquad b_{n,\ k} = a\binom{n+k}{2k+1} + b\binom{n+k}{2k},$$

which generate two lower triangles

Table 1.1

n/k	0	1	2	3	4
0	a				
1	a	b			
2	a	$a+2b$	b		
3	a	$3a+3b$	$a+4b$	b	
4	a	$6a+4b$	$5a+10b$	$a+6b$	b

Table 1.2

n/k	0	1	2	3	4
0	b				
1	$a+b$	b			
2	$2a+b$	$a+3b$	b		
3	$3a+b$	$4a+6b$	$a+5b$	b	
4	$4a+b$	$10a+10b$	$6a+15b$	$a+7b$	b

Theorem 1.1 *Let* $A^{(i)}_{n\times n} = (a_{n+k+i+1,\ k+i+1})_{0\le k\le n}$ *and* $B^{(i)}_{n\times n} = (b_{n+k+i,\ k+i})_{0\le k\le n}$, *then*

$$|A^{(i)}_{n\times n}| = |B^{(i)}_{n\times n}| = 2^{\binom{n+1}{2}} b^{n+1}.$$

Proof. We only prove the second statement. Note that

$$b_{n+k+i,\ k+i} = a\binom{2n+2k+2i}{n-1} + b\binom{2n+2k+2i}{n}$$

is a polynomial in k of degree n with the coefficient of the highest term $\frac{2^n b}{n!}$, according to the Tepper identity [2, 8],

$$\sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\frac{(\alpha-k)^r}{n!} = \begin{cases} 0 & \text{if } 0\le r<n, \\ 1 & \text{if } r=n. \end{cases}$$

We have

$$\begin{aligned} & \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} b_{n+k+i,\ k+i} \\ = & \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\left(a\binom{2n+2k+2i}{n-1} + b\binom{2n+2k+2i}{n}\right) \\ = & \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\left(\frac{2^n b}{n!}k^n + \textit{ the terms of lower orders}\right) \\ = & \ 2^n b. \end{aligned}$$

Then the results hold by trivial computations on determinant. □

In the special case $a = b = 1$,

$$a_{n,\ k} = \binom{n+k}{2k}, \qquad b_{n,\ k} = \binom{n+k+1}{2k+1}.$$

Tables 1.1 and 1.2 reduce to the triangles DFF [3, 11] and DFFz [3, 4] respectively.

Table 1.3 (DFF)

n/k	0	1	2	3	4
0	1				
1	1	1			
2	1	3	1		
3	1	6	5	1	
4	1	10	15	7	1

Table 1.4 ($DFFz$)

n/k	0	1	2	3	4
0	1				
1	2	1			
2	3	4	1		
3	4	10	6	1	
4	5	20	21	8	1

Table 1.5

n/k	0	1	2	3	4
0	1				
1	1	$\underline{1}$			
2	1	$\underline{2}$	1		
3	1	$\underline{3}$	3	$\underline{1}$	
4	1	$\underline{4}$	6	$\underline{4}$	1

It is well-known that F_{n+1} is the number of $(0,1)-$sequences of length n without successive ones. It is easy to show that $a_{n,\ k}$ $(b_{n,\ k})$ is the number of such sequences of length $2n-1(2n)$ containing exact $n-k$ ones, which illustrates that the row sums of Table 1.3 (Table 1.4) are equal to the Fibonacci numbers with even(old) subscripts [3, 4]. In fact Tables 1.3 and 1.4 can be obtained directly from the classical Pascal triangle displayed in Table 1.5, where Table 1.3 is just the even columns and Table 1.4 the odd columns of the Pascal triangle. Note that the un-bared and bared numbers in Pascal triangles are respectively the numbers of the diagonals of Tables 1.3 and 1.4, which illustrates that the sums of the diagonals are the powers of 2 [3, 4].

Theorem 1.2 *Let* $A=(a_{n,\ k})_{0\le k\le n}$ *and* $B=(b_{n,\ k})_{0\le k\le n}$*, then we have*

$$A^{-1} = \left((-1)^{n-k}\frac{2k+1}{2n+1}\binom{2n+1}{n-k}\right)_{0\le k\le n} \tag{1.4}$$

$$B^{-1} = \left((-1)^{n-k}\frac{k+1}{n+1}\binom{2n+2}{n-k}\right)_{0\le k\le n}, \tag{1.5}$$

where A^{-1} is the inverse of the matrix A.

Proof. It suffices to prove (1.4) that

$$\sum_{k=0}^{n}(-1)^{m-k}\frac{2m+1}{2k+1}\binom{2k+1}{k-m}\binom{n+k}{2k}=\delta_{mn}, \tag{1.6}$$

where δ_{mn} is the Kronecker symbol.
It is easy to see that (1.6) holds for $m\ge n$. In case $m<n$, we have

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^{m-k}\frac{2m+1}{2k+1}\binom{2k+1}{k-m}\binom{n+k}{2k}\\
&= \sum_{k=0}^{n}(-1)^{m-k}\frac{2m+1}{m+k+1}\binom{2k}{k+m}\binom{n+k}{2k}\\
&= \sum_{k=0}^{n}(-1)^{m-k}\frac{2m+1}{m+k+1}\binom{n+k}{m+k}\binom{n-m}{k-m}\\
&= \sum_{k=0}^{n}(-1)^{m+k}\frac{2m+1}{n-m}\binom{n+k}{m+k+1}\binom{n-m}{k-m}\\
&= \sum_{k=0}^{n}\frac{2m+1}{m-n}\binom{m-n}{m+k+1}\binom{n-m}{n-k}\\
&= \frac{2m+1}{m-n}\binom{0}{m+n+1}=0.
\end{aligned}$$

Then (1.4) holds and (1.5) follows in the same way. □

In the special case $a=2,\ b=1$, we have

$$a_{n,\ k} = \binom{n+k}{2k}+\binom{n+k-1}{2k}=\frac{2n}{n+k}\binom{n+k}{2k},$$

$$b_{n,\ k} = \binom{n+k+1}{2k+1}+\binom{n+k}{2k+1}=\frac{2n+1}{2k+1}\binom{n+k}{2k},$$

with $a_{0,\ 0} = 2$. And Table 1.1 and 1.2 yield the triangles,

Table 1.6

n/k	0	1	2	3	4
0	2				
1	2	1			
2	2	4	1		
3	2	9	6	1	
4	2	16	20	8	1

Table 1.7

n/k	0	1	2	3	4
0	1				
1	3	1			
2	5	5	1		
3	7	14	7	1	
4	9	30	27	9	1

It is also well-known that L_n is the number of $(0, 1)-$sequences of length n without successive ones, where the first and last components of the sequences are considered to be adjacent. It is easy to show that $a_{n,\ k}$ ($b_{n,\ k}$) is the number of such sequences of length $2n$ ($2n-1$) containing exact $n-k$ ones, which illustrates the row sums of Table 1.6 (Table 1.7) are equal to the Lucas numbers with even (odd) subscripts.

Theorem 1.3 *Let* $A = (a_{n+1,\ k+1})_{0\leq k\leq n}$ *and* $B = (b_{n+1,\ k+1})_{0\leq k\leq n}$*, then we have*

$$\begin{aligned} A^{-1} &= \left((-1)^{n-k}\binom{2n+2}{n-k}\right)_{0\leq k\leq n} \\ B^{-1} &= \left((-1)^{n-k}\binom{2n+1}{n-k}\right)_{0\leq k\leq n}. \end{aligned}$$

Proof. This proof is similar to the proof of Theorem (1.2) so it is omitted. □

2. The Generalized Pell and Pell-Lucas Polynomials

Let us define a sequence of polynomials $\{P_n(y)\}$ by the recurrence relation

$$P_{n+1}(y) = (1+y)P_n(y) + y^2 P_{n-1}(y), \qquad n \geq 1, \tag{2.1}$$

where $P_0(y) = 1,\ P_1(y) = 1 + y$, which generates the Pell sequence $\{P_n\}$ when $y = 1$.

Define

$$P_n(y) = \sum_{k=0}^{n} P_{n,\ n-k}y^k, \quad P(x, y) = \sum_{n\geq 0} P_n(y)x^n. \tag{2.2}$$

By (2.1) and (2.2) it is easy to derive

$$P(x, y) = \frac{1}{1 - x - xy - x^2y^2}, \tag{2.3}$$

and

$$\begin{aligned} P_{n,\ k} &= [x^n y^{n-k}]P(x, y) = [x^n y^{n-k}]\sum_{r\geq 0}\frac{x^r}{(1 - xy - x^2y^2)^{r+1}} \\ &= [x^n y^{n-k}]\sum_{r,\ m\geq 0}\ \sum_{i_0+i_1+\cdots+i_r=m}\ \prod_{j=0}^{r} F_{i_j}x^{m+r}y^m \\ &= \sum_{i_0+i_1+\cdots+i_k=n-k}\ \prod_{j=0}^{k} F_{i_j}, \end{aligned} \tag{2.4}$$

which satisfies the recurrence $P_{n+1,\ k} = P_{n,\ k} + P_{n,\ k-1} + P_{n-1,\ k}$, with the initial conditions $P_{0,\ 0} = P_{1,\ 0} = P_{1,\ 1} = 1$.

From (2.3), we can deduce another formula for $P_{n,\ k}$,

$$\begin{aligned} P_{n,\ k} &= [x^n y^{n-k}]P(x,y) = [x^n y^{n-k}]\sum_{r\geq 0}\frac{(xy)^r(1+xy)^r}{(1-x)^{r+1}} \\ &= [x^n y^{n-k}]\sum_{r,\ m\geq 0}\sum_{i=0}^{r}\binom{r}{i}\binom{r+m}{m}x^{m+r+i}y^{r+i} \\ &= \sum_{r+i=n-k}\binom{r}{i}\binom{r+k}{k} \\ &= \sum_{i=0}^{[(n-k)/2]}\binom{n-i}{i}\binom{n-2i}{k}. \end{aligned} \tag{2.5}$$

By (2.4) and (2.5), we have the following interesting identity.

$$\sum_{i_0+i_1+\cdots+i_k=n-k}\prod_{j=0}^{k}F_{i_j} = \sum_{i=0}^{[(n-k)/2]}\binom{n-i}{i}\binom{n-2i}{k}.$$

Now we give a combinatorial interpretation for $P_{n,\ k}$, that is the following:

Theorem 2.1 *For any integer $n,\ k \geq 0$, $P_{n-k+1,\ k}$ is the number of $(0,1,2)-$sequences of length n with k 2's but without subsequences* $11,\ 12,\ 21,\ 22$.

Proof. Let $S_{n,\ k}$ be the desired number. Consider the last component x_n of such sequences in three cases, i.e., $x_n = 0,\ 1$ or 2, we have

$$S_{n+1,\ k} = S_{n,\ k} + S_{n-1,\ k} + S_{n-1,\ k-1}, \qquad (n \geq 1),$$

with the initial conditions $S_{0,\ 0} = 1,\ S_{1,\ 0} = 2,\ S_{1,\ 1} = 1$. It is easy to verify that $P_{n-k+1,\ k}$ also satisfies this recurrence with the same initial values, so it must equal $S_{n,\ k}$. □

Notice that $P_{n,\ 0} = F_n$, and $P_{n,\ k}$ leads to the triangle

Table 2.1

n/k	0	1	2	3	4	5
0	1					
1	1	1				
2	2	2	1			
3	3	5	3	1		
4	5	10	9	4	1	
5	8	20	22	14	5	1

Theorem 2.2 *Let $P^{(i)}_{n\times n} = (P_{n+k+i,\ k+i})_{0\leq k\leq n}$, then*

$$|P^{(i)}_{n\times n}| = 1. \tag{2.6}$$

Proof. It suffices to show that

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P_{n+k+i,\ k+i} = 1, \tag{2.7}$$

Note that

$$P_{n+k+i,\ k+i} = [x^{n+k+i}y^n]P(x,y) = [x^{n+k+i}]\frac{x^{k+i}}{(1-x-x^2)^{k+i+1}} = [x^n]\frac{1}{(1-x-x^2)^{k+i+1}}.$$

Then by the Cauchy Residue Theorem (for details see [2]), we get

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P_{n+k+i,\ k+i} &= \sum_{k=0}^{n}\underset{u}{res}\frac{(u-1)^n}{u^{k+1}}\underset{x}{res}\frac{(1-x-x^2)^{-k-i-1}}{x^{n+1}}\\ &= \underset{x}{res}\frac{(1-x-x^2)^{-i-1}}{x^{n+1}}(\frac{1}{1-x-x^2}-1)^n\\ &= \underset{x}{res}\frac{(1+x)^n}{x(1-x-x^2)^{n+i+1}}\\ &= 1.\end{aligned}$$

Thus the result holds. □

Now let us define another sequence of polynomials $\{Q_n(y)\}$ by the recurrence relation

$$Q_{n+1}(y) = (1+y)Q_n(y) + y^2Q_{n-1}(y), \qquad n \geq 1, \tag{2.8}$$

where $Q_0(y) = 1,\ Q_1(y) = 1+2y$, which generates the Pell-Lucas sequence $\{2Q_n\}$ when $y = 1$.

Define

$$Q_n(y) = \sum_{k=0}^{n}Q_{n,\ n-k}y^k, \quad Q(x,y) = \sum_{n\geq 0}Q_n(y)x^n. \tag{2.9}$$

By (2.8) and (2.9) it is easy to derive

$$Q(x,y) = \frac{1+xy}{1-x-xy-x^2y^2}, \tag{2.10}$$

and

$$\begin{aligned}Q_{n,\ k} &= [x^ny^{n-k}]Q(x,y) = [x^ny^{n-k}]\sum_{r\geq 0}\frac{(1+xy)x^r}{(1-xy-x^2y^2)^{r+1}}\\ &= [x^ny^{n-k}]\sum_{r,\ m\geq 0}\ \sum_{i_0+i_1+\cdots+i_r=m}F_{i_0+1}\prod_{j=1}^{r}F_{i_j}x^{m+r}y^m\\ &= \sum_{i_0+i_1+\cdots+i_k=n-k}F_{i_0+1}\prod_{j=1}^{k}F_{i_j},\end{aligned} \tag{2.11}$$

which satisfies the recurrence $Q_{n+1,\ k} = Q_{n,\ k} + Q_{n,\ k-1} + Q_{n-1,\ k}$, with the initial conditions $Q_{0,\ 0} = 1,\ Q_{1,\ 0} = 2,\ Q_{1,\ 1} = 1$.

From (2.10), we can deduce another formula for $Q_{n,\ k}$,

$$\begin{aligned}Q_{n,\ k} &= [x^ny^{n-k}]Q(x,y) = [x^ny^{n-k}]\sum_{r\geq 0}\frac{(xy)^r(1+xy)^{r+1}}{(1-x)^{r+1}}\\ &= [x^ny^{n-k}]\sum_{r,\ m\geq 0}\sum_{i=0}^{r+1}\binom{r+1}{i}\binom{r+m}{m}x^{m+r+i}y^{r+i}\\ &= \sum_{r+i=n-k}\binom{r+1}{i}\binom{r+k}{k}\\ &= \sum_{i=0}^{[(n-k)/2]}\binom{n-k-i+1}{i}\binom{n-i}{k}.\end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we obtain the following identity.

$$\sum_{i_0+i_1+\cdots+i_k=n-k} F_{i_0+1} \prod_{j=1}^{k} F_{i_j} = \sum_{i=0}^{[(n-k)/2]} \binom{n-k-i+1}{i}\binom{n-i}{k}.$$

Notice that $Q_{n,\ 0} = F_{n+1}$, and (2.11) implies $Q_{n,\ k} = P_{n,\ k} + P_{n-1,\ k}$. We can display $\{Q_{n,\ k}\}$ in a triangle

Table 2.2

n/k	0	1	2	3	4	5
0	1					
1	2	1				
2	3	3	1			
3	5	7	4	1		
4	8	15	12	5	1	
5	13	30	31	18	6	1

Theorem 2.3 *Let* $Q^{(i)}_{n\times n} = (Q_{n+k+i,\ k+i})_{0\le k\le n}$, *then*

$$|Q^{(i)}_{n\times n}| = 1. \tag{2.13}$$

Proof. This proof is similar to the proof of Theorem (2.2) so it is omitted. □

Considering the sums and alternating sums of the diagonals of Table 2.2, we obtain the following.

Theorem 2.4 *For any integers* $n \ge 2k \ge 0$, *we get*

$$\sum_{k\ge 0} Q_{n-k,\ k} = 2^n, \tag{2.14}$$

$$\sum_{k\ge 0} (-1)^k Q_{n-k,\ k} = 2 - \delta_{0n}. \tag{2.15}$$

Proof. It suffices to show (2.14). By (2.12) and the Cauchy Residue Theorem, we have

$$\begin{aligned}
\sum_{k\ge 0} Q_{n-k,\ k} &= \sum_{k\ge 0}\sum_{r\ge 0} \binom{r+1}{n-2k-r}\binom{r+k}{k} \\
&= \sum_{k\ge 0}\sum_{r\ge 0} \operatorname*{res}_x \frac{(1+x)^{r+1}}{x^{n-2k-r+1}} \operatorname*{res}_y \frac{(1-y)^{-r-1}}{y^{k+1}} \\
&= \sum_{r\ge 0} \operatorname*{res}_x \frac{(1+x)^{r+1}}{x^{n-r+1}} (1-x^2)^{-r-1} \\
&= \sum_{r\ge 0} \operatorname*{res}_x \frac{x^r(1-x)^{-r-1}}{x^{n+1}} \\
&= \operatorname*{res}_x \frac{(1-2x)^{-1}}{x^{n+1}} = 2^n.
\end{aligned}$$

Then (2.14) holds and (2.15) follows in the same way. □

3. The Generalized Jacobsthal and Jaco-Lucas Polynomials

Let us define a sequence of polynomials $\{J_n(y)\}$ by the recurrence relation

$$J_{n+1}(y) = J_n(y) + (1+y)J_{n-1}(y), \qquad n \geq 1, \tag{3.1}$$

where $J_0(y) = J_1(y) = 1$, which generates the Jacobsthal sequence $\{2J_n\}$ when $y = 1$. Other references related to Jacobsthal and Jaco-Lucas Polynomials, see [1, 5, 6, 9, 10].

Define

$$J_n(y) = \sum_{k=0}^{n} J_{n,\ k} y^k, \quad J(x,y) = \sum_{n\geq 0} J_n(y) x^n. \tag{3.2}$$

By (3.1) and (3.2) it is easy to derive

$$J(x,y) = \frac{1}{1 - x - x^2 - x^2 y}, \tag{3.3}$$

and

$$\begin{aligned} J_{n,\ k} &= [x^n y^k] J(x,y) = [x^n y^k] \sum_{r\geq 0} \frac{x^{2r} y^r}{(1-x-x^2)^{r+1}} \\ &= [x^n y^k] \sum_{r,\ m\geq 0} \sum_{i_0+i_1+\cdots+i_r=m} \prod_{j=0}^{r} F_{i_j} x^{m+2r} y^r \\ &= \sum_{i_0+i_1+\cdots+i_k=n-2k} \prod_{j=0}^{k} F_{i_j}, \end{aligned} \tag{3.4}$$

which satisfies the recurrence $J_{n+1,\ k} = J_{n,\ k} + J_{n-1,\ k} + J_{n-1,\ k-1}$, with the initial conditions $J_{0,\ 0} = J_{1,\ 0} = 1,\ \ J_{1,\ 1} = 0$.

From (3.3), we can deduce another formula for $J_{n,\ k}$,

$$\begin{aligned} J_{n,\ k} &= [x^n y^k] J(x,y) = [x^n y^k] \sum_{r\geq 0} \frac{x^{2r}(1+y)^r}{(1-x)^{r+1}} \\ &= [x^n y^k] \sum_{r,\ m\geq 0} \sum_{i=0}^{r} \binom{r}{i} \binom{r+m}{m} x^{m+2r} y^i \\ &= \sum_{m+2r=n} \binom{r}{k} \binom{r+m}{m} \\ &= \sum_{i=0}^{[n/2]} \binom{n-i}{i} \binom{i}{k}. \end{aligned} \tag{3.5}$$

By (3.4) and (3.5), we have the following interesting identity.

$$\sum_{i_0+i_1+\cdots+i_k=n-2k} \prod_{j=0}^{k} F_{i_j} = \sum_{i=0}^{[n/2]} \binom{n-i}{i} \binom{i}{k}.$$

Note that by (2.4) and (3.4), $J_{n,\ k} = P_{n-k,\ k}$, so $J_{n+1,\ k}$ also counts the number of $(0,1,2)$-sequences of length n with k 2's, but without subsequences $11, 12, 21, 22$. Also $J_{2n,\ k}$ and $J_{2n+1,\ k}$

produce the triangles

Table 3.1

n/k	0	1	2	3	4
0	1				
1	2	1			
2	5	5	1		
3	13	20	9	1	
4	34	71	51	14	1

Table 3.2

n/k	0	1	2	3	4
0	1				
1	3	2			
2	8	10	3		
3	21	38	22	4	
4	55	130	111	40	5

Theorem 3.1 *For any integer $n, k \geq 0$, we have*

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}J_{n+2k+2i,\ k+i} \quad = \quad 1.$$

Proof. This proof is similar to the proof of Theorem (2.2), so it is omitted. □

Now let us define another sequence of polynomials $\{JL_n(y)\}$ by the recurrence relation

$$JL_{n+1}(y) = JL_n(y) + (1+y)JL_{n-1}(y), \qquad n \geq 1, \tag{3.6}$$

where $JL_0(y) = 2,\ JL_1(y) = 1$, which generates the Jaco-Lucas sequence $\{JL_n\}$ when $y = 1$.

Define

$$JL_n(y) = \sum_{k=0}^{n} JL_{n,\ k}y^k, \quad JL(x,y) = \sum_{n\geq 0} JL_n(y)x^n. \tag{3.7}$$

By (3.6) and (3.7) it is easy to deduce

$$JL(x,y) = \frac{2-x}{1-x-x^2-x^2y}, \tag{3.8}$$

and

$$\begin{aligned} JL_{n,\ k} &= [x^ny^k]JL(x,y) = [x^ny^k]\sum_{r\geq 0}\frac{(2-x)x^{2r}y^r}{(1-x-x^2)^{r+1}} \\ &= [x^ny^k]\sum_{r,\ m\geq 0}\ \sum_{i_0+i_1+\cdots+i_r=m} L_{i_0}\prod_{j=1}^{r}F_{i_j}x^{m+2r}y^r \\ &= \sum_{i_0+i_1+\cdots+i_k=n-2k} L_{i_0}\prod_{j=1}^{k}F_{i_j}, \end{aligned} \tag{3.9}$$

which satisfies the recurrence $JL_{n+1,\ k} = JL_{n,\ k} + JL_{n,\ k-1} + JL_{n-1,\ k}$, with the initial conditions $JL_{0,\ 0} = 2,\ JL_{1,\ 0} = 1,\ JL_{1,\ 1} = 0$.

From (3.8), we can deduce another formula for $JL_{n,\ k}$ for $n \geq 1$,

$$\begin{aligned}
JL_{n,\ k} &= [x^n y^k] JL(x,y) = [x^n y^k] \sum_{r\geq 0} \frac{(2-x)x^{2r}(1+y)^r}{(1-x)^{r+1}} \\
&= [x^n y^k] \sum_{r,\ m\geq 0} \sum_{i=0}^{r} \binom{r}{i}\binom{r+m}{m} x^{m+2r} y^i (2-x) \\
&= \sum_{m+2r=n} \binom{2r}{k}\binom{r+m}{m} - \sum_{m+2r+1=n} \binom{r}{k}\binom{r+m}{m} \\
&= \sum_{i=0}^{[n/2]} \left(\binom{n-i}{i} + \binom{n-i-1}{i-1}\right)\binom{i}{k} \\
&= \sum_{i=0}^{[n/2]} \frac{n}{n-i}\binom{n-i}{i}\binom{i}{k}. \qquad (3.10)
\end{aligned}$$

By (3.9) and (3.10), we obtain the following identity for $n \geq 1, k \geq 0$,

$$\sum_{i_0+i_1+\cdots+i_k=n-2k} L_{i_0} \prod_{j=1}^{k} F_{i_j} = \sum_{i=0}^{[n/2]} \frac{n}{n-i}\binom{n-i}{i}\binom{i}{k}.$$

Notice that $JL_{n,\ 0} = L_n$, and $JL_{2n,\ k}, JL_{2n+1,\ k}$ lead to the triangles.

Table 3.3

n/k	0	1	2	3	4
0	2				
1	3	2			
2	7	8	2		
3	18	30	15	2	
4	47	104	80	24	2

Table 3.4

n/k	0	1	2	3	4
0	1				
1	4	3			
2	11	15	5		
3	29	56	35	7	
4	76	189	171	66	9

Theorem 3.2 *For any integer $n, k \geq 0$, we have*

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} JL_{n+2k+2i,\ k+i} = 2.$$

Proof. This proof is similar to the proof of Theorem (2.2) so it is omitted. □

Similar to Theorem (2.1), we can also give a combinatorial interpretation for $JL_{n,\ k}$, that is the following:

Theorem 3.3 *For any integer $n,\ k \geq 0$, $JL_{n,\ k}$ is the number of $(0,1,2)-$sequences of length n with k 2's, but without subsequences 11, 12, 21, 22, where the first and last components of the sequences are considered to be adjacent.*

Proof. Let $T_{n,\ k}$ be the desired number. Consider the last component x_n of such sequences in three cases, i.e., $x_n = 0,\ 1$ or 2, we have

$$T_{n+1,\ k} = J_{n+1,\ k} + J_{n-1,\ k} + J_{n-1,\ k-1}, \qquad (n \geq 1),$$

with the initial values $T_{0,\ 0} = 2,\ T_{1,\ 0} = 1,\ T_{1,\ 1} = 0$.

By (3.3) and (3.8), we have $JL_{n+1,\ k} = 2J_{n+1,\ k} - J_{n,\ k} = J_{n+1,\ k} + J_{n-1,\ k} + J_{n-1,\ k-1}$, and $JL_{0,\ 0} = 2,\ JL_{1,\ 0} = 1,\ JL_{1,\ 1} = 0$, so $JL_{n,\ k}$ must coincide with $T_{n,\ k}$. $\square$

Acknowledgments This work was done under the auspices of the National 973 Project on Mathematical Mechanization and the National Science Foundation of China. The author would like to my advisor Leetsch C. Hsu for helpful comments. Thanks also to the referees for their careful reading of the manuscript and valuable suggestions.

References

[1] G. B. Djordjevic, "Generalized Jacobsthal Polynomials", *The Fibonacci Quarterly* **38.3** (2000): 239-243.

[2] G. P. Egorychev, "Integral Representation and the Computation of Combinatorial Sums", Translations of mathematical monographs, Vol. 59, American mathematical society, (1984).

[3] G. Ferri, M. Faccio and A. D'Amico, "A New Numerical Triangle Showing Links with Fibonacci Numbers", *The Fibonacci Quarterly* **29.4** (1991): 316-320.

[4] G. Ferri, M. Faccio and A. D'Amico, "Fibonacci Numbers and Ladder Network Impedance", *The Fibonacci Quarterly* **30.1** (1992): 62-67.

[5] A. F. Horadam, "Rodriques' Formulas for Jacobsthal-Type Polynomials", *The Fibonacci Quarterly* **35.4** (1997): 361-370.

[6] A. F. Horadam and P. Filipponi, "Derivative Sequences of Jacobsthal and Jacobsthal-Lucas Polynomials", *The Fibonacci Quarterly* **35.4** (1997): 352-357.

[7] R. A. Jeannin, "A Generalization of Morgan-Voyce Polynomials", *The Fibonacci Quarterly* **32.3** (1994): 228-231.

[8] F. J. Papp, "Another Proof of Tepper's Inequality", *Math. Magazine***45** (1972): 119-121.

[9] M. N. S. Swamy, "Some Further Properties of Andre-Jeannin and Their Companion Polynomials", *The Fibonacci Quarterly* **38.2** (2000): 114-122.

[10] M. N. S. Swamy, "A Generalization of Jacobsthal Polynomials", *The Fibonacci Quarterly* **37.2** (1999): 141-144.

[11] Z. W. Trzaska, "Modified Numerical Triangle and the Fibonacci Sequence", *The Fibonacci Quarterly* **29.4** (1991): 316-320.

AMS Classification Numbers: 11B39, 11B83