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Identities from weighted Motzkin paths

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Abstract

Based on a weighted version of the bijection between Dyck paths and 2-Motzkin paths, we find combinatorial interpretations of two identities related to the Narayana polynomials and the Catalan numbers. These interpretations answer two questions posed recently by Coker.

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1. Introduction

In answer to two problems proposed by Coker [5], we find combinatorial interpretations of two identities on the Narayana polynomials and the Catalan numbers, by using a weighted version of the well-known bijection between Dyck paths and 2-Motzkin paths. The Catalan numbers are defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geqslant 0.$$

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The Narayana numbers are defined by

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}, \quad n \geqslant 1,$$

with $N(0,0) = 1$ and $N(0,k) = 1$ for $k \geqslant 1$. The Narayana numbers are listed as sequence $A001263$ in [15], see also [8,13,16,17,22]. The Narayana polynomials are given by

$$\mathcal{N}_n(x) = \sum_{k=0}^{n-1} N(n,k)x^k, \quad n \geqslant 1,$$

which have been studied by Bonin, Shapiro, Simion [2], Coker [5], and Sulanke [18,19].

We will be concerned with the following two combinatorial identities due to Coker [5]. For $n \geqslant 1$,

$$\sum_{k=0}^{n-1} \frac{1}{n}\binom{n}{k}\binom{n}{k+1}x^k = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} x^k (1+x)^{n-2k-1}, \tag{1.1}$$

$$\sum_{k=0}^{n-1} \frac{1}{n}\binom{n}{k}\binom{n}{k+1}x^{2k}(1+x)^{2(n-1-k)} = \sum_{k=0}^{n-1} C_{k+1}\binom{n-1}{k} x^k (1+x)^k. \tag{1.2}$$

The above identities are derived by using generating functions, and Coker proposed the problems of finding combinatorial interpretations of these two identities. Our work was motivated by the work of Chen, Deutsch and Elizalde [4] on plane trees and 2-Motzkin paths. However, our combinatorial proofs of (1.1) and (1.2) in Section 3 are based on the bijection between Dyck paths and 2-Motzkin paths, which was first discovered by Delest and Viennot [6], together with the fact that the numbers of evenly positioned up steps on Dyck paths of length $2n$ are distributed with respect to the Narayana numbers as described in Lemma 3.3.

2. Coker's problems

The aforementioned two identities arose from the study of multiple Dyck paths. Recall that a *multiple Dyck path* is a path that starts at the origin, never runs below the horizontal axis, and uses steps in the set $\{(h,0)\colon h > 0\} \cup \{(0,h)\colon h > 0\}$. Coker [5] proposed the following problems.

Problem 2.1. Find a bijective proof of the identity

$$\sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1}4^{n-k} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} 4^k 5^{n-2k-1}. \tag{2.1}$$

Problem 2.2. Find a combinatorial interpretation of the identity

$$\sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1}x^{2k}(1+x)^{2n-2k} = x^2 \sum_{k=0}^{n-1} C_{k+1}\binom{n-1}{k} x^k (1+x)^k. \tag{2.2}$$

The first identity is a special case of (1.1). Note that identity (1.1) can be derived from the following identity due to Simion and Ullman [14], see also [3]:

$$\frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \sum_{r=0}^{k-1}\binom{n-1}{2r}\binom{n-2r-1}{k-1-r}C_r. \tag{2.3}$$

The identity (1.1) has many consequences as pointed out by Coker [5]. For example, it implies the classical identity of Touchard [20] when $x=1$,

$$C_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} 2^{n-2k-1},$$

and implies the following identity on the little Schröder numbers s_n when $x=2$, see [12,19],

$$s_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} 2^k 3^{n-2k-1}.$$

Coker's interest in the evaluation of $\mathcal{N}_n(t)$ at $t=4$ lies in the fact that $\mathcal{N}_n(4)$ equals the number d_n of multiple Dyck paths of length $2n$. The first few values of d_n for $n \geqslant 0$ are as follows:

$$1,\ 1,\ 5,\ 29,\ 185,\ 1257,\ 8925,\ 65445,$$

which form the sequence $A059231$ in [15]. Coker [5] proved this fact from the well-known interpretation of Narayana numbers as counting Dyck paths of length $2n$ with $k+1$ peaks. The enumeration of multiple Dyck paths has also been studied independently by Sulanke [18] and Woan [21].

Identity (2.2) was established from the enumeration of multiple Dyck paths of length $2n$ with a given number of steps. Let $\lambda_{n,j}$ be the number of multiple Dyck paths of length $2n$ with j steps, and $\mathcal{P}_n(x)$ be the polynomial

$$\mathcal{P}_n(x) = \sum_{j=2}^{2n} \lambda_{n,j} x^j.$$

Coker [5] derived the following formula:

$$\mathcal{P}_n(x) = \sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} x^{2k}(1+x)^{2n-2k},$$

which can be restated as

$$\mathcal{P}_n(x) = x^{2n}\mathcal{N}_n\big((1+x^{-1})^2\big).$$

On the other hand, $\mathcal{P}_n(x)$ can be considered as a variant of the polynomial $\mathcal{R}_n(x)$ which was defined by Denise and Simion [7]. Then (2.2) can be deduced from the formula

$$\mathcal{R}_n(x) = \sum_{k=0}^{n-1} (-1)^k C_{k+1} \binom{n-1}{k} x^k (1-x)^k,$$

and the relation

$$\mathcal{P}_n(x) = x^2 \mathcal{R}_n(-x).$$

The combinatorial interpretations of the above identities will be given in the next section.

3. Lattice path proofs

In this section, we present combinatorial interpretations of (1.1) and (1.2) by using a weighted version of the bijection between Dyck paths and 2-Motzkin paths. In general, for a nonnegative integer c, a *c-Motzkin path* is a lattice path starting at $(0,0)$, ending at $(n,0)$, and never going below the x-axis, with possible steps $(1,1)$, $(1,0)$ and $(1,-1)$, where the level steps, or horizontal steps, can be colored by one of c colors. When $c=1$, we have a common Motzkin path and we use U, D, and H to denote an up step $(1,1)$, a down step $(1,-1)$ and a level step $(1,0)$, respectively. When $c=0$, there are no level steps allowed and such paths reduce to Dyck paths. When $c=2$, a level step may be colored by B or R, where B and R stand for a blue and a red step, respectively. When $c=3$, the level steps are colored with B, R and G, where G denotes the third color green. The *length* of a path is defined to be the number of steps. The notion of 2-Motzkin paths may have originated in the work of Delest and Viennot [6] and has been studied by others, including [1,9].

Let $\mathcal{D}_n$ denote the set of Dyck paths of length $2n$; it is well known that $|\mathcal{D}_n| = C_n$. Let $\mathcal{M}_n$ denote the set of Motzkin paths of length n, and let $\mathcal{CM}_n$ denote the set of 2-Motzkin paths of length n. For a Dyck path $P = p_1 p_2 \ldots p_{2n}$, we say that a step p_i is in an *even position* if i is even. Let $\mathrm{EU}(P)$ denote the number of U steps in even positions on a Dyck path P. From [6,10, 11,13,16,22] one can find that the statistic EU is distributed by the Narayana numbers:

Lemma 3.3. *For $n \geqslant 1$, the number of Dyck paths P of length $2n$ with $\mathrm{EU}(P) = k$ is given by the Narayana number $N(n,k)$.*

Here we recall a well-known bijection between Dyck paths and 2-Motzkin paths, first introduced by Delest and Viennot [6]. Define

$$\Psi : \mathcal{D}_n \to \mathcal{CM}_{n-1},$$

where $P = p_1 p_2 \ldots p_{2n} \in \mathcal{D}_n$ is mapped to $Q = q_1 q_2 \ldots q_{n-1} \in \mathcal{CM}_{n-1}$ such that

$$\begin{aligned} p_{2i} p_{2i+1} &= UU \quad \text{if and only if} \quad q_i = U, \\ &= DD \quad \text{if and only if} \quad q_i = D, \\ &= UD \quad \text{if and only if} \quad q_i = B, \\ &= DU \quad \text{if and only if} \quad q_i = R. \end{aligned}$$

From the above bijection we see that for $n \geqslant 1$, the number of 2-Motzkin paths of length $n-1$ equals the Catalan number C_n.

For a 2-Motzkin path P, we use $\mathrm{UB}(P)$ to denote the total number of U and B steps on P. Then we have the following relation concerning the Narayana numbers and the statistic UB.

Lemma 3.4. *For $n \geqslant 1$, the number of 2-Motzkin paths P of length $n-1$ with $\mathrm{UB}(P) = k$ is given by the Narayana number $N(n,k)$.*

Combinatorial proof of identity (1.1). As usual, the weight of a path is the product of the weights of its steps, and the weight of a set of paths is the sum of the weights of the paths. For the left-hand side of (1.1), let us consider the set $\mathcal{CM}_{n-1}$, where we assign the weight x to each U or B step and the weight 1 to any other step. Then, by Lemma 3.4 the left-hand side equals the weight of $\mathcal{CM}_{n-1}$.

For the right-hand side of (1.1), we consider the weight of the subset of $\mathcal{CM}_{n-1}$ consisting of paths having exactly k up steps. The weight of this subset equals

$$C_k \binom{n-1}{n-1-2k} x^k (1+x)^{n-1-2k},$$

since (i) there are $(1+x)^{n-1-2k}$ ways to arrange the bi-colored level steps among themselves reflecting the weight assignment that a blue step has weight x and a red step has weight 1, (ii) there are $\binom{n-1}{n-1-2k}$ ways to intersperse the level steps in a Dyck path of length $2k$ to form a path of $\mathcal{CM}_{n-1}$, and (iii) there are C_k such Dyck paths. This completes the proof. □

Combinatorial proof of identity (1.2). For the left-hand side of (1.2), if we assign the weight x^2 to each U or B step and the weight $(1+x)^2$ to any other step, then the left-hand side equals the weight of $\mathcal{CM}_{n-1}$.

For the right-hand side, we first let $S(k)$ denote any subset of $\mathcal{CM}_{n-1}$ where each path has k up steps and has the up and down steps in given positions. Since the U's and D's can be matched on any path, and since $x^2 \cdot (1+x)^2 = (x(1+x))^2$, there is no change in the total weight if we reassign the weight $x(1+x)$ to all U and D steps. Thus the weight of $S(k)$ is

$$\big(x(1+x)\big)^{2k}\big(x^2 + (1+x)^2\big)^{n-1-2k},$$

since a blue step has weight x^2 and a red step has weight $(1+x)^2$.

Let $\mathcal{TM}_{n-1}$ denote the set of 3-Motzkin paths of length $n-1$ having level steps B, R, and G. Assign the weight $x(1+x)$ to each of the U, D, B, and R steps and the weight 1 to each G step. Let $S'(k)$ denote any subset of $\mathcal{TM}_{n-1}$ where each path has k up steps and has the up and down steps in given positions. Similarly, the weight of $S'(k)$ equals

$$\big(x(1+x)\big)^{2k}\big(1 + x(1+x) + x(1+x)\big)^{n-1-2k}.$$

Since $S(k)$ and $S'(k)$ have the same weight, it remains to show that the weight of $\mathcal{TM}_{n-1}$ coincides with the right-hand side of (1.2). To construct a path of $\mathcal{TM}_{n-1}$ with $(n-1-k)$ G steps, we may insert the G steps into bi-colored paths of $\mathcal{CM}_k$ where all the U, D, B, and R steps have the same weight $x(1+x)$. Since there are $\binom{n-1}{n-1-k} = \binom{n-1}{k}$ ways to insert the G steps and since $|\mathcal{CM}_k| = C_{k+1}$, the weight of the subset of $\mathcal{TM}_{n-1}$ consisting of paths with

$(n-1-k)$ G steps equals $C_{k+1}\binom{n-1}{k}x^k(1+x)^k$, which is the summand of the right-hand side of (1.2). This completes the proof. □

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