The Statistic “number of udu’s” in Dyck Paths

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ABSTRACT: In the present paper we consider the statistic “number of udu’s” in Dyck paths. The enumeration of Dyck paths according to semilength and various other parameters has been studied in several papers. However, the statistic “number of udu’s” has been considered only recently. Let $\mathcal{D}_n$ denote the set of Dyck paths of semilength n and let $T_{n,\ k}$, $L_{n,\ k}$, $H_{n,\ k}$ and $W_{n,\ k}^{(r)}$ denote the number of Dyck paths in $\mathcal{D}_n$ with k udu’s, with k udu’s at low level, at high level, and at level $r \geq 2$, respectively. We derive their generating functions, their recurrence relations and their explicit formulas. A new setting counted by Motzkin numbers is also obtained. Several combinatorial identities are given and other identities are conjectured.
KEYWORDS: Dyck paths, Motzkin numbers, Identities

1. Introduction

A *Dyck path* is a lattice path in the first quadrant, which begins at the origin $(0,0)$, ends at $(2n,0)$ and consists of steps $(1,1)$ (called rises), $(1,-1)$ (called falls). We can encode each rise by the letter u (for up), each fall by the letter d (for down), obtaining the encoding of Dyck path by a so-called *Dyck word*. We will refer to n as the *semilength* of the path. Let $\mathcal{D}_n$ denote the set of Dyck paths of semilength n and let ε denote the empty path. It is well known that the cardinality of $\mathcal{D}_n$ is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, having generating function $C(x) = \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$, which satisfies the relation

$$xC(x)^2 - C(x) + 1 = 0. \tag{1.1}$$

A *peak* in a Dyck path is an occurrence of ud, a *valley* is an occurrence of du. By the level of a peak or of a valley we mean the level of the intersection point of its two steps. By a *return step* we mean a d step ending at level zero. Dyck paths that have exactly one return step are said to be *primitive* [14]. If α and β are Dyck path, then we define $\hat{\alpha}= u\alpha d$ as the elevation of α and $\alpha\beta$ as the concatenation of α and β.

A *Fine path* is a Dyck path with no peaks at level one. They are counted by the Fine numbers F_n [3], having generating function $F(x) = \sum_{n\geq 0} F_n x^n = \frac{1-\sqrt{1-4x}}{x(3-\sqrt{1-4x})}$, which satisfies the relation

$$F(x) \;=\; \frac{C(x)}{1+xC(x)} = \frac{1}{1+x-xC(x)}. \tag{1.2}$$

As is well known, the Motzkin numbers M_n [1, 5, 7, 8] and the Catalan numbers are intimately related [5]; namely,

$$M_n = \sum_{k\geq 0}^{[n/2]} \binom{n}{2k} C_k, \qquad C_{n+1} = \sum_{k\geq 0}^{n} \binom{n}{k} M_{n-k}, \tag{1.3}$$

which imply

$$C(x) = 1 + \frac{x}{1-x} M(\frac{x}{1-x}), \tag{1.4}$$

where $M(x) = \sum_{n\geq 0} M_n x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$, which satisfies the relation

$$x^2 M(x)^2 + (x-1)M(x) + 1 = 0. \tag{1.5}$$

Note that (1.4) can be derived also by a straightforward verification from (1.1) and (1.5).

The enumeration of Dyck paths according to semilength and various other parameters has been studied in several papers [2, 9, 10, 11, 22]. However, the statistic "number of *udu*'s" has been considered only recently [13](it is also mentioned briefly in [4]).

The organization of this paper is as follows. In the next Section we find an explicit formula for the number of Dyck paths in $\mathcal{D}_n$ with k *udu*'s, and obtain a new combinatorial interpretation of the second expression of (1.3). In Section 3 we study Dyck paths in $\mathcal{D}_n$ with k *udu*'s at low and high level. We obtain the explicit expressions for these two cases and give a new setting counted by Motzkin numbers. In Section 4 we study Dyck paths in $\mathcal{D}_n$ with k *udu*'s at level $r \geq 2$. We derive the corresponding recurrence relations and generating functions. It turns out that the Fibonacci-like polynomials [21] occur in the latter. We also conjecture two families of interesting identities.

2. Enumeration of Dyck paths according to number of *udu*'s

Let $\mathcal{T}_{n,\ k}$ denote the set of all Dyck paths in $\mathcal{D}_n$ with k *udu*'s and let $T_{n,\ k}$ be the cardinality of the set $\mathcal{T}_{n,\ k}$. In this section we find an explicit formula for $T_{n,\ k}$ and obtain a new combinatorial interpretation of the second identity in (1.3). Namely,

Theorem 2.1 *For any integer $n,\ k \geq 0$,*

$$T_{n+1,\ k} = \binom{n}{k} M_{n-k}.$$

Proof. Consider any Dyck path D in $\mathcal{T}_{n+1,\ 0}$ ($n \geq 2$), i.e., a Dyck path with no *udu* (or *udu*-avoiding). Let $D = uPdQ$ be the first return decomposition of D. Then, both P and Q are *udu*-avoiding and, moreover, if $P = \epsilon$, then $Q = \epsilon$. Consequently, for $n \geq 2$, we cannot have $P = \epsilon$. Then

$$T_{n+1,\ 0} = T_{n,\ 0} + \sum_{r=1}^{n-1} T_{r,\ 0} T_{n-r,\ 0},$$

and $T_{0,\ 0} = T_{1,\ 0} = T_{2,\ 0} = 1$. Since $T_{n+1,\ 0}$ and the Motzkin numbers M_n satisfy the same recurrence relation with the same initial condition, we obtain

$$T_{n+1,\ 0} = M_n. \tag{2.1}$$

Now, consider a Dyck path $D \in \mathcal{T}_{n-k+1,\ 0}$ and consider the $n-k+1$ endpoints of its $n-k+1$ rise steps. From these we select k points, repetitions allowed, and at each of the selected points we insert a valley du. Clearly, we obtain a member of $\mathcal{T}_{n+1,\ k}$. Conversely, every Dyck path $D \in \mathcal{T}_{n+1,\ k}$ can be obtained in this manner from a member of $\mathcal{T}_{n-k+1,\ 0}$(delete the k valleys du that follow immediately a u step). Since the number of r-subsets of $[n]$ with repetitions allowed is $\binom{n+r-1}{r}$, we have $T_{n+1,\ k} = \binom{n}{k} T_{n-k+1,\ 0}$ or, taking into account (2.1),

$$T_{n+1,\ k} = \binom{n}{k} M_{n-k}.$$

This is the desired result. $\square$

The first few values of $T_{n,\ k}$ are given in Table 1.

Table 1: The first few values of $T_{n,\ k}$

$n\backslash k$	0	1	2	3	4	5	6
1	1						
2	1	1					
3	2	2	1				
4	4	6	3	1			
5	9	16	12	4	1		
6	21	45	40	20	5	1	
7	51	126	135	80	30	6	1

Theorem 2.2 *The bivariate generating function $T(x,z)$ for Dyck paths according to the number of udu's (marked by x), and semilength (marked by z) is given by*

$$T(x,z) = \frac{1+z-xz-\sqrt{1-2(x+1)z+(x^2+2x-3)z^2}}{2z} \tag{2.2}$$

$$= 1+\frac{z}{1-xz}M(\frac{z}{1-xz}) = C(\frac{z}{1+z-xz}). \tag{2.3}$$

Proof. An equation for $T(x,z)$ is obtained from the "first return decomposition" of a Dyck path D: $D = uPdQ$, where P, Q are Dyck paths. The *udu*'s of P and Q are *udu*'s also in D; in addition, D has a *udu* whenever $P = \epsilon$ and $Q \neq \epsilon$. Making use of the so-called symbolic method (for details see [16]), one obtain

$$T(x,z) = 1+z[(T(x,z)-1)^2+(T(x,z)-1)+x(T(x,z)-1)+1]. \tag{2.4}$$

Solving this for $T(x,z)$, we obtain (2.2). Rewriting the above equation as

$$T(x,z) = 1+\frac{z}{1+z-xz}T(x,z)^2,$$

and taking into account (1.4), we obtain (2.3) at once. $\square$

Remark 1 Applying the Lagrange inversion theorem to (2.4), we reobtain $[x^k z^{n+1}]T(x,z) = \binom{n}{k}M_{n-k}$.

Setting $x = -1$ in (2.2) we have

$$T(-1,z) = 1+\sum_{n\geq 0}\sum_{k=0}^{n}(-1)^k T_{n+1,\ k} z^{n+1} \tag{2.5}$$

$$= 1+\frac{1-\sqrt{1-4z^2}}{2z} = 1+\sum_{n\geq 0} C_n z^{2n+1}. \tag{2.6}$$

Comparing the coefficients of z^n in (2.5) and (2.6), we obtain

Theorem 2.3 *For any positive integer n,*

$$\sum_{k\ even} T_{2n,\ k} = \sum_{k\ odd} T_{2n,\ k},$$

$$\sum_{k\ even} T_{2n-1,\ k} = \sum_{k\ odd} T_{2n-1,\ k} + C_{n-1}.$$

3. Enumeration of Dyck paths according to number of *udu*'s at low and high levels

We say that a *udu* is at low (high) level if its peak is at level (greater than) one. In this section we consider the following statistics on Dyck paths: number of low(high) *udu*'s. Let $L_{n,\ k}$ ($H_{n,\ k}$) denote the number of Dyck paths in $\mathcal{D}_n$ with k *udu*'s at low (high) level. Define $L_{0,\ 0} = H_{0,\ 0} = 1$; clearly, $L_{n,\ k} = H_{n,\ k} = 0$ for $k \geq n > 0$.

Consider any Dyck path $D = uPdQ$, where $P,\ Q$ are Dyck paths. The $udu's$ of P and Q are $udu's$ also in D; in addition, D has a *udu* whenever P is the empty path and Q is any nonempty path. Let $G(x, y, z)$ be the trivariate generating function, where x and y mark the number of *udu* at low and high level, respectively, and z marks the semilength of the Dyck path. Then $G(x, x, z) = T(x, z)$, the generating function of $T_{n,\ k}$. Making use of the so-called symbolic method, $G(x, y, z)$ should satisfy the equation

$$G(x, y, z) = 1 + z[(T(y, z) - 1)(G(x, y, z) - 1) + (T(y, z) - 1) + x(G(x, y, z) - 1) + 1],$$

from where, we obtain

$$G(x, y, z) = \frac{1 + z - xz}{1 + z - xz - zT(y, z)}. \tag{3.1}$$

Setting $y = x$ in (3.1) and solving this for $T(x, z)$, we obtain (2.2) again.

Define the generating function $L(x, z)$ of the $L_{n,\ k}$'s by

$$L_k(z) = \sum_{n \geq k} L_{n,\ k} z^n, \quad L(x, z) = \sum_{k \geq 0} L_k(z) x^k.$$

Setting $y = 1$ in $G(x, y, z)$ (i.e. we consider only the number of *udu* at low level), we obtain

$$L(x, z) = G(x, 1, z) = \frac{1 + z - xz}{1 + z - xz - zT(1, z)}.$$

Taking into account $T(1, z) = C(z)$ and (1.2), we obtain

$$L(x, z) = 1 + \frac{zC(z)}{1 + z - zC(z) - xz} = 1 + \frac{zC(z)F(z)}{1 - xzF(z)}.$$

Thus

$$L_k(z) = [x^k]L(x, z) = \delta_{0k} + z^{k+1}F^{k+1}(z)C(z), \tag{3.2}$$

where δ_{0k} is the Kronecker symbol.

Making use of the known series expansion of $C^p(z)F^q(z)$ [2, Eq.(B.7)], we find an explicit expression for $L_{n,\ k}$; namely,

Theorem 3.1 *For any integers $n > k \geq 0$, we have*

$$L_{n,\ k} = [z^n]L_k(z) = \sum_{i \geq 0} \frac{2i + 1}{n - k} \binom{k + i}{k} \binom{2n - 2k - 2i - 2}{n - k - 1}.$$

The $L_{n,\ k}$'s form a Riordan array, namely $(zF(z)C(z), F(z))$ [12, 18]. The first few values of $L_{n,\ k}$ are given in Table 2.

Table 2: The first few values of $L_{n,\ k}$

$n\backslash k$	0	1	2	3	4	5	6
1	1						
2	1	1					
3	3	1	1				
4	8	4	1	1			
5	24	11	5	1	1		
6	75	35	14	6	1	1	
7	243	113	47	17	7	1	1

In the special case $k = 0$, from (3.2) and the easily verifiable identity $zF(z)C(z) = C(z) - F(z)$, we have

$$L_0(z) \quad = \quad 1 + C(z) - F(z) = (1+z)F(z),$$

from where

$$L_{n,\ 0} = F_n + F_{n-1} = C_n - F_n, \quad (n \geq 1).$$

Now considering the diagonal difference in the Riordan array, we have the following identity.

Theorem 3.2 *For any integers $n, k \geq 0$,*

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}L_{m+k+1+i,\ k+i} = \begin{cases} 1 & \text{if} \quad m = 2n, \\ 2n+1 & \text{if} \quad m = 2n+1. \end{cases}$$

Proof. By the Cauchy Residue Theorem (for details see [6]), we have

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}L_{m+k+1+i,\ k+i} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m}(-1)^{n-i}\binom{n}{i}\binom{k+i+j}{j}\frac{2j+1}{m+1}\binom{2m-2j}{m} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{\infty}\underset{u}{res}\frac{(u-1)^n}{u^{i+1}}\underset{v}{res}\frac{(1-v)^{-k-i-1}}{v^{j+1}}\underset{w}{res}\frac{(1-2w)(1-w)^{-m-2}}{w^{m-2j+1}} \\
&= \sum_{i=0}^{\infty}\underset{w}{res}\frac{(1-2w)(1-w)^{-m-2}}{w^{m+1}}(1-w^2)^{-k-i-1}\underset{u}{res}\frac{(u-1)^n}{u^{i+1}} \\
&= \underset{w}{res}\frac{(1-2w)(1-w)^{-m-2}}{w^{m-2n+1}}(1-w^2)^{-n-k-1} \\
&= \begin{cases} 1 & \text{if} \quad m = 2n, \\ -\binom{-2n-3}{1} - 2 & \text{if} \quad m = 2n+1. \end{cases} \\
&= \begin{cases} 1 & \text{if} \quad m = 2n, \\ 2n+1 & \text{if} \quad m = 2n+1. \end{cases}
\end{aligned}$$

Hence this result holds. □

Several interesting identities involving some numbers related to Dyck paths have been found by the same method [20].

Define the generating function $H(y, z)$ of the $H_{n,\ k}$'s by

$$H_k(z) = \sum_{n \geq k} H_{n,\ k} z^n, \quad H(y, z) = \sum_{k \geq 0} H_k(z) y^k.$$

Setting $x = 1$ in $G(x, y, z)$ (i.e. we consider only the number of *udu*'s at high level), we obtain

$$H(y, z) = G(1, y, z) = \frac{1}{1 - zT(y, z)},$$

and now from $H(y, z) = 1 + zT(y, z)H(y, z)$ we can derive the recurrence of $H_{n,\ k}$ and $H_k(z)$,

$$H_{n+1,\ k} = \sum_{j=0}^{k} \sum_{i=j}^{n-2k+2j} T_{i,\ j} H_{n-i,\ k-j}, \tag{3.3}$$

and

$$H_k(z) = \begin{cases} M(z) & \text{if } \ k = 0, \\ zM(z)\left(\sum_{i=0}^{k-1} H_i(z) T^*_{k-i}(z)\right) & \text{if } \ k \geq 1, \end{cases} \tag{3.4}$$

where $T^*_k(z)$ is the column generating function of the $T_{n,\ k}$'s, namely

$$T^*_k(z) = \sum_{n \geq 0} \binom{n+k}{k} M_n z^{n+k+1} = \frac{z^{k+1}}{k!} (z^k M(z))^{\langle k \rangle}, \tag{3.5}$$

and $f(z)^{\langle k \rangle}$ denotes the k-th order derivative of $f(z)$.

In the special case $k = 0$ in (3.4), we obtain the explicit formula

$$H_{n,\ 0} \quad = \quad M_n. \tag{3.6}$$

Remark 2 In [5], R. Donaghey and L. W. Shapiro gave 14 representative settings for Motzkin numbers. Other settings can be found in [19]. In [17] J. M. Sen also gave another one implicitly, namely (2.1). However, we notice that (3.6) is an entirely new setting.

In the special case $k = 1$, since from (3.4) and (3.5) one can express both $T^*_1(z)$ and $H_1(z)$ in terms of $M(z)$, it is straightforward to show that

$$zH_1(z) \quad = \quad T^*_1(z) + (z - 1)M(z) + 1.$$

Now taking into account that $[z^{n+1}]T^*_1(z) = nM_{n-1}$, we obtain the explicit formula

$$H_{n,\ 1} = nM_{n-1} + M_n - M_{n+1}, \quad (n \geq 1). \tag{3.7}$$

More generally, we obtain

Theorem 3.3 *For any integers $n, k \geq 0$,*

$$H_{n,\ k} = \delta_{0k} + \sum_{r=k}^{n-2} \sum_{j=0}^{n-r-2} (-1)^{r+k} \binom{r}{k} \binom{j+r}{r} B_{n-r-1,\ j+1}, \tag{3.8}$$

where $B_{n,\ i} = \frac{n-i+1}{n+1}\binom{n+i}{i}$ stands for the well-known sequence of ballot numbers.

Proof.

$$\begin{aligned}
H_{n,\,k} &= [z^n y^k]\frac{1}{1-zT(y,z)} = [z^n y^k]\{1+\sum_{i\geq 0} z^{i+1}T(y,z)^{i+1}\} \\
&= [z^n y^k]\{1+\sum_{i\geq 0} z^{i+1}C(\frac{z}{1+z-yz})^{i+1}\} && \text{by (2.3)} \\
&= [z^n y^k]\{1+\sum_{i\geq 0} z^{i+1}\sum_{j\geq 0}\frac{i+1}{i+j+1}\binom{2j+i}{j}\frac{z^j}{(1+z-yz)^j}\} && \text{by } [2, Eq.(B.7)] \\
&= [z^n y^k]\frac{1}{1-z} + [z^n y^k]\sum_{i\geq 0} z^{i+1}\sum_{j\geq 0}\frac{i+1}{i+j+2}\binom{2j+i+2}{j+1}\frac{z^{j+1}}{(1+z-yz)^{j+1}} \\
&= \delta_{0k} + [z^n y^k]\sum_{i\geq 0}\sum_{j\geq 0}\frac{i+1}{i+j+2}\binom{2j+i+2}{j+1}\sum_{r\geq 0}\binom{-j-1}{r}\sum_{m=0}^{r}(-1)^m\binom{r}{m}y^m z^{i+j+r+2} \\
&= \delta_{0k} + [z^n y^k]\sum_{i,j,m\geq 0}\sum_{r\geq m}(-1)^{m+r}\binom{r}{m}\binom{j+r}{r}\frac{i+1}{i+j+2}\binom{2j+i+2}{j+1}y^m z^{i+j+r+2} \\
&= \delta_{0k} + \sum_{r=k}^{n-2}\sum_{i+j=n-r-2}(-1)^{k+r}\binom{r}{k}\binom{j+r}{r}\frac{i+1}{i+j+2}\binom{2j+i+2}{j+1} \\
&= \delta_{0k} + \sum_{r=k}^{n-2}\sum_{j=0}^{n-r-2}(-1)^{k+r}\binom{r}{k}\binom{j+r}{r}\frac{n-r-j-1}{n-r}\binom{n-r+j}{j+1} \\
&= \delta_{0k} + \sum_{r=k}^{n-2}\sum_{j=0}^{n-r-2}(-1)^{k+r}\binom{r}{k}\binom{j+r}{r}B_{n-r-1,\,j+1}.
\end{aligned}$$

This is the desired result. □

The first few values of $H_{n,\,k}$ are given in Table 3.

Table 3: The first few values of $H_{n,\,k}$

$n\backslash k$	0	1	2	3	4	5
1	1					
2	2					
3	4	1				
4	9	4	1			
5	21	15	5	1		
6	51	50	24	6	1	
7	127	161	98	35	7	1

Taking into account (3.6), (3.7) and (3.8), we have

Corollary 3.4 *For any integer $n \geq 1$,*

$$\begin{aligned}
\sum_{r=0}^{n-2}\sum_{j=0}^{n-r-2}(-1)^r\binom{j+r}{r}B_{n-r-1,\,j+1} &= M_n - 1, \\
\sum_{r=1}^{n-2}\sum_{j=0}^{n-r-2}(-1)^{r-1}r\binom{j+r}{r}B_{n-r-1,\,j+1} &= nM_{n-1} + M_n - M_{n+1}.
\end{aligned}$$

Now considering the diagonal difference in Table 3, we have the following identity.

Theorem 3.5 *For any integers $n \geq 0, t \geq 1$, we have*

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}H_{n+t+i+2,\ t+i} = M_n.$$

Proof. Recall that the well-known Tepper identity [6, 15] states as follows

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}(x-i)^m = \begin{cases} 0 & \text{if } \ 0 \leq m < n, \\ n! & \text{if } \ m = n. \end{cases} \tag{3.9}$$

Then we have

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}H_{n+t+i+2,\ t+i} \\
&= \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\sum_{r=t+i}^{n+t+i}\sum_{j=0}^{n+t+i-r}(-1)^{r+t+i}\binom{r}{t+i}\binom{j+r}{j}B_{n+t+i+1-r,\ j+1} \\
&= \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\sum_{r=0}^{n}\sum_{j=0}^{n-r}(-1)^{r}\binom{r+t+i}{t+i}\binom{j+t+i+r}{j}B_{n-r+1,\ j+1} \\
&= \sum_{r=0}^{n}(-1)^{n-r}\sum_{j=0}^{r}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\binom{n-r+t+i}{t+i}\binom{j+t+i+n-r}{j}B_{r+1,\ j+1} \\
&= \sum_{r=0}^{n}\frac{(-1)^{n-r}}{(n-r)!}\sum_{j=0}^{r}\frac{1}{j!}B_{r+1,\ j+1}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\frac{(t+i+j+n-r)!}{(t+i)!} \\
&= \sum_{r=0}^{n}\frac{(-1)^{n-r}}{(n-r)!}\frac{n!}{r!}B_{r+1,\ r+1} \qquad \text{by (3.9)} \\
&= \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}C_{r+1} \\
&= M_n,
\end{aligned}$$

where the last equality follows by applying the Möbius inversion to the second identity of (1.3). Hence we complete the proof. □

4. Enumeration of Dyck paths according to number of udu's at level $r \geq 2$

In this section we study Dyck paths with k udu's at level $r \geq 2$. We say that a udu in a Dyck path is at level r if its peak is at level r. Let $W_{n,\ k}^{(r)}$ denote the number of Dyck paths in $\mathcal{D}_n$ with k udu's at level r. Define $W_{0,\ 0}^{(r)} = 1$. Obviously, $W_{n,\ k}^{(r)} = 0$ for $n \leq k + r - 1$ and $k \neq 0$; $W_{n,\ k}^{(1)} = L_{n,\ k}$ and $W_{n,\ 0}^{(r)} = C_n$ for $1 \leq n \leq r$.

For the investigation of the number of $udu's$ at level r, we again utilize the fruitful symbolic method. Similar to the analysis of Section 3, let $W^{(r)}(x, z)$ be the corresponding generating function, where x marks the number of udu at level r, and z marks the semilength of the Dyck path. Obviously $W^{(1)}(x,z) = L(x,z) = \frac{1+z-xz}{1+z-xz-zC(z)}$, and "the first return decomposition" yields

$$W^{(r)}(x,z) = 1 + zW^{(r-1)}(x,z)W^{(r)}(x,z), \tag{4.1}$$

leading to the recurrence relation

$$W^{(r)}(x,z) = \frac{1}{1 - zW^{(r-1)}(x,z)}.$$

Making use of the initial value $W^{(1)}(x,z)$, by induction on r, we obtain

Theorem 4.1 *For any integer $r \geq 2$,*

$$W^{(r)}(x,z) = \frac{(1+z-xz)f_{r-1}(z) - zf_{r-2}(z)C(z)}{(1+z-xz)f_r(z) - zf_{r-1}(z)C(z)},$$

where $f_{i+1}(z) = f_i(z) - zf_{i-1}(z)$ with $f_0(z) = f_1(z) = 1$ is a Fibonacci-like polynomial.

For example for $r = 3$ we obtain

$$W^{(3)}(x,z) = \frac{1 - 2z^2 - 2xz + 2xz^2 + \sqrt{1-4z}}{1 - z - 4z^2 - 2xz + 4xz^2 + (1-z)\sqrt{1-4z}}.$$

Now define the column generating function for $W^{(r)}_{n,\ k}$ as,

$$W^{(r)}_k(z) \quad = \quad \sum_{n \geq k} W^{(r)}_{n,\ k} z^n.$$

By (4.1), we can derive $W^{(r)}_k(z)$ and the recurrence for $W^{(r)}_{n,\ k}$, that is,

$$W^{(r)}_{n+1,\ k} = \sum_{j=0}^{k} \sum_{i=j}^{n-1} W^{(r-1)}_{i,\ j} W^{(r)}_{n-i,\ k-j},$$

and

$$W^{(r)}_k(z) = \begin{cases} \dfrac{1}{1 - zW^{(r-1)}_0(z)} & \text{if } k = 0, \\ zW^{(r)}_0(z)\left(\sum_{j=0}^{k} W^{(r)}_j(z) W^{(r-1)}_{k-j}(z)\right) & \text{if } k \geq 1, \end{cases} \tag{4.2}$$

where, by definition, $W^{(1)}_0(z) = L_0(z)$.

From (4.2) we can compute the expression of $W^{(r)}_k(z)$ for fixed k and r. Unfortunately, the explicit expression for $W^{(r)}_{n,\ k}$ seems much more complicated. But we conjecture the following interesting diagonal difference identities for any integer $n \geq 0$, $k \geq 0$ and $r \geq 2$:

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} W^{(2)}_{m+k+2+i,\ k+i} = \begin{cases} 2^n & \text{if } m = 2n, \\ 2^n(2n+3) & \text{if } m = 2n+1, \end{cases} \tag{4.3}$$

and for $r \geq 3$

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} W^{(r)}_{m+k+2+i,\ k+i} = \begin{cases} 2^n & \text{if } m = 2n, \\ 2^n(\frac{5}{2}n + 2r - 1) & \text{if } m = 2n+1. \end{cases} \tag{4.4}$$

We have checked by Maple that (4.3) and (4.4) hold for $r \leq 10$, $n \leq 10$ and $k \leq 5$. $\square$

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