# SYMPLECTIC GRAPHS AND THEIR AUTOMORPHISMS

ZHONGMING TANG AND ZHE-XIAN WAN

Abstract. The general symplectic graph $Sp(2\nu, q)$ is introduced. It is shown that $Sp(2\nu, q)$ is strongly regular. Its parameters are computed, its chromatic number and group of graph automorphisms are also determined.

## 1. Introduction

Let $\mathbb{F}_q$ be a finite field of any characteristic and $\nu \geq 1$ an integer. Let

$$\mathbb{F}_q^{(2\nu)} = \{(a_1, \ldots, a_{2\nu}) : a_i \in \mathbb{F}_q, i = 1, \ldots, 2\nu\}.$$

be the $2\nu$-dimensional row vector space over $\mathbb{F}_q$. For any $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_q^{(2\nu)}$, we denote the subspace of $\mathbb{F}_q^{(2\nu)}$ generated by $\alpha_1, \ldots, \alpha_n$ by $[\alpha_1, \ldots, \alpha_n]$. Thus, if $\alpha \neq 0 \in \mathbb{F}_q^{(2\nu)}$ then $[\alpha]$ is an one dimensional subspace of $\mathbb{F}_q^{(2\nu)}$ and $[\alpha] = [k\alpha]$ for any $k \in \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$.

Let $K$ be a $2\nu \times 2\nu$ nonsingular alternate matrix over $\mathbb{F}_q$. The *symplectic graph* relative to $K$ over $\mathbb{F}_q$ is the graph with the set of one dimensional subspaces of $\mathbb{F}_q^{(2\nu)}$ as its vertex set and with the adjacency defined by

$$[\alpha] \sim [\beta] \text{ if and only if } \alpha K {}^t\beta \neq 0, \text{ for any } \alpha \neq 0, \beta \neq 0 \in \mathbb{F}_q^{(2\nu)},$$

where $[\alpha] \sim [\beta]$ means that $[\alpha]$ and $[\beta]$ are adjacent. Since any two $2\nu \times 2\nu$ nonsingular alternate matrices over $\mathbb{F}_q$ are cogredient, any two symplectic graphs relative to two different $2\nu \times 2\nu$ nonsingular alternate matrices over $\mathbb{F}_q$ are isomorphic. Thus we can assume that

$$K = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}_{2\nu \times 2\nu}.$$

and consider only the symplectic graph relative to the above $K$ over $\mathbb{F}_q$, which will be denoted by $Sp(2\nu, q)$.

When $q = 2$, the special case $Sp(2\nu, 2)$ of the graph $Sp(2\nu, q)$ was studied previously by Rotman [4], Rotman and Weichsel [5], Godsil and Royle [2, 3], etc. In the present paper we study the general case $Sp(2\nu, q)$. In Section 2, we show that

*Key words and phrases.* symplectic graphs, chromatic numbers, automorphisms.
Both Authors are Supported by the National Natural Science Foundation of China.

$Sp(2\nu, q)$ is strongly regular and compute its parameters. We also prove that the chromatic number of $Sp(2\nu, q)$ is $q^\nu + 1$. Section 3 is devoted to discuss the group of automorphisms $\mathrm{Aut}(Sp(2\nu, q))$ of the graph. The structure of this group depends on $q$ and $\nu$. When $q = 2$, $\mathrm{Aut}(Sp(2\nu, 2))$ is isomorphic to the symplectic group of degree $2\nu$ over $\mathbb{F}_2$. When $q > 2$, $\mathrm{Aut}(Sp(2\nu, q))$ is the product of two subgroups which are identified clearly (cf. Theorem 3.4).

## 2. Strongly Regularity and Chromatic Numbers of Symplectic Graphs

For any subspace $V$ of $\mathbb{F}_q^{(2\nu)}$, we denote the subspace of $\mathbb{F}_q^{(2\nu)}$ formed by all $\beta \in \mathbb{F}_q^{(2\nu)}$ such that $\alpha K {}^t\beta = 0$ for all $\alpha \in V$ by $V^\perp$. Then $[\alpha] \sim [\beta]$ if and only if $\beta \notin [\alpha]^\perp$.

Denote the vertex set of the graph $Sp(2\nu, q)$ by $V(Sp(2\nu, q))$. We first show that $Sp(2\nu, q)$ is strongly regular.

**Theorem 2.1.** *$Sp(2\nu, q)$ is a strongly regular graph with parameters*

$$\left(\frac{q^{2\nu}-1}{q-1}, q^{2\nu-1}, q^{2\nu-2}(q-1), q^{2\nu-2}(q-1)\right)$$

*and eigenvalues $q^{2\nu-1}, q^{\nu-1}$ and $-q^{\nu-1}$.*

**Proof**. As $|\mathbb{F}_q^{(2\nu)}| = q^{2\nu}$, it follows that $|V(Sp(2\nu, q))| = \frac{q^{2\nu}-1}{q-1}$. For any $[\alpha] \in V(Sp(2\nu, q))$, since $\dim([\alpha]^\perp) = 2\nu - 1$, we see that the degree of $[\alpha]$ which is just the number of one dimensional subspaces $[\beta]$ such that $\beta \notin [\alpha]^\perp$, is $\frac{q^{2\nu}-q^{2\nu-1}}{q-1} = q^{2\nu-1}$.

Let $[\alpha], [\beta]$ be any two different vertices of $Sp(2\nu, q)$ which are adjacent with each other or not. Then $\dim([\alpha, \beta]^\perp) = 2\nu - 2$. Note that a vertex $[\gamma]$ is adjacent with both $[\alpha]$ and $[\beta]$ is equivalent to that $\gamma \notin [\alpha]^\perp \cup [\beta]^\perp$. But

$$|[\alpha]^\perp \cup [\beta]^\perp| = |[\alpha]^\perp| + |[\beta]^\perp| - |[\alpha, \beta]^\perp|.$$

Hence the number of vertices which are adjacent with both $[\alpha]$ and $[\beta]$ is $\frac{q^{2\nu}-2q^{2\nu-1}+q^{2\nu-2}}{q-1}$ $= q^{2\nu-2}(q-1)$. Therefore $Sp(2\nu, q)$ is a strongly regular graph with parameter

$$\left(\frac{q^{2\nu}-1}{q-1}, q^{2\nu-1}, q^{2\nu-2}(q-1), q^{2\nu-2}(q-1)\right).$$

By the same arguments as in [3, Section 10.2], we get that the eigenvalues of $Sp(2\nu, q)$ are $q^{2\nu-1}, q^{\nu-1}$ and $-q^{\nu-1}$. □

Let $n \geq 2$. We say that a graph $X$ is *$n$-partite* if there are subsets $X_1, \ldots, X_n$ of the vertex set $V(X)$ of $X$ such that $V(X) = X_1 \cup \cdots \cup X_n$, where $X_i \cap X_j = \emptyset$ for all $i \neq j$, and that there is no edge of $X$ joining two vertices of the same subset. We are going to show that $Sp(2\nu, q)$ is $(q^\nu + 1)$-partite. We need some results about subspaces of $\mathbb{F}_q^{(2\nu)}$. A subspace $V$ of $\mathbb{F}_q^{(2\nu)}$ is called *totally isotropic* if $V \subseteq V^\perp$. Then totally isotropic subspaces of $\mathbb{F}_q^{(2\nu)}$ are of dimension $\leq \nu$ and there exist totally isotropic subspaces of dimension $\nu$ which are called *maximal totally isotropic subspaces*, cf. [6, Corollary 3.8].

The following lemma is due to Dye[1].

**Lemma 2.2.** *There exist maximal totally isotropic subspaces $V_i$, $i = 1, \ldots, q^\nu + 1$, of $\mathbb{F}_q^{(2\nu)}$ such that*

$$\mathbb{F}_q^{(2\nu)} = V_1 \cup \cdots \cup V_{q^\nu+1},$$

*where $V_i \cap V_j = \{0\}$ for all $i \neq j$.*

**Proposition 2.3.** *$Sp(2\nu, q)$ is $(q^\nu + 1)$-partite. That is, there exist subsets $X_1, \ldots, X_{q^\nu+1}$ of $V(Sp(2\nu, q))$ such that*

$$V(Sp(2\nu, q)) = X_1 \cup \cdots \cup X_{q^\nu+1},$$

*where $X_i \cap X_j = \emptyset$ for all $i \neq j$, and there is no edge of $Sp(2\nu, q)$ joining two vertices of the same subset. Moreover, the subsets $X_1, \ldots, X_{q^\nu+1}$ can be so chosen that for any two disinct indices $i$ and $j$, every $\alpha \in X_i$ is adjacent with exactly $q^{\nu-1}$ vertices in $X_j$.*

**Proof**. Let $\mathbb{F}_q^{(2\nu)} = V_1 \cup \cdots \cup V_{q^\nu+1}$ as in 2.2. Set $X_i = \{[\alpha] : \alpha \neq 0 \in V_i\}$, $i = 1, \ldots, q^\nu + 1$. Then

$$V(Sp(2\nu, q)) = X_1 \cup \cdots \cup X_{q^\nu+1},\ X_i \cap X_j = \emptyset,\ \text{for all}\ \ i \neq j.$$

As $V_i$ is totally isotropic, we see that there is no edge joining any two vertices in $X_i$. Thus $Sp(2\nu, q)$ is $(q^\nu + 1)$-partite. For any $i \neq j$, let $[\alpha] \in X_i$. Since $V_j$ is maximal totally isotropic of dimension $\nu$, it follows that $\alpha \notin V_j = V_j^\perp$ and $\dim([\alpha]^\perp \cap V_j) = \dim([\alpha, V_j]^\perp) = \nu - 1$. Note that, for any $[\beta] \in X_j$, $[\beta]$ is adjacent with $[\alpha]$ if and only if $\beta \in V_j \setminus ([\alpha]^\perp \cap V_j)$. Hence the number of vertices in $X_j$ which is adjacent with $[\alpha]$ is $\frac{q^\nu-1}{q-1} - \frac{q^{\nu-1}-1}{q-1} = q^{\nu-1}$. □

Now we can compute the chromatic number of $Sp(2\nu, q)$.

**Theorem 2.4.** $\chi(Sp(2\nu, q)) = q^\nu + 1$.

**Proof**. By 2.3, we see that $\chi(Sp(2\nu, q)) \leq q^\nu + 1$. Note that $\chi(Sp(2\nu, q))$ is the minimal $n$ such that $Sp(2\nu, q)$ is $n$-partite. Suppose that $Sp(2\nu, q)$ is $n$-partite. Then there exist subsets $Y_1, \ldots, Y_n$ of $V(Sp(2\nu, q))$ such that

$$V(Sp(2\nu, q)) = Y_1 \cup \cdots \cup Y_n,\ Y_i \cap Y_j = \emptyset,\ \text{for all}\ \ i \neq j,$$

and there is no edge joining any two vertices in the same $Y_i$ for $i = 1, \ldots, n$. We want to show that $n \geq q^\nu + 1$. Suppose that $n < q^\nu + 1$. From the above equality, we have $\sum_{i=1}^n |Y_i| = \frac{q^{2\nu}-1}{q-1} = (\frac{q^\nu-1}{q-1})(q^\nu + 1)$. Then there exists some $i$ such that $|Y_i| > \frac{q^\nu-1}{q-1}$. Let $W_i$ be the subspace of $\mathbb{F}_q^{(2\nu)}$ generated by all $\alpha$ such that $[\alpha] \in Y_i$. Then $W_i$ is a totally isotropic subspace, hence $\dim W_i \leq \nu$. This turns out $|Y_i| \leq \frac{q^\nu-1}{q-1}$, a contradiction. Hence $\chi(Sp(2\nu, q)) = q^\nu + 1$. □

## 3. Automorphisms of Symplectic Graphs

We recall that a $2\nu \times 2\nu$ matrix $T$ is called a *symplectic matrix* (or *generalized symplectic matrix*) of order $2\nu$ over $\mathbb{F}_q$ if $TK^tT = K$ (or $TK^tT = kK$ for some $k \in \mathbb{F}_q^*$, respectively). The set of symplectic matrices (or generalized symplectic matrices) of order $2\nu$ over $\mathbb{F}_q$ forms a group with respect to the matrix multiplication, which is called the *symplectic group* (or *generalized symplectic group, respectively,*) of degree $2\nu$ over $\mathbb{F}_q$ and denoted by $Sp_{2\nu}(\mathbb{F}_q)$ (or $GSp_{2\nu}(\mathbb{F}_q)$). The center of $Sp_{2\nu}(\mathbb{F}_q)$ consists of the identity matrix $E$ and $-E$, and the factor group $Sp_{2\nu}(\mathbb{F}_q)/\{E, -E\}$ is called the *projective symplectic group* of degree $2\nu$ over $\mathbb{F}_q$ and denoted by $PSp_{2\nu}(\mathbb{F}_q)$. The center of $GSp_{2\nu}(\mathbb{F}_q)$ consists of all $kE$, where $k \in \mathbb{F}_q^*$, and the factor group of $GSp_{2\nu}(\mathbb{F}_q)$ with respect to its center is called the *projective generalized symplectic group* of degree $2\nu$ over $\mathbb{F}_q$ and denoted by $PGSp_{2\nu}(\mathbb{F}_q)$. Clearly, $PGSp_{2\nu}(\mathbb{F}_q) \cong PSp_{2\nu}(\mathbb{F}_q)$, and when $q = 2$, $GSp_{2\nu}(\mathbb{F}_2) = Sp_{2\nu}(\mathbb{F}_2)$.

**Proposition 3.1.** *Let $T$ be a $2\nu \times 2\nu$ nonsingular matrix over $\mathbb{F}_q$ and*

$$\begin{array}{rccc} \sigma_T: & V(Sp(2\nu,q)) & \to & V(Sp(2\nu,q)) \\ & [\alpha] & \mapsto & [\alpha T]. \end{array}$$

*Then*

(1) *$T \in GSp_{2\nu}(\mathbb{F}_q)$ if and only if $\sigma_T \in \mathrm{Aut}(Sp(2\nu,q))$. In particular, when $q = 2$, $T \in Sp_{2\nu}(\mathbb{F}_2)$ if and only if $\sigma_T \in \mathrm{Aut}(Sp(2\nu,2))$*

(2) *For any $T_1, T_2 \in GSp_{2\nu}(\mathbb{F}_q)$, $\sigma_{T_1} = \sigma_{T_2}$ if and only if $T_1 = kT_2$ for some $k \in \mathbb{F}_q$;*

**Proof**. It is clear that $\sigma_T$ is an one-one correspondence from $V(Sp(2\nu,q))$ to itself.

(1) First assume $T \in GSp_{2\nu}(\mathbb{F}_q)$. Then $TK^tT = kK$ for some $k \in \mathbb{F}_q^*$. For any $[\alpha], [\beta] \in V(Sp(2\nu,q))$, since $\alpha K^t\beta = k^{-1}(\alpha T)K^t(\beta T)$, $[\alpha] \sim [\beta]$ if and only if $\sigma_T([\alpha]) \sim \sigma_T([\beta])$, hence $\sigma_T \in \mathrm{Aut}(Sp(2\nu,q))$.

Conversely, assume $\sigma_T \in \mathrm{Aut}(Sp(2\nu,q))$. Then, for any $\alpha, \beta \neq 0 \in \mathbb{F}_q^{(2\nu)}$, $\alpha K^t\beta = 0$ if and only if $\alpha(TK^tT)^t\beta = 0$. Hence, for any $\alpha \neq 0 \in \mathbb{F}_q^{(2\nu)}$, the two systems of linear equations $(\alpha K)^tX = 0$, $(\alpha TK^tT)^tX = 0$ have the same solutions. But $\mathrm{rank}(\alpha K) = \mathrm{rank}(\alpha TK^tT) = 1$, we see that $\alpha K = k(\alpha TK^tT)$ for some $k \in \mathbb{F}_q^*$, which depends on $\alpha$. Take $\alpha = (1,0,\ldots,0)$,$(0,1,\ldots,0)$, $\ldots$, $(0,0,\ldots,1)$, we get that $K = \mathrm{diag}(k_1,k_2,\ldots,k_{2\nu})TK^tT$, for some $k_1,k_2,\ldots,k_{2\nu} \in \mathbb{F}_q^*$. Take $\alpha = (1,1,\ldots,1)$, we see that $k_1 = k_2 = \ldots = k_{2\nu}$, hence $K = k_1TK^tT$.

(2) It is clear that $\sigma_{T_1} = \sigma_{T_2}$ if $T_1 = kT_2$ for some $k \in \mathbb{F}_q^*$. Conversely, suppose that $\sigma_{T_1} = \sigma_{T_2}$. Then, for any $\alpha \neq 0 \in \mathbb{F}_q^{(2\nu)}$, $\alpha T_1 = k\alpha T_2$ for some $k \in \mathbb{F}_q^*$. Take $\alpha = (1,0,\ldots,0)$,$(0,1,\ldots,0)$, and so on as above, we see that $T_1 = kT_2$ for some $k \in \mathbb{F}_q^*$. □

By 3.1, every generalized symplectic matrix in $GSp_{2\nu}(\mathbb{F}_q)$ induces an automorphism of $Sp(2\nu,q)$ and two generalized symplectic matrices $T_1$ and $T_2$ induce the same automorphism of $Sp(2\nu,q)$ if and only if $T_1 = kT_2$ for some $k \in \mathbb{F}_q$. Thus $PSp_{2\nu}(\mathbb{F}_q)$ can be regarded as a subgroup of $\mathrm{Aut}(Sp(2\nu,q))$.

**Proposition 3.2.** *$Sp(2\nu, q)$ is vertex transitive and edge transitive.*

**Proof**. For any $[\alpha], [\beta] \in V(Sp(2\nu, q))$, there exists $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that $\alpha T = \beta$ by [6, Lemma 3.11]. Then $\sigma_T \in \mathrm{Aut}(Sp(2\nu, q))$ such that $\sigma_T([\alpha]) = [\beta]$. Hence $Sp(2\nu, q)$ is vertex transitive.

Let $[\alpha_1], [\alpha_2], [\beta_1], [\beta_2] \in V(Sp(2\nu, q))$ such that $[\alpha_1] \sim [\alpha_2]$ and $[\beta_1] \sim [\beta_2]$. We may assume that $\alpha_1 K^t \alpha_2 = \beta_1 K^t \beta_2$. Then, by [6, Lemma 3.11] again, there exists $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that $\alpha_1 T = \beta_1$ and $\alpha_2 T = \beta_2$. Then $\sigma_T \in \mathrm{Aut}(Sp(2\nu, q))$ such that $\sigma_T([\alpha_1]) = [\beta_1]$ and $\sigma_T([\alpha_2]) = [\beta_2]$. Hence $Sp(2\nu, q)$ is edge transitive. □

When $q = 2$, we have the following

**Proposition 3.3.** $\mathrm{Aut}(Sp(2\nu, 2)) \cong Sp_{2\nu}(\mathbb{F}_2)$.

**Proof**. Let

$$\begin{array}{rccc} \sigma: & Sp_{2\nu}(\mathbb{F}_2) & \to & \mathrm{Aut}(Sp(2\nu, 2)) \\ & T & \mapsto & \sigma_T. \end{array}$$

Then, by 3.1, $\sigma$ is an injection. Clearly, $\sigma$ preserves the operation. It remains to show that, for any $\tau \in \mathrm{Aut}(Sp(2\nu, 2))$, there exists a $T \in Sp_{2\nu}(\mathbb{F}_2)$ such that $\tau = \sigma_T$.

Note that, for any $\alpha \neq 0 \in \mathbb{F}_2^{(2\nu)}$, we have that $[\alpha] = \{0, \alpha\}$. We will denote the uniquely defined element $\tau([\alpha]) \setminus \{0\}$ by $\tau(\alpha)$ and set $\tau(0) = 0$. Then from $\tau \in \mathrm{Aut}(Sp(2\nu, 2))$ we see that $\alpha K^t \beta = \tau(\alpha) K^t(\tau(\beta))$ for any $\alpha, \beta \in \mathbb{F}_2^{(2\nu)}$ (not necessarily non-zero). Fix any $\alpha \in \mathbb{F}_2^{(2\nu)}$. Let $\beta_1, \beta_2 \in \mathbb{F}_2^{(2\nu)}$. Then

$$\begin{aligned} \alpha K^t \beta_1 &= \tau(\alpha) K^t(\tau(\beta_1)), \\ \alpha K^t \beta_2 &= \tau(\alpha) K^t(\tau(\beta_2)). \end{aligned}$$

Thus

$$\alpha K^t(\beta_1 + \beta_2) = \tau(\alpha) K^t(\tau(\beta_1) + \tau(\beta_2)).$$

But

$$\alpha K^t(\beta_1 + \beta_2) = \tau(\alpha) K^t(\tau(\beta_1 + \beta_2)),$$

hence

$$\tau(\alpha) K^t(\tau(\beta_1 + \beta_2) + \tau(\beta_1) + \tau(\beta_2)) = 0.$$

This is true for any $\alpha \in \mathbb{F}_2^{(2\nu)}$, it follows that $\tau(\beta_1 + \beta_2) + \tau(\beta_1) + \tau(\beta_2) = 0$, i.e., $\tau(\beta_1 + \beta_2) = \tau(\beta_1) + \tau(\beta_2)$. Set

$$T = \begin{pmatrix} \tau(1, 0, \ldots, 0) \\ \tau(0, 1, \ldots, 0) \\ \vdots \\ \tau(0, 0, \ldots, 1) \end{pmatrix}.$$

Then $\tau(\alpha) = \alpha T$ for any $\alpha \in \mathbb{F}_2^{(2\nu)}$. Thus $T$ is nonsingular. By 3.1 $T \in Sp_{2\nu}(\mathbb{F}_2)$ and $\tau = \sigma_T$ as required. □

From now on, we assume that $q > 2$. In $\mathbb{F}_q^{(2\nu)}$, let us set

$$\begin{aligned} e_1 &= (1,0,0,0,\ldots,0,0), \\ f_1 &= (0,1,0,0,\ldots,0,0), \\ e_2 &= (0,0,1,0,\ldots,0,0), \\ f_2 &= (0,0,0,1,\ldots,0,0), \\ &\cdots\cdots \\ e_\nu &= (0,0,0,0,\ldots,1,0), \\ f_\nu &= (0,0,0,0,\ldots,0,1). \end{aligned}$$

Then $e_i, f_i$, $i = 1, \ldots, \nu$, form a basis of $\mathbb{F}_q^{(2\nu)}$ and $e_i K^t f_i = 1$, $e_i K^t e_j = 0$, $f_i K^t f_j = 0$, $i, j = 1, \ldots, \nu$, and $e_i K^t f_j = 0$, $i \neq j$, $i, j = 1, \ldots, \nu$.

In order to describe $\mathrm{Aut}(Sp(2\nu, q))$ for any prime power $q$, we need some definition from group theory. Let $\varphi$ be the natural action of $\mathrm{Aut}(\mathbb{F}_q)$ on the group $\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*$ ($\nu$ in number) defined by

$$\varphi(\pi)((k_1, \ldots, k_\nu)) = (\pi(k_1), \ldots, \pi(k_\nu)), \text{ for all } \pi \in \mathrm{Aut}(\mathbb{F}_q) \text{ and } k_1, \ldots, k_\nu \in \mathbb{F}_q^*,$$

then the semi-direct product of $\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*$ by $\mathrm{Aut}(\mathbb{F}_q)$ corresponding to $\varphi$, denoted by $(\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*) \rtimes_\varphi \mathrm{Aut}(\mathbb{F}_q)$, is the group consisting of all elements of the form $(k_1, \ldots, k_\nu, \pi)$, where $k_1, \ldots, k_\nu \in \mathbb{F}_q^*$ and $\pi \in \mathrm{Aut}(\mathbb{F}_q)$, with multiplication defined by

$$(k_1, \ldots, k_\nu, \pi)(k_1', \ldots, k_\nu', \pi') = (k_1\pi(k_1'), \ldots, k_\nu\pi(k_\nu'), \pi\pi').$$

Then the main result about $\mathrm{Aut}(Sp(2\nu, q))$ is as follows.

**Theorem 3.4.** *Regard $PSp_{2\nu}(\mathbb{F}_q)$ as a subgroup of $\mathrm{Aut}(Sp(2\nu, q))$ and let $E$ be the subgroup of $\mathrm{Aut}(Sp(2\nu, q))$ defined as follows*

$$E = \{\sigma \in \mathrm{Aut}(Sp(2\nu, q)) : \sigma([e_i]) = [e_i], \sigma([f_i]) = [f_i], i = 1, \ldots, \nu\}.$$

*Then*

(1) $\mathrm{Aut}(Sp(2\nu, q)) = PSp_{2\nu}(\mathbb{F}_q) \cdot E$;
(2) *If $\nu = 1$, then $E$ is isomorphic to the symmetric group on $q - 1$ elements;*
(3) *If $\nu > 1$, then*

$$E \cong \underbrace{(\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*)}_{\nu} \rtimes_\varphi \mathrm{Aut}(\mathbb{F}_q).$$

**Proof**. (1) Let $\tau \in \mathrm{Aut}(Sp(2\nu, q))$. Suppose that $\tau([e_i]) = [e_i']$, $\tau([f_i]) = [f_i']$, $i = 1, \ldots, \nu$. Then $e_i' K^t f_i' \neq 0$, $e_i' K^t e_j' = 0$, $f_i' K^t f_j' = 0$, $i, j = 1, \ldots, \nu$ and $e_i' K^t f_j' = 0$, $i \neq j$, $i, j = 1, \ldots, \nu$. We may choose $e_i', f_i'$, $i = 1, \ldots, \nu$, such that $e_i' K^t f_i' = 1$,$i =$

$1, \ldots, \nu$. Let

$$A = \begin{pmatrix} e_1 \\ f_1 \\ e_2 \\ f_2 \\ \vdots \\ e_\nu \\ f_\nu \end{pmatrix}, \ A' = \begin{pmatrix} e_1' \\ f_1' \\ e_2' \\ f_2' \\ \vdots \\ e_\nu' \\ f_\nu' \end{pmatrix}.$$

Then $AK^tA = K = A'K^tA'$. Thus, by [6, Lemma 3.11], there exists $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that $A = A'T$, i.e., $e_i'T = e_i$, $f_i'T = f_i$, $i = 1, \ldots, \nu$. Set $\tau_1 = \sigma_T\tau$. Then $\tau_1([e_i]) = [e_i]$, $\tau_1([f_i]) = [f_i]$, $i = 1, \ldots, \nu$, hence $\tau_1 \in E$. Thus $\tau \in PSp_{2\nu}(\mathbb{F}_q) \cdot E$. It follows that $\mathrm{Aut}(Sp(2\nu, q)) = PSp_{2\nu}(\mathbb{F}_q) \cdot E$.

(2) When $\nu = 1$, it is clear that $E$ is isomorphic to the symmetric group on the $q-1$ vertices of $Sp(2, q)$ since $Sp(2, q)$ is a complete graph.

(3) Suppose that $\nu > 1$. Firstly, let us write out some elements of $E$. Let $k_1, \ldots, k_\nu \in \mathbb{F}_q^*$ and $\pi \in \mathrm{Aut}(\mathbb{F}_q)$. Let $\sigma_{(k_1,\ldots,k_\nu,\pi)}$ be the map which takes any vertex $[a_1, a_2, a_3, a_4, \ldots, a_{2\nu-1}, a_{2\nu}]$ of $Sp(2\nu, q)$ to the vertex

$$[\pi(a_1), k_1\pi(a_2), k_2\pi(a_3), k_1k_2^{-1}\pi(a_4), \ldots, k_\nu\pi(a_{2\nu-1}), k_1k_\nu^{-1}\pi(a_{2\nu})].$$

Then it is clear that $\sigma_{(k_1,\ldots,k_\nu,\pi)}$ is well-defined. Furthermore, it is easy to see that $\sigma_{(k_1,\ldots,k_\nu,\pi)}$ is injective, but the vertex set of $Sp(2\nu, q)$ is finite, $\sigma_{(k_1,\ldots,k_\nu,\pi)}$ is a bijection from $V(Sp(2\nu, q))$ to itself. Let $\alpha = [a_1, a_2, a_3, a_4, \ldots, a_{2\nu-1}, a_{2\nu}]$, $\beta = [a_1', a_2', a_3', a_4', \ldots, a_{2\nu-1}', a_{2\nu}']$ be two vertices of $Sp(2\nu, q)$. If $\alpha \not\sim \beta$, then, by definition,

$$(a_1a_2' - a_2a_1') + (a_3a_4' - a_4a_3') + \ldots + (a_{2\nu-1}a_{2\nu}' - a_{2\nu}a_{2\nu-1}') = 0,$$

which implies that

$$\begin{aligned}&(\pi(a_1)k_1\pi(a_2') - \pi(a_2)k_1\pi(a_1')) + (k_2\pi(a_3)k_1k_2^{-1}\pi(a_4') - k_1k_2^{-1}\pi(a_4)k_2\pi(a_3'))\\ &+ \ldots + (k_\nu\pi(a_{2\nu-1})k_1k_\nu^{-1}\pi(a_{2\nu}') - k_1k_\nu^{-1}\pi(a_{2\nu})k_\nu\pi(a_{2\nu-1}')) = 0,\end{aligned}$$

i.e., $\sigma_{(k_1,\ldots,k_\nu,\pi)}(\alpha) \not\sim \sigma_{(k_1,\ldots,k_\nu,\pi)}(\beta)$. Since the edges set of $Sp(2\nu, q)$ is finite, $\alpha \not\sim \beta$ if and only if $\sigma_{(k_1,\ldots,k_\nu,\pi)}(\alpha) \not\sim \sigma_{(k_1,\ldots,k_\nu,\pi)}(\beta)$. Hence $\sigma_{(k_1,\ldots,k_\nu,\pi)} \in \mathrm{Aut}(Sp(2\nu, q))$. Note that $\sigma_{(k_1,\ldots,k_\nu,\pi)}([e_i]) = [e_i]$, $\sigma_{(k_1,\ldots,k_\nu,\pi)}([f_i]) = [f_i]$, $i = 1, \ldots, \nu$, hence, $\sigma_{(k_1,\ldots,k_\nu,\pi)} \in E$.

If we define a map $h$ as $(k_1, \ldots, k_\nu, \pi) \mapsto \sigma_{(k_1,\ldots,k_\nu,\pi)}$, then it is easy to verify that $h$ is a group homomorphism from $(\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*) \times_\varphi \mathrm{Aut}(\mathbb{F}_q)$ to $E$. It is also easy to see that if $(k_1, \ldots, k_\nu, \pi) \neq (k_1', \ldots, k_\nu', \pi')$ then $\sigma_{(k_1,\ldots,k_\nu,\pi)} \neq \sigma_{(k_1',\ldots,k_\nu',\pi')}$. Thus, to show that $h$ is a group isomorphism, it remains to show that every element of $E$ is of the form $\sigma_{(k_1,\ldots,k_\nu,\pi)}$.

Suppose that $\sigma \in E$. Note that if $\sigma([a_1, a_2, \ldots, a_{2\nu}]) = [b_1, b_2, \ldots, b_{2\nu}]$, then $a_{2i-1} \neq 0$ if and only if $[a_1, a_2, \ldots, a_{2\nu}] \sim [f_i]$ and $a_{2i} \neq 0$ if and only if $[a_1, a_2, \ldots, a_{2\nu}] \sim [e_i]$, and similar results are also true for $b_i$. But $\sigma([e_i]) = [e_i]$ and $\sigma([f_i]) = [f_i]$, it follows that $a_i = 0$ if and only if $b_i = 0$. For any vertex $[a_1, a_2, \ldots, a_{2\nu}]$, if $a_1 = \cdots = a_{i-1} = 0$ and $a_i \neq 0$ then $[a_1, a_2, \ldots, a_{2\nu}]$ can be uniquely written as $[0, \ldots, 0, 1, a_{i+1}', \ldots, a_{2\nu}']$ and $\sigma([a_1, a_2, \ldots, a_{2\nu}])$ can be uniquely written as

$[0,\ldots,0,1,b'_{i+1},\ldots,b'_{2\nu}]$. Let us show how to determine $b'_{i+1},\ldots,b'_{2\nu}$ from $a'_{i+1},\ldots,a'_{2\nu}$. We will use frequently the fact that, for any vertices $[\alpha],[\beta]$, if $[\alpha]\not\sim[\beta]$ then $\sigma([\alpha])\not\sim\sigma([\beta])$.

In the following, we will denote $[a_1,a'_1,a_2,a'_2,\ldots,a_\nu,a'_\nu]$ by $\sum_{i=1}^{\nu}a_i[e_i]+\sum_{i=1}^{\nu}a'_i[f_i]$, for example, $[a,b,0,\ldots,0]$ is denoted by $a[e_1]+b[f_1]$. Since $\sigma$ is a bijection from $V(Sp(2\nu,q))$ to itself, we have permutations $\pi_i$, $i=2,\ldots,2\nu$, of $\mathbb{F}_q$ with $\pi(0)=0$ such that

$$\begin{aligned}\sigma([e_1]+a_{2i-1}[e_i]) &= [e_1]+\pi_{2i-1}(a_{2i-1})[e_i],\\ \sigma([e_1]+a_{2i}[f_i]) &= [e_1]+\pi_{2i}(a_{2i})[f_i].\end{aligned}$$

We firstly consider the cases $\sigma([0,1,a_3,\ldots,a_{2\nu}])$ and $\sigma([1,a_2,a_3,\ldots,a_{2\nu}])$. Let $\sigma([0,1,a_3,\ldots,a_{2\nu}])=[0,1,a'_3,\ldots,a'_{2\nu}]$ and $j\geq 1$. If $a_{2j+1}\neq 0$, then, from $[0,1,a_3,\ldots,a_{2\nu}]\not\sim[e_1]+a_{2j+1}^{-1}[f_{j+1}]$ we have $[0,1,a'_3,\ldots,a'_{2\nu}]\not\sim[e_1]+\pi_{2j+2}(a_{2j+1}^{-1})[f_{j+1}]$, hence, $a'_{2j+1}=\pi_{2j+2}(a_{2j+1}^{-1})^{-1}$. If $a_{2j+2}\neq 0$, then from $[0,1,a_3,\ldots,a_{2\nu}]\not\sim[e_1]-a_{2j+2}^{-1}[e_{j+1}]$ we have $[0,1,a'_3,\ldots,a'_{2\nu}]\not\sim[e_1]+\pi_{2j+1}(-a_{2j+2}^{-1})[e_{j+1}]$, hence, $a'_{2j+2}=-\pi_{2j+1}(-a_{2j+2}^{-1})^{-1}$. Thus

$$\sigma([0,1,a_3,\ldots,a_{2\nu}])=[0,1,a'_3,\ldots,a'_{2\nu}], \tag{1}$$

where $a'_{2j+1}=\pi_{2j+2}(a_{2j+1}^{-1})^{-1}$ if $a_{2j+1}\neq 0$ and $a'_{2j+2}=-\pi_{2j+1}(-a_{2j+2}^{-1})^{-1}$ if $a_{2j+2}\neq 0$.

For the case $\sigma([1,a_2,a_3,\ldots,a_{2\nu}])$. Let $\sigma([1,a_2,a_3,\ldots,a_{2\nu}])=[1,a''_2,a''_3,\ldots,a''_{2\nu}]$. From $[1,a_2,a_3,\ldots,a_{2\nu}]\not\sim[e_1]+a_2[f_1]$ we get $[1,a''_2,a''_3,\ldots,a''_{2\nu}]\not\sim[e_1]+\pi_2(a_2)[f_1]$, hence, $a''_2=\pi_2(a_2)$. Let $j\geq 1$. If $a_{2j+1}\neq 0$, then, from $[1,a_2,a_3,\ldots,a_{2\nu}]\not\sim[f_1]-a_{2j+1}^{-1}[f_{j+1}]$ and $\sigma([f_1]-a_{2j+1}^{-1}[f_{j+1}])=[f_1]-\pi_{2j+1}(a_{2j+1})^{-1}[f_{j+1}]$ as been shown above, we have $[1,a''_2,a''_3,\ldots,a''_{2\nu}]\not\sim[f_1]-\pi_{2j+1}(a_{2j+1})^{-1}[f_{j+1}]$, hence, $a''_{2j+1}=\pi_{2j+1}(a_{2j+1})$. Similarly, if $a_{2j+2}\neq 0$, then from $[1,a_2,a_3,\ldots,a_{2\nu}]\not\sim[f_1]+a_{2j+2}^{-1}[e_{j+1}]$ we have $[1,a''_2,a''_3,\ldots,a''_{2\nu}]\not\sim[f_1]+\pi_{2j+2}(a_{2j+2})^{-1}[e_{j+1}]$, hence, $a''_{2j+2}=\pi_{2j+2}(a_{2j+2})$. Thus, for any $a_2,a_3,\ldots,a_{2\nu}\in\mathbb{F}_q$,

$$\sigma([1,a_2,a_3,\ldots,a_{2\nu}])=[1,\pi_2(a_2),\pi_3(a_3),\ldots,\pi_{2\nu}(a_{2\nu})]. \tag{2}$$

Then, let $i\geq 2$, we discuss the general cases $\sigma([0,\ldots,0,1,a_{2i+1},\ldots,a_{2\nu}])$ and $\sigma([0,\ldots,0,1,a_{2i},\ldots,a_{2\nu}])$. The above results of case $i=1$ will be used. Let $\sigma([0,\ldots,0,1,a_{2i+1},\ldots,a_{2\nu}])=[0,\ldots,0,1,a'_{2i+1},\ldots,a'_{2\nu}]$ and $j\geq i$. If $a_{2j+1}\neq 0$, then, from

$$[0,\ldots,0,1,a_{2i+1},\ldots,a_{2\nu}]\not\sim[e_1]+[e_i]+a_{2j+1}^{-1}[f_{j+1}]$$

and $\sigma([e_1]+[e_i]+a_{2j+1}^{-1}[f_{j+1}])=[e_1]+\pi_{2i-1}(1)[e_i]+\pi_{2j+2}(a_{2j+1}^{-1})[f_{j+1}]$ as been shown above, we have

$$[0,\ldots,0,1,a'_{2i+1},\ldots,a'_{2\nu}]\not\sim[e_1]+\pi_{2i-1}(1)[e_i]+\pi_{2j+2}(a_{2j+1}^{-1})[f_{j+1}],$$

hence, $a'_{2j+1}=\pi_{2i-1}(1)\pi_{2j+2}(a_{2j+1}^{-1})^{-1}$. Similarly, if $a_{2j+2}\neq 0$, then from

$$[0,\ldots,0,1,a_{2i+1},\ldots,a_{2\nu}]\not\sim[e_1]+[e_i]-a_{2j+2}^{-1}[e_{j+1}]$$

we have

$$[0,\ldots,0,1,a'_{2i+1},\ldots,a'_{2\nu}]\not\sim[e_1]+\pi_{2i-1}(1)[e_i]+\pi_{2j+1}(-a_{2j+2}^{-1})[e_{j+1}],$$

hence, $a'_{2j+2} = -\pi_{2i-1}(1)\pi_{2j+1}(-a_{2j+2}^{-1})^{-1}$. Thus,

$$\sigma([0,\ldots,0,1,a_{2i+1},\ldots,a_{2\nu}]) = [0,\ldots,0,1,a'_{2i+1},\ldots,a'_{2\nu}], \tag{3}$$

where $a'_{2j+1} = \pi_{2i-1}(1)\pi_{2j+2}(a_{2j+1}^{-1})^{-1}$ if $a_{2j+1} \neq 0$ and $a'_{2j+2} = -\pi_{2i-1}(1)\pi_{2j+1}(-a_{2j+2}^{-1})^{-1}$ if $a_{2j+2} \neq 0$.

Finally, for the case $\sigma([0,\ldots,0,1,a_{2i},\ldots,a_{2\nu}])$. Let $\sigma([0,\ldots,0,1,a_{2i},\ldots,a_{2\nu}]) = [0,\ldots,0,1,a''_{2i},\ldots,a''_{2\nu}]$. From

$$[0,\ldots,0,1,a_{2i},\ldots,a_{2\nu}] \not\sim [e_1] + [e_i] + a_{2i}[f_i]$$

we get

$$[0,\ldots,0,1,a''_{2i},\ldots,a''_{2\nu}] \not\sim [e_1] + \pi_{2i-1}(1)[e_i] + \pi_{2i}(a_{2i})[f_i],$$

hence, $a''_{2i} = \pi_{2i-1}(1)^{-1}\pi_{2i}(a_{2i})$. Let $j \geq i$. If $a_{2j+1} \neq 0$, then from

$$[0,\ldots,0,1,a_{2i},\ldots,a_{2\nu}] \not\sim [f_i] - a_{2j+1}^{-1}[f_{j+1}]$$

we have

$$[0,\ldots,0,1,a''_{2i},\ldots,a''_{2\nu}] \not\sim [f_i] - \pi_{2i-1}(1)^{-1}\pi_{2j+1}(a_{2j+1})^{-1}[f_{j+1}],$$

hence, $a''_{2j+1} = \pi_{2i-1}(1)^{-1}\pi_{2j+1}(a_{2j+1})$. If $a_{2j+2} \neq 0$, then from

$$[0,\ldots,0,1,a_{2i},\ldots,a_{2\nu}] \not\sim [f_i] + a_{2j+2}^{-1}[e_{j+1}]$$

we have

$$[0,\ldots,0,1,a''_{2i},\ldots,a''_{2\nu}] \not\sim [f_i] + \pi_{2i-1}(1)\pi_{2j+2}(a_{2j+2})^{-1}[e_{j+1}],$$

hence, $a''_{2j+2} = \pi_{2i-1}(1)^{-1}\pi_{2j+2}(a_{2j+2})$. Thus, for any $a_{2i}, a_{2i+1}, \ldots, a_{2\nu} \in \mathbb{F}_q$,

$$\begin{aligned} &\sigma([0,\ldots,0,1,a_{2i},a_{2i+1},\ldots,a_{2\nu}]) \\ = \ &[0,\ldots,0,1,\pi_{2i-1}(1)^{-1}\pi_{2i}(a_{2i}),\pi_{2i-1}(1)^{-1}\pi_{2i+1}(a_{2i+1}),\ldots,\pi_{2i-1}(1)^{-1}\pi_{2\nu}(a_{2\nu})]. \end{aligned} \tag{4}$$

Having represented $\sigma$ by $\pi_i$, $i = 2,\ldots,2\nu$, let us discuss some properties of $\pi_i$.

**Lemma 3.5.** (1) *For any $i \geq 1$ and $a \in \mathbb{F}_q$,*

$$\pi_{2i+1}(1)\pi_{2i+2}(a) = \pi_{2i+2}(1)\pi_{2i+1}(a) = \pi_2(a);$$

(2) *For any $i \geq 2$ and $a, b \in \mathbb{F}_q$,*

$$\begin{aligned} \pi_i(a+b) &= \pi_i(a) + \pi_i(b); \\ \pi_i(-a) &= -\pi_i(a); \\ \pi_i(ab) &= \pi_i(a)\pi_i(b)\pi_i(1)^{-1}; \\ \pi_i(a^{-1}) &= \pi_i(a)^{-1}\pi_i(1)^2 \text{ if } a \neq 0. \end{aligned}$$

**Proof**. (1) We may assume that $a \neq 0$. Since $[e_1]+a[e_{i+1}]+a[f_{i+1}] \not\sim [e_{i+1}]+[f_{i+1}]$, it follows that $\sigma([e_1] + a[e_{i+1}] + a[f_{i+1}]) \not\sim \sigma([e_{i+1}] + [f_{i+1}])$, but

$$\begin{aligned} \sigma([e_1] + a[e_{i+1}] + a[f_{i+1}]) &= [e_1] + \pi_{2i+1}(a)[e_{i+1}] + \pi_{2i+2}(a)[f_{i+1}], \\ \sigma([e_{i+1}] + [f_{i+1}]) &= [e_{i+1}] + \pi_{2i+1}(1)^{-1}\pi_{2i+2}(1)[f_{i+1}], \end{aligned}$$

we have that

$$\pi_{2i+1}(1)^{-1}\pi_{2i+2}(1)\pi_{2i+1}(a) - \pi_{2i+2}(a) = 0,$$

i.e.,

$$\pi_{2i+1}(1)\pi_{2i+2}(a) = \pi_{2i+2}(1)\pi_{2i+1}(a).$$

Similarly, since $[e_1] + a[f_1] + [e_{i+1}] \not\sim [e_1] + a[f_{i+1}]$, we have that $[e_1] + \pi_2(a)[f_1] + \pi_{2i+1}(1)[e_{i+1}] \not\sim [e_1] + \pi_{2i+2}(a)[f_{i+1}]$, hence, $\pi_{2i+1}(1)\pi_{2i+2}(a) = \pi_2(a)$.

(2) From $[e_1] + (a+b)[f_1] + [e_2] \not\sim [e_1] + a[f_1] + b[f_2]$ we have that

$$[e_1] + \pi_2(a+b)[f_1] + \pi_3(1)[e_2] \not\sim [e_1] + \pi_2(a)[f_1] + \pi_4(b)[f_2].$$

Then $\pi_2(a) - \pi_2(a+b) + \pi_3(1)\pi_4(b) = 0$, but $\pi_3(1)\pi_4(b) = \pi_2(b)$, hence, $\pi_2(a+b) = \pi_2(a) + \pi_2(b)$. It turns out from (1) that this equality holds for all $i \geq 2$. Thus $\pi_i(-a) = -\pi_i(a)$ as $\pi_i(0) = 0$.

For multiplication, let $i \geq 1$, from $[e_1] + b[e_{i+1}] + ab[f_{i+1}] \not\sim [e_{i+1}] + a[f_{i+1}]$ we get that

$$[e_1] + \pi_{2i+1}(b)[e_{i+1}] + \pi_{2i+2}(ab)[f_{i+1}] \not\sim [e_{i+1}] + \pi_{2i+1}(1)^{-1}\pi_{2i+2}(a)[f_{i+1}],$$

hence, $\pi_{2i+1}(b)\pi_{2i+1}(1)^{-1}\pi_{2i+2}(a) - \pi_{2i+2}(ab) = 0$, but $\pi_{2i+1}(b)\pi_{2i+1}(1)^{-1} = \pi_{2i+2}(b)\pi_{2i+2}(1)^{-1}$. Thus

$$\pi_{2i+2}(ab) = \pi_{2i+2}(a)\pi_{2i+2}(b)\pi_{2i+2}(1)^{-1}.$$

It follows from $\pi_{2i+1}(1)\pi_{2i+2}(a) = \pi_{2i+2}(1)\pi_{2i+1}(a)$ and $\pi_{2i+1}(1)\pi_{2i+2}(1) = \pi_{2i+2}(1)\pi_{2i+1}(1)$ that the abve equality also holds for $2i+1$. It remains to consider $\pi_2$. We have

$$\begin{aligned} \pi_2(ab) &= \pi_3(1)\pi_4(ab) \\ &= \pi_3(1)\pi_4(1)^{-1}\pi_4(a)\pi_4(b) \\ &= \pi_3(1)^{-1}\pi_4(1)^{-1}\pi_2(a)\pi_2(b) \\ &= \pi_2(a)\pi_2(b)\pi_2(1)^{-1}. \end{aligned}$$

Finally, if $a \neq 0$, then from $\pi_i(1) = \pi_i(aa^{-1}) = \pi_i(a)\pi_i(a^{-1})\pi_i(1)^{-1}$ we obtain that $\pi_i(a^{-1}) = \pi_i(a)^{-1}\pi_i(1)^2$, then the proof of lemma is complete. □

We continue the proof of the theorem. Let us denote the identity automorphism on $\mathbb{F}_q$ by $\pi_1$. Then when $i = 1$, (3) reduces to (1) and (4) reduces to (2). Therefore (3) and (4) hold for all $i$, where $1 \leq i \leq \nu$. By the above lemma, for any $i \geq 1$, we can rewrite (3) in the form of (4) as follows. In (3), for any $j \geq i$, we have

$$\begin{aligned} a'_{2j+1} &= \pi_{2i-1}(1)\pi_{2j+2}(a_{2j+1}^{-1})^{-1} \\ &= \pi_{2i-1}(1)\pi_{2j+2}(a_{2j+1})\pi_{2j+2}(1)^{-2} \\ &= \pi_{2i-1}(1)\pi_{2j+1}(1)^{-1}\pi_{2j+2}(1)^{-1}\pi_{2j+1}(a_{2j+1}) \\ &= \pi_{2i-1}(1)\pi_2(1)^{-1}\pi_{2j+1}(a_{2j+1}) \\ &= \pi_{2i}(1)^{-1}\pi_{2j+1}(a_{2j+1}), \end{aligned}$$

and

$$\begin{aligned} a'_{2j+2} &= -\pi_{2i-1}(1)\pi_{2j+1}(-a_{2j+2}^{-1})^{-1} \\ &= \pi_{2i-1}(1)\pi_{2j+1}(a_{2j+2}^{-1})^{-1} \\ &= \pi_{2i-1}(1)\pi_{2j+1}(a_{2j+2})\pi_{2j+1}(1)^{-2} \\ &= \pi_{2i-1}(1)\pi_{2j+1}(1)^{-1}\pi_{2j+2}(1)^{-1}\pi_{2j+2}(a_{2j+2}) \\ &= \pi_{2i-1}(1)\pi_{2}(1)^{-1}\pi_{2j+2}(a_{2j+2}) \\ &= \pi_{2i}(1)^{-1}\pi_{2j+2}(a_{2j+2}). \end{aligned}$$

Hence, for any $a_{2i+1}, \ldots, a_{2\nu} \in \mathbb{F}_q$,

(5) $\sigma([0, \ldots, 0, 1, a_{2i+1}, \ldots, a_{2\nu}]) = [0, \ldots, 0, 1, \pi_{2i}(1)^{-1}\pi_{2i+1}(a_{2i+1}), \ldots, \pi_{2i}(1)^{-1}\pi_{2\nu}(a_{2\nu})]$,

which is of the same form as (4).

Now let $k_1 = \pi_2(1)$, $\pi = k_1^{-1}\pi_2$, $k_2 = \pi_3(1)$, $k_3 = \pi_5(1), \ldots, k_\nu = \pi_{2\nu-1}(1)$. Then $\pi \in \mathrm{Aut}(\mathbb{F}_q)$, $\pi_2 = k_1\pi$, $\pi_3 = k_2\pi, \pi_4 = k_1k_2^{-1}\pi, \ldots, \pi_{2\nu-1} = k_\nu\pi, \pi_{2\nu} = k_1k_\nu^{-1}\pi$. Assembling (4) and (5), we obtain

$$\begin{aligned} &\sigma([a_1, a_2, a_3, a_4, \ldots, a_{2\nu-1}, a_{2\nu}]) \\ = \ &[\pi(a_1), k_1\pi(a_2), k_2\pi(a_3), k_1k_2^{-1}\pi(a_4), \ldots, k_\nu\pi(a_{2\nu-1}), k_1k_\nu^{-1}\pi(a_{2\nu})]. \end{aligned}$$

Hence $\sigma = h(k_1, \ldots, k_\nu, \pi)$, as required. □

**Corollary 3.6.** *When* $\nu = 1$,

$$|\mathrm{Aut}(Sp(2, q))| = q(q^2 - 1) \cdot (q - 2)!,$$

*and when* $\nu \geq 2$,

$$|\mathrm{Aut}(Sp(2\nu, q))| = q^{\nu^2} \prod_{i=1}^{\nu}(q^{2i} - 1) \cdot [\mathbb{F}_q : \mathbb{F}_p].$$

**Proof**. Note that $PSp_{2\nu}(\mathbb{F}_q) \cap E$ consists of $\sigma$ which is reduced from some matrix of the form $\mathrm{diag}(k_1, l_1, k_2, l_2, \ldots\ldots, k_\nu, l_\nu)$, with $k_il_i = 1$, $i = 1, \ldots, \nu$. Thus $|PSp_{2\nu}(\mathbb{F}_q) \cap E| = \frac{1}{2}(q-1)^\nu$. Hence

$$\begin{aligned} |\mathrm{Aut}(Sp(2\nu, q))| &= \frac{|PSp_{2\nu}(\mathbb{F}_q)||E|}{|PSp_{2\nu}(\mathbb{F}_q) \cap E|} \\ &= \frac{\frac{1}{2}q^{\nu^2}\prod_{i=1}^{\nu}(q^{2i}-1) \cdot |E|}{\frac{1}{2}(q-1)^\nu}. \end{aligned}$$

Thus, when $\nu = 1$, $|\mathrm{Aut}(Sp(2,q))| = q(q^2-1)\cdot(q-2)!$, and when $\nu \geq 2$,

$$\begin{aligned} & |\mathrm{Aut}(Sp(2\nu,q))| \\ = & \frac{\frac{1}{2}q^{\nu^2}\prod_{i=1}^{\nu}(q^{2i}-1)\cdot(q-1)^{\nu}\cdot|\mathrm{Aut}(\mathbb{F}_q)|}{\frac{1}{2}(q-1)^{\nu}} \\ = & q^{\nu^2}\prod_{i=1}^{\nu}(q^{2i}-1)\cdot|\mathrm{Aut}(\mathbb{F}_q)| \\ = & q^{\nu^2}\prod_{i=1}^{\nu}(q^{2i}-1)\cdot[\mathbb{F}_q:\mathbb{F}_p], \end{aligned}$$

as is well-known that $|\mathrm{Aut}(\mathbb{F}_q)| = [\mathbb{F}_q : \mathbb{F}_p]$ where $p = \mathrm{char}(\mathbb{F}_q)$. □

## References


[1] R. H. Dye, Partitions and their stabilizers for line complexes and quadrics, *Annali di Matematica Pura ed Applicata* **114**(1977), 173-194.

[2] C. Godsil and G. Royle, Chromatic number and the 2-rank of a graph, *J. Combin. Theory Ser.B* **81**(2001), 142-149.

[3] C. Godsil and G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics Vol. 207, Springer-Verlag, 2001.

[4] J.J. Rotman, Projective planes, graphs, and simple algebras, *J. Algebra* **155**(1993), 267-289.

[5] J.J. Rotman and P.M. Weichsel, Simple Lie algebras and graphs, *J. Algebra* **169**(1994), 775-790.

[6] Z. Wan, *Geometry of Classical Groups over Finite Fields, 2nd edition*, Science Press, Beijing/New York, 2002.



Zhongming Tang
Department of Mathematics
Suzhou University
Suzhou 215006
P. R. China
E-mail: zmtang@suda.edu.cn

Zhe-xian Wan
Academy of Mathematics and System Sciences
Chinese Academy of Science
Beijing 100080
P. R. China
E-mail: wan@amss.ac.cn