Linear Time Algorithms to the Minimum All-Ones Problem for Unicyclic and Bicyclic Graphs *

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Abstract

In this paper, we give graph-theoretic algorithms of linear time to the Minimum All-Ones Problem for unicyclic and bicyclic graphs. These algorithms are based on a graph-theoretic algorithm of linear time to the Minimum All-Ones Problem with Restrictions for trees.

Keywords: (Minimum) All-Ones Problem; graph-theoretic algorithm; linear time

AMS subject classification(2000): 05C85, 05C70, 90C27, 68Q25, 68R10

1 Introduction

The term *All-Ones Problem* was introduced by Sutner, see [10]. It has applications in linear cellular automata, see [11] and the references therein. The problem is cited as follows: suppose each of the square of an $n \times n$ chessboard is equipped with an indicator light and a button. If the button of a square is pressed, the light of that square will change from off to on and vice versa; the same happens to the lights of all the edge-adjacent squares. Initially all lights are off. Now, consider the following questions: is it possible to press a sequence of buttons in such a way that in the end all lights are on ? This is referred as the *All-Ones Problem*. If there is such a solution, how to find a such way ? And finally, how to find such a way that presses as few buttons as possible ? This is referred as the *Minimum All-Ones Problem*. All the above questions can be asked for arbitrary graphs. Here and in what follows, we consider connected simple undirected graphs only. One can deal with disconnected graphs component by component. For all terminology and notations on graphs, we refer to [7]. An equivalent version of the All-Ones Problem was proposed by Peled in [8], where it was called the *Lamp Lighting Problem*. The rule of the All-Ones Problem is called σ^+-rule on graphs, which means that a button lights not only its neighbors but also its own light. If a button lights only its neighbors but not its own light, this rule on graphs is called σ-rule.

In graph-theoretic terminology, a solution to the All-Ones Problem with σ^+-rule can be stated as follows: given a graph $G = (V, E)$, where V and E denotes the vertex-set and the edge-set of

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G, respectively. A subset X of V is a solution if and only if for every vertex v of G the number of vertices in X adjacent to or equal to v is odd. Such a subset X is called an *odd parity cover* in [11]. So, the All-Ones Problem can be formulated as follows: given a graph $G = (V, E)$, does a subset X of V exist such that for all vertex $v \in V - X$, the number of vertices in X adjacent to v is odd, while for all vertex $v \in X$, the number of vertices in X adjacent to v is even ? If there exists a solution, how to find a one with minimum cardinality ?

There have been many publications on the All-Ones Problem, see Sutner [12,13], Barua and Ramakrishnan [1] and Dodis and Winkler [3]. Using linear algebra, Sutner [11] proved that it is always possible to light every lamp in any graphs by σ^+-rule. Lossers [6] gave another beautiful proof also by using linear algebra. A graph-theoretic proof was given by Erikisson et al [4]. So, the existence of solutions of the All-Ones Problem for general graphs was solved already. Sutner [10] proposed the question whether there is a graph-theoretic method to find a solution to the All-Ones Problem for trees. Galvin [5] solved this question by giving a graph-theoretic algorithm of linear time. In [9], Sutner proved that the Minimum All-Ones Problem is NP-complete for arbitrary graphs. So, it becomes an interesting problem to find graph-theoretic algorithms of polynomial time to the Minimum All-Ones Problem for some special classes of graphs. In [2] we gave a linear time algorithm for trees, which is based on the idea of Galvin algorithm [5] to the All-Ones Problem for trees. In his algorithm, the nodes of a rooted tree, drawn like a family tree with the root at the top, will be divided into three classes: outcasts, oddballs and rebels. The classification is defined inductively, from the bottom up, as follows:

• All of the childless nodes or leaves are rebels.

• A node, other than a leaf, is called an *rebel* if it has no oddball children and an even number of its children are rebels.

• A node is called an *oddball* if it has no oddball children and an odd number of its children are rebels.

• A node is called an *outcast* if at least one of its children is an oddball.

We sometimes simply call a node *r-type, b-type or o-type* if it belongs to the rebel class, the oddball class or the outcast class.

These notations and terminology will be used in the sequel.

2 Algorithm to the Minimum All-Ones Problem with Restrictions for Trees

In order to solve the Minimum All-Ones Problems for unicyclic and bicyclic graphs, we need to introduce and solve the Minimum All-Ones Problem with Restrictions for trees, which is an interesting problem on its own. First we need to solve the following problem.

For a matrix $M_{2\times n} = (m_{ij})_{2\times n}$, $i \in \{0, 1\}$, $j \in \{1, 2, \cdots, n\}$, $m_{ij} \in Z^+ \cup \{\infty\}$, the *Minimum Odd Sum Problem with Restrictions* is defined as

$$\min \sum_{j=1}^{n} m_{0j}x_{0j} + m_{1j}x_{1j}$$

$$\begin{cases} \sum_{j=1}^{n} x_{1j} = 1 \mod 2 \\ x_{0j} + x_{1j} = 1, j = 1, 2, \cdots, n \\ x_{ij} \in \{0, 1\}, i \in \{0, 1\} \end{cases}$$

Here we suppose that "∞" is bigger than any $k \in Z^+$, and $\forall k \in Z^+$, $k + \infty = \infty$.

We omit the description and the proof of the algorithm for the Minimum Odd Sum Problem with Restrictions. The time complexity of the algorithm is linear.

Replacing $\sum_{j=1}^{n} x_{1j} = 1 \mod 2$ in the Minimum Odd Sum Problem with Restrictions by $\sum_{j=1}^{n} x_{1j} = 0 \mod 2$, we then get a new problem, called the *Minimum Even Sum Problem with Restrictions*. It can be solved in the same way as above. The details are omitted.

For convenience, we say that the truth value of a node u in G, denoted by $tv(u)$, is 1, if it belongs to the solution to the All-Ones Problem for G; and 0, otherwise.

The so-called "the Minimum All-Ones Problem with Restrictions for trees" is to find a solution to the Minimum All-Ones Problem for trees under the condition that the truth values of some nodes have been assigned. For this problem, our algorithm uses induction on the number of layers of a tree and the algorithms for the Minimum Odd or Even Sum with Restrictions as subprocess. The details are omitted.

3 Algorithm for Unicyclic Graphs

First, we recall that a unicyclic graph is a connected graph with a unique cycle. So, we can regard a unicyclic graph as a cycle attached with each node a rooted tree, called a *suspended tree*. Note that the depth of a suspended tree can be 0. For simplicity, we say that a node t in the cycle has the same type as the type of the root t of the suspended tree. Based on the algorithm with restrictions for trees in Section 2, we give a graph-theoretic algorithm of linear time to the Minimum All-Ones Problem for unicyclic graphs.

Algorithm for Unicyclic Graphs

Case 1. If none of the nodes on the cycle with length q is an outcast then we use the following way to get all the possible truth values of all nodes on the cycle.

(1) Fix an order on the cycle. Assume that the truth value of the 1st node is x, the truth value of the 2nd node is y. x and y will be completely determined in the end.

(2) Suppose that the truth value of the i-th node is a_i, a function of x and y. Next, determine the truth value of the $(i+1)$-th node in two cases: If the i-th node is a rebel, then the truth value of the $(i+1)$-th node is $a_{i+1} = (1 - a_i - a_{i-1}) \mod 2$; else, the truth value is $a_{i+1} = a_{i-1}$. Repeat this step until $i = q$.

(3) After we get the truth value of a_q for the q-th node, the following equalities hold.

$$\begin{cases} a_{q-1} = x & \text{if the } q\text{-th node is an oddball} \\ a_{q-1} + a_q + x = 1 \bmod 2 & \text{if the } q\text{-th node is a rebel} \end{cases}$$

$$\begin{cases} a_q = y & \text{if the 1-st node is an oddball} \\ a_q + x + y = 1 \bmod 2 & \text{if the 1-st node is a rebel} \end{cases}$$

By solving these equalities, we get at most 4 possible values of x and y, which determine all possible truth values of each node on the cycle. Suppose that all the possible $k(1 \leq k \leq 4)$ group of truth values of the nodes on the cycle are z_{j1}, z_{j2}, ..., z_{jq}, $(1 \leq j \leq k)$. According to each group of z_{j1}, z_{j2}, ..., z_{jq}, the unicyclic graph G has a solution to the All-Ones Problem. Conversely, any solution to the All-Ones Problem for the unicyclic graph G has a restriction on the nodes on the cycle, which must coincide with one of the t groups z_{j1}, z_{j2}, ..., z_{jq}, $(1 \leq j \leq k)$.

Subcase 1.1 v_i is a rebel. If $z_{ji} = 1$. Let $C_j(T_i)$ be the minimum solution to the All-Ones Problem for T_i. If $z_{ji} = 0$. Let $C_j(T_i)$ be the minimum solution to the Quasi-All-Ones Problem for T_i.

Subcase 1.2 v_i is an oddball. If $z_{ji} = 1$, then let $C_j(T_i)$ be the minimum solution to the All-Ones Problem for T_i with the restriction that the truth value of the root is 1. If $z_{ji} = 0$, then let $C_j(T_i)$ be the minimum solution to the All-Ones Problem for T_i with the restriction that the truth value of the root is 0.

After getting all the $C_j(T_i)$, $1 \leq i \leq q$, we can see that

$$C_j(G) = \bigcup_{1 \leq i \leq q} C_j(T_i), \;\; 1 \leq j \leq k$$

are all the possible minimum solutions to the All-Ones Problem for the unicyclic graph G. So the minimum one in the k solutions is the minimum solution to the All-Ones Problem for G.

Case 2. There is at least one of the nodes on the cycle is an outcast. Suppose this outcast node is u. We fix an order to the nodes on the cycle. Suppose the node before u on the cycle is v, the node after u on the cycle is w. Then we cut the edges between u and v and u and w. The unicyclic graph G will be changed into two trees. One has the root u, denoted by T_u, the other is the remaining part of G excluding T_u, denoted by T'_u. We can easily verify that the minimum solution to the All-Ones Problem for G, denoted by $C(G)$, must be one of the following four possible solutions:

$$C(T'_u|tv(v) = i, tv(w) = j) \bigcup C_{[(i+j+1) \bmod 2]}(T_u), \;\; i, j \in \{0, 1\},$$

where $C_{[1]}(T_u)$ means the minimum solution to the All-Ones Problem for the suspended tree T_u with the root u, $C_{[0]}(T_u)$ means the minimum solution to the Quasi-All-Ones Problem for T_u.

Then the minimum one of the four possible solutions will be the minimum solution to the All-Ones Problem for unicyclic graphs.

Summing up the above, we get that the algorithm outputs a solution to the Minimum All-Ones Problem for unicyclic graphs, and the time complexity is linear.

For bicyclic graphs, the analysis of the Minimum All-Ones Problem would be similar but more detailed and complicated. As an abstract, we have to omit its details. We can use the same technique for tricyclic graphs, quadrucyclic graphs, etc. But, unfortunately, we cannot employ it efficiently for graphs with more and more cycles. We have point out that in [9], Sutner proved that the Minimum All-Ones Problem is NP-complete for general graphs.

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