

Removable Edges in a Cycle of a 4-Connected Graph *

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Abstract

Let G be a 4-connected graph. For an edge e of G , we do the following operations on G : first, delete the edge e from G , resulting the graph $G - e$; second, for all the vertices x of degree 3 in $G - e$, delete x from $G - e$ and then completely connect the 3 neighbors of x by a triangle. If multiple edges occur, we use single edges to replace them. The final resultant graph is denoted by $G \ominus e$. If $G \ominus e$ is still 4-connected, then e is called a *removable edge* of G . In this paper, we investigate the problem on how many removable edges there are in a cycle of a 4-connected graph, and give examples to show that our results are in some sense best possible.

Key Words: 4-Connected graph, Removable edge, Edge-vertex-cut fragment, Edge-vertex-cut atom

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1 Introduction

All graphs considered here are simple and finite. For notations and terminology not given here, we refer the reader(s) to [1]. In this paper we shall study the removable edges in a cycle of a 4-connected graph. First of all, we give the definition of a removable edge for a 4-connected graph. Let G be a 4-connected graph and e an edge of G . Consider the graph $G - e$ obtained by deleting the edge e from G . If $G - e$ has vertices of degree 3, we do the following operations on $G - e$. For all

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vertices x of degree 3 in $G - e$, delete x from $G - e$ and then completely connect the three neighbors of x by a triangle. If multiple edges occur, we use single edges to replace them. The final resultant graph is denoted by $G \ominus e$. Note that if there is no vertex of degree 3 in $G - e$, then $G \ominus e$ is simply the graph $G - e$.

Definition 1.1. For a 4-connected graph G and an edge e of G , if $G \ominus e$ is still 4-connected, then the edge e is called *removable*; otherwise, it is called *unremovable*. The set of all removable edges of G is denoted by $E_R(G)$; whereas the set of unremovable edges of G is denoted by $E_N(G)$.

Definition 1.2. A 2-cyclic graph G of order n is defined to be the square of the cycle C_n , namely, G can be obtained from C_n by adding edges between all pairs of vertices of C_n which are at distance 2 in C_n .

The aim to introduce the concept of removable edges in 4-connected graphs is to find new method to construct 4-connected graphs and to prove some properties of 4-connected graphs inductively. In [2], Yin proved that there always exist removable edges in 4-connected graphs G unless G is a 2-cyclic graph of order 5 or 6. He showed that a 4-connected graph can be obtained from a 2-cyclic graph by the following four operations: (i) adding edges, (ii) splitting vertices, (iii) adding vertices and removing edges, and (iv) extending vertices. He also obtained a lower bound for the number of removable edges and contractible edges in a 4-connected graph G . In this paper we shall investigate how many removable edges there are in a cycle of a 4-connected graph G , and give examples to show that our results are best possible in some sense.

For convenience we introduce the following notations. Without specific statement, in the sequel G always denotes a 4-connected graph. The vertex set and edge set of G is denoted, respectively, by $V(G)$ and $E(G)$. The order and size of G is denoted, respectively, by $|G|$ and $|E(G)|$. For $x \in V(G)$, we simply write $x \in G$. The neighborhood of $x \in G$ is denoted by $\Gamma_G(x)$ and the degree of x is denoted by $d(x)$. If x and y are the two end-vertices of an edge e , we write $e = xy$. For a nonempty subset F of $E(G)$, or N of $V(G)$, the induced subgraph by F or N in G is denoted by $[F]$ or $[N]$. Let $A, B \subset V(G)$ such that $A \neq \emptyset \neq B$ and $A \cap B = \emptyset$, define $[A, B] = \{xy \in E(G) \mid x \in A, y \in B\}$. If H is a subgraph of G , we say that G contains H . For a subset S of $V(G)$, $G - S$ denotes the graph obtained by deleting all the vertices in S from G together with all the incident edges. If $G - S$ is disconnected, we say that S is a vertex-cut of G . If $|S| = s$ for such an S , we say that S is an s -vertex-cut. For $e = xy \in E(G)$ and $S \subset V(G)$ such that $|S| = 3$, if $G - e - S$ has exactly two (connected) components, say A and B , such that $|A| \geq 2$ and $|B| \geq 2$, then we say that (e, S) is a *separating pair* and $(e, S; A, B)$ is a *separating group*, in which A and B are called the *edge-vertex-cut fragments*. If, moreover, $|A| = 2$, then A is called an *edge-vertex-cut atom*. For an edge-vertex-cut atom A , let $A = \{x, z\}$ and $S = \{a, b, c\}$, if $ax, bx \in E(G), cx \notin E(G)$, then A is called a *1-edge-vertex-cut atom*; whereas if $ax, bx, cx \in E(G)$, then A is called a *2-edge-vertex-cut atom*. It is easy to see that if A is an edge-vertex-cut atom, then A is either a 1-edge-vertex-cut atom or a 2-edge-vertex-cut atom. Let $E_0 \subset E_N(G)$ such that $E_0 \neq \emptyset$ and let $(xy, S; A, B)$ be a separating group of G such that $x \in A$ and $y \in B$. If $xy \in E_0$, then A and B are called *E_0 -edge-vertex-cut fragments*. An E_0 -edge-vertex-cut fragment is called an *E_0 -edge-vertex-cut end-fragment* of G if it does not contain any

other E_0 -edge-vertex-cut fragment of G as a proper subset. It is easy to see that any E_0 -edge-vertex-cut fragment of G contains a such end-fragment. Similarly, if $|A| = 2$, then A is called an E_0 -edge-vertex-cut atom.

2 Some Known Results

In the sequel we shall use the following results on the existence of removable edges in 4-connected graphs, which were obtained by Yin in [2].

Theorem 2.1. Let G be a 4-connected graph with $|G| \geq 7$. An edge e of G is unremovable if and only if there is a separating pair (e, S) , or a separating group $(e, S; A, B)$ in G .

Theorem 2.2. Let G be a 4-connected graph with $|G| \geq 8$ and let $(xy, S; A, B)$ be a separating group of G such that $x \in A$, $y \in B$ and $|A| \geq 3$. Then, every edge in $\{\{x\}, S\}$ is removable.

Corollary 2.3. Let G be a 4-connected graph with $|G| \geq 8$. Then, every 3-cycle of G contains at least one removable edge.

Theorem 2.4. Let G be a 4-connected graph with $|G| \geq 7$. If for an unremovable edge xy , i.e., $xy \in E_N(G)$, there is a separating group $(xy, S; A, B)$, then all the edge in $E([S])$ are removable, i.e., $E([S]) \subset E_R(G)$.

3 Notations and Terminology for Subgraphs With Special Structures

For convenience we introduce the following definitions for subgraphs of G with special structures.

Definition 3.1. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$ and $E(H) = \{ax_1, ax_2, ax_3, ax_4, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1v_1, x_2v_2, x_3v_3, x_4v_4\}$. If H satisfies the following conditions

- (i) $d(a) = d(x_i) = 4$ for $i = 1, 2, 3, 4$,
- (ii) $ax_1, ax_2, ax_3, ax_4 \in E_N(G)$ and $x_1x_2, x_2x_3, x_3x_4, x_4x_1 \in E_R(G)$,

then H is called a *helm*. The vertices a, x_i for $i = 1, 2, 3, 4$ of a helm H are called *inner vertices* of H .

Definition 3.2. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{a, b, x_1, x_2, \dots, x_{l+3}\}$ and $E(H) = \{x_1x_2, x_2x_3, \dots, x_{l+2}x_{l+3}, ax_2, ax_3, \dots, ax_{l+2}, bx_2, bx_3, \dots, bx_{l+2}\}$, where $l \geq 1$. If H satisfies the following conditions

- (i) $x_ix_{i+1} \in E_N(G)$ for $i = 1, 2, \dots, l+2$,
- (ii) $ax_j, bx_j \in E_R(G)$ for $j = 2, 3, \dots, l+2$,
- (iii) $d(x_j) = 4$ for $j = 2, 3, \dots, l+2$,

then H is called an l -bi-fan.

An l -bi-fan H is said to be *maximal* if $\Gamma_G(x_1) \neq \{a, b, x_2, u\}$ and $\Gamma_G(x_{l+3}) \neq \{a, b, x_{l+2}, v\}$ for any $u, v \in G$. The vertices of an l -bi-fan or a maximal l -bi-fan H satisfying the condition (iii) are called *inner vertices* of H .

Definition 3.3. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, \dots, x_{l+2}, y_1, y_2, \dots, y_{l+2}\}$ and $E(H) = E_1(H) \cup E_2(H)$ where $E_1(H) = \{x_1x_2, x_2x_3, \dots, x_{l+1}x_{l+2}, y_1y_2, y_2y_3, \dots, y_{l+1}y_{l+2}\}$ and $E_2(H) = \{y_1x_2, x_2y_2, y_2x_3, \dots, y_lx_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}\}$. Then, H is called an l -belt if the following conditions are satisfied

- (i) $E_1(H) \subset E_N(H)$ and $E_2(H) \subset E_R(H)$,
- (ii) $d(x_i) = d(y_j) = 4$ for $i = 2, 3, \dots, l+1; j = 2, 3, \dots, l+1$.

An l -belt H is said to be *maximal* if $\Gamma_G(y_1) \neq \{x_1, x_2, y_2, u\}$ and $\Gamma_G(x_{l+2}) \neq \{x_{l+1}, y_{l+1}, y_{l+2}, v\}$ for any $u, v \in G$. The vertices of an l -belt or a maximal l -belt H satisfying the condition (ii) are called *inner vertices* of H .

Definition 3.4. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, \dots, x_{l+2}, x_{l+3}, y_1, y_2, \dots, y_{l+2}\}$ and $E(H) = E_1(H) \cup E_2(H)$ where $E_1(H) = \{x_1x_2, x_2x_3, \dots, x_{l+1}x_{l+2}, x_{l+2}x_{l+3}, y_1y_2, y_2y_3, \dots, y_{l+1}y_{l+2}\}$ and $E_2(H) = \{y_1x_2, x_2y_2, y_2x_3, \dots, y_lx_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}, x_{l+2}y_{l+2}\}$. Then, H is called an l -co-belt if the following conditions are satisfied

- (i) $E_1(H) \subset E_N(H)$ and $E_2(H) \subset E_R(H)$,
- (ii) $d(x_i) = d(y_j) = 4$ for $i = 2, 3, \dots, l+1, l+2; j = 2, 3, \dots, l+1$.

An l -co-belt H is said to be *maximal* if $\Gamma_G(y_1) \neq \{x_1, x_2, y_2, u\}$ and $\Gamma_G(y_{l+2}) \neq \{x_{l+2}, y_{l+1}, x_{l+3}, v\}$ for any $u, v \in G$. The vertices of an l -co-belt or a maximal l -co-belt H satisfying the condition (ii) are called *inner vertices* of H .

Definition 3.5. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and $E(H) = \{x_1x_2, x_2x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$. Then, H is called a W -framework if the following conditions are satisfied

- (i) $x_i x_{i+1} \in E_N(G)$ for $i = 1, 2$,
- (ii) $d(x_2) = d(y_2) = d(y_3) = 4$,
- (iii) $y_2y_3, x_1y_2, x_2y_2, x_2y_3, x_3y_3 \in E_R(G)$.

The vertex x_2 of a W -framework H is called the *inner vertex* of H .

Definition 3.6. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and $E(H) = \{x_1x_2, x_2x_3, x_1x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$. Then, H is called a W' -framework if the following conditions are satisfied

- (i) $x_i x_{i+1} \in E_N(G)$ for $i = 1, 2$,

- (ii) $d(x_2) = d(x_3) = d(y_2) = d(y_3) = 4$ and $d(x_1) \geq 5$,
- (iii) $y_2y_3, x_1y_2, x_2y_3, x_3y_3, x_1x_3 \in E_R(G), x_2y_2 \in E_N(G)$.

The vertices x_2, x_3 of a W' -framework H are called *inner vertices* of H .

After we have done the above preparations, we can state and prove our main results in the next section.

4 The Main Results

In this section we shall consider the problem on how many removable edges there are in a cycle of a 4-connected graph G . Before we give our main results, we need to show some lemmas.

Lemma 4.1. Let G be a 4-connected graph, $(xy, S; A, B)$ be a separating group of G such that $x \in A, y \in B, S = \{a, b, c\}$ and A be a 1-edge-vertex atom, say, $A = \{x, z\}$. Then, one of the following conclusions holds:

- (i) $ax, bx, zx \in E_R(G)$.
- (ii) $ax \in E_N(G), d(x) = d(z) = 4, bx, zx, az \in E_R(G), zc \in E_N(G)$.
- (iii) $ax \in E_N(G), ay \in E_R(G)$. And, if $d(a) = 4, d(y) \geq 5$, then $az, zb, zx, by, ay \in E_R(G), bx \in E_N(G)$. If $d(a) \geq 5, d(y) = 4$, then $by, bx, bz, az \in E_R(G), zx \in E_N(G)$. If $d(a) = d(y) = 4$, then $az, bz, by \in E_R(G), bx, zx \in E_N(G)$. If $d(a) \geq 5, d(y) \geq 5$, then $az, zx, bx, by \in E_R(G)$.
- (iv) $ax, bx, ac, bc \in E_R(G), zx, zc \in E_N(G), \{za, zb\} \cap E_N(G) \neq \emptyset, d(x) = d(c) = d(z) = 4$. If $za \in E_N(G)$, then the following conclusion holds: $d(b) = 4$, and if $d(a) = 4$, then $bz \in E_N(G)$; if $d(a) \geq 5$, then $bz \in E_R(G)$ holds. If $bz \in E_N(G)$, then the following conclusion holds: $d(a) = 4$, and if $d(b) = 4$, then $az \in E_N(G)$; if $d(b) \geq 5$, then $az \in E_R(G)$.
- (v) $ax, bx, az, bz \in E_R(G), xz \in E_N(G), d(x) = d(z) = 4$.
- (vi) $bx \in E_N(G), by \in E_R(G)$. And, if $d(a) = 4, d(y) \geq 5$, then $bz, za, zx, ay, by \in E_R(G), ax \in E_N(G)$. If $d(b) \geq 5, d(y) = 4$, then $ay, ax, az, bz \in E_R(G), zx \in E_N(G)$. If $d(b) = d(y) = 4$, then $bz, az, ay \in E_R(G), ax, zx \in E_N(G)$. If $d(b) \geq 5, d(y) \geq 5$, then $bz, zx, ax, ay \in E_R(G)$.

Proof. If $ax, bx, zx \in E_R(G)$, then the conclusion (i) holds. So, we may assume that $\{ax, bx, zx\} \cap E_N(G) \neq \emptyset$. Next we will distinguish the following cases to proceed the proof.

Case 1. $ax \in E_N(G)$.

Then, we take the corresponding separating group $(ax, T; C, D)$ such that $x \in C, a \in D$, and so, $x \in A \cap C, y \in B \cap (C \cup T)$. Let

$$\begin{aligned}
X_1 &= (C \cap S) \cup (S \cap T) \cup (A \cap T), \\
X_2 &= (A \cap T) \cup (S \cap T) \cup (S \cap D), \\
X_3 &= (D \cap S) \cup (S \cap T) \cup (B \cap T), \\
X_4 &= (B \cap T) \cup (S \cap T) \cup (C \cap S).
\end{aligned}$$

Subcase 1.1. $y \in B \cap C$.

Since $|A| = 2$ and A is a connected subgraph of G , we have that $A \cap D = \emptyset$. First, we claim that $A \cap T \neq \emptyset$. Otherwise, $A \cap T = \emptyset$, and so $|A \cap C| = 2$. Since $a \in S \cap D$, we have that $|X_1| \leq 2$. Then, $X_1 \cup \{x\}$ is a vertex-cut of G with cardinality less than 4, a contradiction. Hence, $A \cap T = \{z\}$. Second, we claim that $S \cap T = \emptyset$. Otherwise, $S \cap T \neq \emptyset$, and a contradiction will be deduced as follows. If $B \cap T = \emptyset$, since B is a connected subgraph of G , then we have that $B \cap D = \emptyset$. Then, $B = B \cap C$, and so $|S \cap T| = 2$. Noticing that $a \in S \cap D$ and $|S| = 3$, we have that $S \cap C = \emptyset$. From $|B| \geq 2$ we know that $|B \cap C| \geq 2$. Then, it is easy to see that $\{y\} \cup (S \cap T)$ is a vertex-cut of G with cardinality less than 4, a contradiction. So, $B \cap T \neq \emptyset$, and so $|S \cap T| = 1$. Noticing that $|T| = 3$, we have that $|B \cap T| = 1$. Since X_4 is a vertex-cut of $G - xy$, we have that $|X_4| \geq 3$, and so, $|S \cap C| \geq 1$. Since $S \cap D \neq \emptyset$, by noticing that $|S| = 3$, we have that $|S \cap D| = 1$, i.e., $S \cap D = \{a\}$. Note that $|X_3| = 3$. Since G is 4-connected, we have that $B \cap D = \emptyset$. Hence, $D = \{a\}$, which contradicts to that $|D| \geq 2$. Therefore, $S \cap T = \emptyset$. Note that $|B \cap T| = 2$. If $|S \cap D| = 1$, by a similar argument we can get that $D = \{a\}$, a contradiction. So, $|S \cap D| \geq 2$. Since $|X_4| \geq 3$, we have that $|S \cap C| \geq 1$. Therefore, $|S \cap C| = 1$ and $|S \cap D| = 2$. Since $bx \in E(G)$, obviously we have $b \in X_1$, and so $S \cap C = \{b\}$. Then, $S \cap D = \{a, c\}$, $\Gamma_G(x) = \{a, b, y, z\}$, $\Gamma_G(z) = \{x, a, b, c\}$. We claim that $xz \in E_R(G)$. Otherwise, $xz \in E_N(G)$, and we take the corresponding separating group $(xz, S'; A', B')$ such that $x \in A', z \in B'$. Since $xzax$ is a 3-cycle of G , we have that $a \in S'$ and $ax \in E_N(G)$. From Theorem 2.2 we know that $|A'| = 2$, say $A' = \{x, v_1\}$. Then, we have that axv_1a is a 3-cycle of G and $v_1 \neq z$, which is impossible to hold in G , and so, $xz \in E_R(G)$. We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, and we take the corresponding separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Obviously, $x \in S'$. Since $ax \in E_N(G)$, from Theorem 2.2 we have that $|A'| = 2$, say $A' = \{a, v_1\}$. Then, axv_1a is a 3-cycle of G and $v_1 \neq z$, which is impossible to hold in G , and so, $az \in E_R(G)$. Let $S' = \{x\} \cup (B \cap T)$, $A' = C \cap (B \cup S)$, $B' = G - bz - S' - A'$, then $(bz, S'; A', B')$ is a separating group of G , and so $bz \in E_N(G)$. We claim that $bx \in E_R(G)$. Otherwise, $bx \in E_N(G)$, and we take the corresponding separating group $(bx, S'; A', B')$ such that $b \in A', x \in B'$. Since $bxzb$ is a 3-cycle of G , we have that $z \in S'$. Since $bz \in E_N(G)$, we have that $|A'| = 2$, say $A' = \{b, v_1\}$. Then, bv_1zb is a 3-cycle of G , and $v_1 \neq x$, which is impossible to hold in G , and hence $bx \in E_R(G)$. Let $S_1 = \{a, b, y\}$, then (zc, S_1) is a separating pair of G , and so, $zc \in E_N(G)$. Obviously, $d(x) = d(z) = 4$. Hence, the conclusion (ii) holds.

Subcase 1.2. $y \in B \cap T$.

Since $xy \in E_N(G)$, from Theorem 2.2 we have that $|C| = 2$. If $|A \cap C| = 2$, then we have that $A = A \cap C = C$. Since $B \cap T \neq \emptyset \neq S \cap D$, we have that $|S \cap T| \leq 2$. It is easy to see that $\{x\} \cup X_1$ is a vertex-cut of G with cardinality less than 4, a contradiction. So, $A \cap C = \{x\}$. Since A and C are connected subgraphs of G , we have that $|S \cap C| = |A \cap T| = 1$ and $B \cap C = \emptyset = A \cap D$. We claim that $S \cap T = \emptyset$. Otherwise, $|S \cap T| = 1$, and so $|B \cap T| = 1$. Note that $|X_3| = 3$. Since G is 4-connected, we have that $B \cap D = \emptyset$, and so $B = B \cap T = \{y\}$, which contradicts to that $|B| \geq 2$. Therefore, $S \cap T = \emptyset$, and so $|B \cap T| = |S \cap D| = 2$. From $\Gamma_G(x) = \{z, b, a, y\}$ we know that $S \cap C = \{b\}$, and so $S \cap D = \{a, c\}$, $A \cap T = \{z\}$.

Let $B \cap T = \{u, y\}$. Next we will discuss the following subsubcases.

Subsubcase 1.2.1. If $ay \notin E(G)$, we claim that $xz \in E_R(G)$. Otherwise, $xz \in E_N(G)$, and we take the corresponding separating group $(xz, S'; A', B')$ such that $z \in A', x \in B'$. Since $azxa$ is a 3-cycle of G , we have that $a \in S'$. Since $ax \in E_N(G)$, from Theorem 2.2 we have that $|B'| = 2$, say $B' = \{x, v_1\}$. Then, axv_1a is a 3-cycle of G . However, $ay \notin E(G)$ and $v_1 \neq z$, which is impossible to hold in G . Hence, $xz \in E_R(G)$. By symmetry, we can show that $bx \in E_R(G)$. We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, and we take the corresponding separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Since $azxa$ is a 3-cycle of G , we have that $x \in S'$. Since $ax \in E_N(G)$, we have that $|A'| = 2$, say $A' = \{a, v_1\}$. Then, axv_1a is a 3-cycle of G , an analogous argument can lead to a contradiction. So, $az \in E_R(G)$. By symmetry, we have that $by \in E_R(G)$. Let $S' = \{a, b, y\}$. Obviously, (zc, S') is a separating pair of G , and so $zc \in E_N(G)$. Hence, the conclusion (ii) holds.

Subsubcase 1.2.2. If $ay \in E(G)$, then from Corollary 2.3 we know that $ay \in E_R(G)$. Then, we consider the following cases.

(1.) If $d(a) \geq 5$ and $d(y) \geq 5$, we claim that $xz \in E_R(G)$. Otherwise, $xz \in E_N(G)$, and we take the corresponding separating group $(xz, S'; A', B')$ such that $x \in A', z \in B'$. Since $azxa$ is a 3-cycle of G , we have that $a \in S'$. Since $ax \in E_N(G)$, from Theorem 2.2 we know that $|A'| = 2$, say $A' = \{x, v_1\}$. Then, axv_1a is a 3-cycle of G . Noticing that $d(v_1) = 4$ and $d(y) \geq 5$, we have that $v_1 \neq y$, which is impossible to hold in G . Hence, $xz \in E_R(G)$. By symmetry, we can show that $bx \in E_R(G)$. We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, and we take the corresponding separating group $(az, S'; A', B')$. Obviously, $x \in S'$, and an analogous argument can lead to a contradiction. So, $az \in E_R(G)$. By symmetry, we have that $by \in E_R(G)$. Hence, the conclusion (iii) holds.

(2.) If $d(a) = 4$ and $d(y) \geq 5$, we let $\Gamma_G(a) = \{x, y, z, v\}$. Let $A' = \{a, x\}, S' = \{v, z, y\}, B' = G - bx - S' - A'$, then $(bx, S'; A', B')$ is a separating group of G , and so $bx \in E_N(G)$. We claim that $bz \in E_R(G)$. Otherwise, $bz \in E_N(G)$, and we take the corresponding separating group $(bz, S'; A', B')$ such that $b \in A', z \in B'$. Noticing that $bzxb$ is a 3-cycle of G , we have $x \in S'$. Since $bx \in E_N(G)$, from Theorem 2.2 we have that $|A'| = 2$, say $A' = \{b, v_1\}$. Then, bxv_1b is a 3-cycle of G . Noticing that $d(y) \geq 5$ and $d(v_1) = 4$, we have that $v_1 \neq y$, which is impossible to hold in G . Therefore, $bz \in E_R(G)$. We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, and we take the separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Obviously, $x \in S'$. Since $ax \in E_N(G)$, from Theorem 2.2 we have that $|A'| = 2$, say $A' = \{a, v_1\}$. Then, axv_1a is a 3-cycle of G and $v_1 \neq z$. Note that $d(v_1) = 4, d(y) \geq 5$, and so, $v_1 \neq y$, which is impossible to hold in G . So, $az \in E_R(G)$. By an analogous argument we can show that $zx \in E_R(G)$. We claim that $by \in E_R(G)$. Otherwise, $by \in E_N(G)$, and we take the separating group $(by, S'; A', B')$ such that $b \in A', y \in B'$. Obviously, $x \in S'$. Since $xy \in E_N(G)$, from Theorem 2.2 we have that $|B'| = 2$, say $B' = \{y, v_1\}$. Then, xyv_1x is a 3-cycle of G . It is easy to see that this is true only if $v_1 = a$. From $\Gamma_G(a) = \{x, y, z, v\}$ we know that $S' = \{x, z, v\}$. Since $d(y) \geq 5$, we have $yz \in E(G)$, which is impossible to hold in G . So, $by \in E_R(G)$. Hence, the conclusion (iii) holds.

(3.) If $d(a) \geq 5$ and $d(y) = 4$. By an analogous argument used in (2.) we can show that the conclusion (iii) holds.

(4.) If $d(a) = d(y) = 4$, we let $\Gamma_G(a) = \{x, y, z, v\}$, $A_1 = \{a, x\}$, $S_1 = \{z, y, v\}$, $B_1 = G - bx - S_1 - A_1$. Then, $(bx, S_1; A_1, B_1)$ is a separating group of G , and so, $bx \in E_N(G)$. By symmetry, we have that $ax, xy, zx \in E_N(G)$. From Corollary 2.3 we have that $az, by, bz \in E_R(G)$. Hence, the conclusion (iii) holds.

If $bx \in E_N(G)$, we may employ a similar argument to show that the conclusion (iv) holds. So, next we may assume that $ax, bx \in E_R(G)$.

Case 2. $xz \in E_N(G)$.

We take the corresponding separating group $(xz, T; C, D)$ such that $x \in C, z \in D$. Then, we have that $x \in A \cap C, z \in A \cap D$. Since $xzax, xzbx$ are two 3-cycles of G , we have that $a, b \in S \cap T$. Since $A \cap D = \{z\}$ and D is a connected subgraph of G as well as $|D| \geq 2$, we can get that $S \cap D \neq \emptyset$. Since $S = \{a, b, c\}$, we have that $S \cap D = \{c\}$. Obviously, $|B \cap T| = 1$.

Subcase 2.1. If $az \in E_N(G)$, from Theorem 2.2 we have that $|D| = 2$, and so $D = \{z, c\}$. It is easy to see that $ac, bc \in E(G)$. From Theorem 2.4 we have that $ac, bc \in E_R(G)$. Obviously, $d(x) = d(c) = d(z) = 4$ and $\Gamma_G(x) = \{z, b, a, y\}$. Let $A_1 = \{x, z\}$, $S_1 = \{y, a, b\}$, $B_1 = G - zc - S_1 - A_1$, then $(zc, S_1; A_1, B_1)$ is a separating group of G , and so $zc \in E_N(G)$. We take the separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Obviously, $x \in S'$. Since $xz \in E_N(G)$, we have that $|B'| = 2$, say $B' = \{z, v_1\}$. Then, xzv_1x is a 3-cycle of G , which is true only if $v_1 = b$, and so $d(b) = 4$. Here, if $d(a) = 4$, let $\Gamma_G(a) = \{x, z, c, v\}$, $A_1 = \{a, z\}$, $S_1 = \{c, x, v\}$ and $B_1 = G - bz - S_1 - A_1$. Then $(bz, S_1; A_1, B_1)$ is a separating group of G , and so $bz \in E_N(G)$. If $d(a) \geq 5$, we claim that $bz \in E_R(G)$. Otherwise, $bz \in E_N(G)$, then we take the corresponding separating group $(bz, S_1; A_1, B_1)$ such that $b \in A_1, z \in B_1$. Obviously, $x \in S_1$. Since $xz \in E_N(G)$, from Theorem 2.2 we have $|B_1| = 2$, say $B_1 = \{z, v_1\}$. Then xv_1zx is a 3-cycle of G . Note that $d(a) \geq 5, d(v_1) = 4$, and so $v_1 \neq a$. Which is impossible to hold in G . So, $bz \in E_R(G)$. Hence, the conclusion (iv) holds.

Subcase 2.2. If $bz \in E_N(G)$, we may employ a similar argument used in Subcase 2.1 to show that the conclusion (iv) holds.

Therefore, we may assume that $az, bz \in E_R(G)$. Obviously, $d(x) = d(z) = 4$, and so the conclusion (v) holds. The proof is now complete. \square

Corollary 4.2. Let G be a 4-connected graph and $(xy, S; A, B)$ be a separating group of G such that $x \in A, y \in B, S = \{a, b, c\}$. Let A be a 1-edge-vertex-cut atom, say $A = \{x, z\}$. If $\{xa, xb, xz\} \cap E_N(G) \neq \emptyset$, then we have that x is an inner vertex of one of the following subgraphs in G : helm, co-belt, belt, W' -framework, W -framework or bi-fan.

Lemma 4.3. Let G be a 4-connected graph, $(xy, S; A, B)$ be a separating group of G , and A be a 2-edge-vertex-cut atom, say $A = \{x, z\}$ and $S = \{a, b, c\}$. Then, $ax, bx, cx, xz \in E_R(G)$.

Proof. By contradiction. We consider the following cases.

(1.) If $ax \in E_N(G)$, we take the corresponding separating group $(ax, T; C, D)$ such that $x \in C, a \in D$. Then, $x \in A \cap C, a \in S \cap D$. Let $X = (D \cap S) \cup (S \cap T) \cup (B \cap T)$. Since $bx, cx \in E(G)$, we can get that $b, c \in S \cap (C \cup T)$, and so $|S \cap D| = 1$. We

claim that $A \cap T \neq \emptyset$. Otherwise, $A \cap T = \emptyset$. Since $|A| = 2$ and A is a connected subgraph of G , we have that $A \cap C = \{x, z\}$. It is easy to see that $\{b, c, x\}$ would be a 3-vertex-cut of G , a contradiction. Therefore, $A \cap T = \{z\}$, $A \cap D = \emptyset$. Obviously, $|X| \geq 3$. Since $|S \cap D| = 1$ and $|D| \geq 2$, we have that $B \cap D \neq \emptyset$, and so $|X| \geq 4$. However, by noticing that $|A \cap T| = 1$, we have that $|(S \cup B) \cap T| = 2$, and so $|X| = 3$, a contradiction.

If $bx \in E_N(G)$ or $cx \in E_N(G)$, we may employ a similar argument. So, next we may assume that $bx, cx \in E_R(G)$.

(2.) If $xz \in E_N(G)$, we take the corresponding separating group $(xz, T; C, D)$ such that $x \in C, z \in D$. Then, we have that $x \in A \cap C, z \in A \cap D$. It is easy to see that $a, b, c \in S \cap T$. Since $|T| = 3$, we have that $y \in B \cap C$. Let $X = (D \cap S) \cup (S \cap T) \cup (B \cap T)$, and so $|X| = 3$. Then, we have that $B \cap D = \emptyset$. Noticing that $D \cap S = \emptyset$, we have that $D = A \cap D = \{z\}$, which contradicts to that $|D| \geq 2$. Therefore, $xz \in E_R(G)$.

From the above arguments, we know that the lemma holds. \square

Now we present our main results. For convenience we denote by \mathfrak{R} the set of all helms, maximal l -bi-fans, maximal l -belts, maximal l -co-belts, W -frameworks and W' -frameworks of a graph G .

Definition 4.4. Let C be a cycle of a 4-connected graph G and H a subgraph of G belonging to \mathfrak{R} . If C contains an inner vertex of H , then we say that C *passes through* H .

Theorem 4.5. Let G be a 4-connected graph and C a cycle of G . If C does not pass through any subgraph of G belonging to \mathfrak{R} , then there are least two removable edges of G in C .

Proof. By contradiction. Assume that C does not pass through any subgraph of G belonging to \mathfrak{R} , and there is at most one removable edge of G in C . Let $F = E(C) \cap E_R(G)$, then $|F| \leq 1$. Denote $E(C) - F$ by E_0 . We take the separating group $(uw, S'; A', B')$ such that $u \in A', w \in B'$ and $uw \in E_0$. From $|F| \leq 1$ we know that $(E(A') \cup ([A', S']) \cap F = \emptyset$ or $(E(B') \cup ([S', B']) \cap F = \emptyset$. Without loss of generality, we may assume that $(E(A') \cup ([A', S']) \cap F = \emptyset$. Since A' is an E_0 -edge-vertex-cut fragment, A' must contain an E_0 -edge-vertex-cut end-fragment as its subgraph, say A . Then, we have that $(E(A) \cup ([A, S]) \cap F = \emptyset$, and we take a separating group $(xy, S; A, B)$ such that $x \in A, y \in B$ with $xy \in E_0$. Next, we will consider $|A|$ by cases.

Case 1. $|A| = 2$. Then, A is a 1-edge-vertex-cut atom or a 2-edge-vertex-cut atom, say, $A = \{x, z\}$. Let $S = \{a, b, c\}$.

Subcase 1.1. If A is a 2-edge-vertex-cut atom, since $xy \in E(C)$ and C is a cycle of G , we have that $\{xa, xb, xc, xz\} \cap E(C) \neq \emptyset$. From Lemma 4.3 we know that $\{xa, xb, xc, xz\} \subset E_R(G)$, which contradicts to that $(E(A) \cup [A, S]) \cap F = \emptyset$.

Subcase 1.2. If A is a 1-edge-vertex-cut atom, by noticing that C is a cycle of G and $([E(A) \cup [A, S]) \cap F = \emptyset$, then obviously $\{xa, xb, xz\} \cap E_N(G) \neq \emptyset$. From Corollary 4.2 we know that x is an inner vertex of one of the subgraphs of G belonging to \mathfrak{R} . Since $xy \in E(C)$, this contradicts to that C does not pass through any subgraph of

G belonging to \mathfrak{R} .

Case 2. $|A| \geq 3$. Then, we will discuss the following subcases.

Subcase 2.1. If there exists an $xz \in E_0 \cap E(A \cup [A, S])$, then obviously $z \notin S$; otherwise, we would have $|A| = 2$, a contradiction to that $|A| \geq 3$. We take the separating group $(xz, S_1; A_1, B_1)$ such that $x \in A_1, z \in B_1$. Then, we have that $x \in A \cap A_1, z \in A \cap B_1$. Let

$$\begin{aligned} X_1 &= (A_1 \cap S) \cup (S \cap S_1) \cup (A \cap S_1), \\ X_2 &= (A \cap S_1) \cup (S \cap S_1) \cup (B_1 \cap S), \\ X_3 &= (B_1 \cap S) \cup (S \cap S_1) \cup (B \cap S_1), \\ X_4 &= (B \cap S_1) \cup (S \cap S_1) \cup (A_1 \cap S). \end{aligned}$$

If $y \in B \cap S_1$, from Theorem 2.2 we have that $|A_1| = 2$, say $A_1 = \{x, v_1\}$. We claim that A_1 is a 1-edge-vertex-cut atom; otherwise, A_1 is a 2-edge-vertex-cut atom, and then, from Lemma 4.3 we have $xy \in E_R(G)$, a contradiction. From Corollary 4.2 we know that x is an inner vertex of some subgraph of G belonging to \mathfrak{R} , a contradiction to the assumption. Therefore, $y \notin B \cap S_1$, and so $y \in A_1 \cap B$. Since $A \cap B_1 \neq \emptyset$, we have that X_2 is a vertex-cut of $G - xz$, and so $|X_2| \geq 3$. By an analogous argument, we can deduce that $|X_4| \geq 3$. Since $|X_2| + |X_4| = |S| + |S_1| = 6$, we can get that $|X_2| = |X_4| = 3$, and so $|A_1 \cap S| = |A \cap S_1|, |B \cap S_1| = |B_1 \cap S|$. We claim that $A \cap B_1 = \{z\}$. Otherwise, $|A \cap B_1| \geq 2$. Then, $(xz, X_2; A \cap B_1, A_1 \cup B)$ is a separating group of G and $xz \in E_0$. It is easy to see that $A \cap B_1$ is an E_0 -edge-vertex-cut fragment contained in A , which contradicts to that A is an E_0 -edge-vertex-cut end-fragment of G . Therefore, $A \cap B_1 = \{z\}$. Since $|B_1| \geq 2$ and B_1 is a connected subgraph of G , we have that $B_1 \cap S \neq \emptyset$.

Subsubcase 2.1.1. If $|B_1 \cap S| = |B \cap S_1| = 3$, then $|X_1| = 0$, and so $\{z, y\}$ would be 2-vertex-cut of G , a contradiction.

Subsubcase 2.1.2. If $|B_1 \cap S| = |B \cap S_1| = 2$, since X_1 is a vertex-cut of $G - xy - xz$, then $|X_1| \geq 2$. Noticing that $|S| = |S_1| = 3$, we have that $|A \cap S_1| = |A_1 \cap S| = 1, S \cap S_1 = \emptyset$. We claim that $A \cap A_1 = \{x\}$. Otherwise, $|A \cap A_1| \geq 2$. Then, $\{x\} \cup X_1$ would be a 3-vertex-cut of G , a contradiction. Let $A \cap S_1 = \{a\}, A_1 \cap S = \{b\}, S \cap B_1 = \{v_1, v_2\}$. From $A \cap B_1 = \{z\}$ we can get that $\Gamma_G(z) = \{x, a, v_1, v_2\}$. We claim that $ab \in E(G)$. Otherwise, $\{x, v_1, v_2\}$ would be a 3-vertex-cut of G , a contradiction. We claim that $av_1, av_2 \in E(G)$. Otherwise, without loss of generality, we may assume that $av_1 \notin E(G)$. Let $A' = \{x, a\}, S' = \{b, z, v_2\}, B' = G - xy - S' - A'$, then $(xy, S'; A', B')$ is a separating group of G . Since $xy \in E_0$, A' is an E_0 -edge-vertex-cut fragment contained in A , which contradicts to that A is an E_0 -edge-vertex-cut end-fragment. So, $av_1, av_2 \in E(G)$, and hence $\Gamma_G(a) = \{x, z, b, v_1, v_2\}$. Let $S_0 = \{x, v_1, v_2\}, A_0 = \{a, z\}, B_0 = G - ab - S_0 - A_0$, then $(ab, S_0; A_0, B_0)$ is a separating group of G , and so $ab \in E_N(G)$.

We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, and we take the corresponding separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Since $axza, av_1za, av_2za$ are 3-cycles of G , we have that $x, v_1, v_2 \in S'$. Since $xz \in E_N(G)$, from Theorem 2.2 we have that $|B'| = 2$, say $B' = \{z, u\}$. Then, $uzxu$ is a 3-cycle of G , which is impossible to hold in G , and so $az \in E_R(G)$.

Since $(E(A) \cup ([A, S])) \cap F = \emptyset$ and C is a cycle of G , we can get that $\{zv_1, zv_2\} \cap E_N(G) \neq \emptyset$. Without loss of generality, we may assume that $zv_1 \in E_N(G)$. We take the separating group $(zv_1, T; C', D')$ such that $z \in C', v_1 \in D'$. Then, we have that $z \in C' \cap B_1, v_1 \in B_1 \cap D'$. Obviously, $a \in S_1 \cap T$. Let

$$\begin{aligned} Y_1 &= (A_1 \cap T) \cup (S_1 \cap T) \cup (C' \cap S_1), \\ Y_2 &= (C' \cap S_1) \cup (S_1 \cap T) \cup (B_1 \cap T), \\ Y_3 &= (B_1 \cap T) \cup (S_1 \cap T) \cup (S_1 \cap D'), \\ Y_4 &= (D' \cap S_1) \cup (S_1 \cap T) \cup (A_1 \cap T). \end{aligned}$$

(1.) If $x \in A_1 \cap C'$, then Y_1 is a vertex-cut of $G - xz$, and so $|Y_1| \geq 3$. By a similar argument, we have that $|Y_3| \geq 3$. Since $|Y_1| + |Y_3| = |S_1| + |T| = 6$, we can conclude that $|Y_1| = |Y_3| = 3$ and $|A_1 \cap T| = |S_1 \cap D'|, |S_1 \cap C'| = |B_1 \cap T|$. Since $a \in S_1$, from Theorem 2.4 we know that $b \notin T \cup S_1$. Since $bx, zv_2 \in E(G)$, we have that $b \in A_1 \cap C'$ and $v_2 \notin D' \cap B_1$. From $\Gamma_G(a) = \{v_1, v_2, z, x, b\}$, we know that $\Gamma_G(a) \cap (B_1 \cap D') = \{v_1\}$. Then, we have that $|A_1 \cap T| = |S_1 \cap D'| = 0, 1$ or 2 .

(1.1.) If $|A_1 \cap T| = |D' \cap S_1| = 2$, then $|S_1 \cap C'| = |B_1 \cap T| = 0$. Since $zv_2 \in E(G)$, we have $v_2 \in B_1 \cap C'$, and hence $\{a, z\}$ would be 2-vertex-cut of G , a contradiction.

(1.2.) If $|A_1 \cap T| = |D' \cap S_1| = 1$, then $|S_1 \cap T| \leq 2$. First, we claim that $B_1 \cap D' = \{v_1\}$. Otherwise, $|B_1 \cap D'| \geq 2$. Then, from $\Gamma_G(a) \cap (B_1 \cap D') = \{v_1\}$, we can conclude that $\{v_1\} \cup (Y_3 - \{a\})$ would be a 3-vertex-cut of G , a contradiction. So, $B_1 \cap D' = \{v_1\}$. Let $D' \cap S_1 = \{u_1\}$. If $A_1 \cap D' \neq \emptyset$, from $\Gamma_G(a) = \{x, z, b, v_1, v_2\}$ we can get that $A_1 \cap D' \cap \Gamma_G(a) = \emptyset$, and so $Y_4 - \{a\}$ would be a vertex-cut of G with cardinality less than 4, a contradiction. Therefore, $A_1 \cap D' = \emptyset$. Then, $au_1 \in E(G)$. However, it is easy to see that $u_1 \notin \{x, z, b, v_1, v_2\}$, a contradiction.

(1.3.) If $|D' \cap S_1| = |A_1 \cap T| = 0$, since D' is a connected subgraph of G , we have that $A_1 \cap D' = \emptyset$. Then, $|D'| = |D' \cap B_1| \geq 2$. Since $\Gamma_G(a) \cap (B_1 \cap D') = \{v_1\}$, by noticing that $|Y_3| = 3$, we have that $\{v_1\} \cup (Y_3 - \{a\})$ would be a 3-vertex-cut of G , a contradiction.

(2.) If $x \in A_1 \cap T$, from Theorem 2.2 we have that $|C'| = 2$. Since C' is a connected subgraph of G , we have that $A_1 \cap C' = \emptyset$. If $S_1 \cap C' \neq \emptyset$, since $a \in S_1 \cap T$, then $|D' \cap S_1| \leq 1$. Noticing that Y_3 is a vertex-cut of $G - zv_1$, we have that $|Y_3| \geq 3$, and so $|B_1 \cap T| = 1, A_1 \cap T = \{x\}$. Obviously, $|Y_4| = 3$, and hence $A_1 \cap D' = \emptyset$, and so $A_1 = \{x\}$, which contradicts to that $|A_1| \geq 2$. So, we have that $S_1 \cap C' = \emptyset$, and so $|B_1 \cap C'| = 2$. Since $A_1 \cap T \neq \emptyset$, obviously, $\{z\} \cup (T - \{x\})$ would be a vertex-cut with cardinality less than 4, a contradiction.

From the above arguments, we can conclude that Subsubcase 2.1.2 does not occur.

Subsubcase 2.1.3. If $|B_1 \cap S| = |B \cap S_1| = 1$, then $|S \cap S_1| \leq 2$. We claim that $|S \cap S_1| < 2$. Otherwise, $|S \cap S_1| = 2$. Then, $A \cap S_1 = \emptyset = S \cap A_1$. If $|A \cap A_1| \geq 2$, then $\{x\} \cup (S \cap S_1)$ would be a vertex-cut of G with cardinality less than 4, a contradiction, and so $A \cap A_1 = \{x\}$. Note that $|X_2| = 3$. If $|A \cap B_1| \geq 2$, then by an argument similar to that used in Subcase 2.1, $A \cap B_1$ would be an E_0 -edge-vertex-cut fragment contained in A , which contradicts to that A is an E_0 -edge-vertex-cut end-fragment. Hence, $A \cap B_1 = \{z\}$, and so $|A| = 2$, which contradicts to that $|A| \geq 3$. Therefore, $|S \cap S_1| \leq 1$, and then $|X_3| \leq 3$, and so $B \cap B_1 = \emptyset$. Since

$A \cap B_1 = \{z\}$, we have that $|B_1| = 2$ and B_1 is a 1-edge-vertex-cut atom of G , say $B_1 = \{z, u\}$. Since C is a cycle and $(E(A) \cup [A, S]) \neq \emptyset$, we have that z is incident with at least two unremovable edges. From Corollary 4.2 we know that z is an inner vertex of some subgraph of G belong to \mathfrak{R} , which contradicts to that C does not pass through any subgraph of G belonging to \mathfrak{R} . The proof is now complete. \square

Theorem 4.6. Let G be a 4-connected graph and C a cycle of G . If C passes through only one subgraph of G belonging to \mathfrak{R} , then there exists at least one removable edge of G in C .

Proof. By contradiction. Assume that $E(C) \subset E_N(G)$. Let C pass through the subgraph H of G that belongs to \mathfrak{R} , see the definitions of H in Definitions 3.1 through 3.6. If H is a maximal l -belt, from the assumption, it is easy to see that $\{x_2x_1, y_l y_{l+1}\} \cap E(C) \neq \emptyset$. If $x_2x_1 \in E(C)$, by letting $S = \{y_{l+2}, x_{l+2}, y_1\}$, $e = x_2x_1$, $B = \{x_2, \dots, x_{l+1}, y_2, \dots, y_{l+1}\}$, $A = G - e - S - B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the maximal l -belt ($l \geq 1$); if $y_l y_{l+1} \in E(C)$, by letting $S = \{x_1, y_1, x_{l+2}\}$, $e = y_{l+1}y_{l+2}$, $B = \{x_2, \dots, x_{l+1}, y_2, \dots, y_{l+1}\}$, $A = G - e - S - B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the maximal l -belt ($l \geq 1$). If H is a maximal l -co-belt, similarly, we have that $\{x_1x_2, y_1y_2\} \cap E(C) \neq \emptyset$, if $x_1x_2 \in E(C)$, by letting $S = \{y_{l+2}, x_{l+3}, y_1\}$, $e = x_2x_1$, $B = \{x_2, \dots, x_{l+2}, y_2, \dots, y_{l+1}\}$, $A = G - e - S - B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the maximal l -co-belt ($l \geq 1$); if $y_1y_2 \in E(C)$, by letting $S = \{y_{l+2}, x_{l+3}, x_2\}$, $e = y_2y_1$, $B = \{x_3, \dots, x_{l+2}, y_2, \dots, y_{l+1}\}$, $A = G - e - S - B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the maximal l -co-belt ($l \geq 1$). If H is a maximal l -bi-fan ($l \geq 1$), by letting $S = \{a, b, x_{l+3}\}$, $e = x_2x_1$, $B = \{x_2, \dots, x_{l+2}\}$, $A = G - e - S - B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the maximal l -bi-fan. If H is a helm, by letting $e = x_1v_1$, $S = \{v_2, v_3, v_4\}$, $B = \{a, x_1, x_2, x_3, x_4\}$, $A = G - e - S - B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the helm. If H is a W -framework, then C must pass through x_1x_2, x_2x_3 . In this case, by letting $e = x_2x_1$, $S = \{x_3, x_4, y_2\}$, $B = \{x_2, y_3\}$, $A = G - e - S - B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the W -framework. If H is a W' -framework, by noticing that $\{x_1x_2, x_2y_2\} \cap E(C) \neq \emptyset$, then if $x_1x_2 \in E(C)$, by letting $S = \{y_2, x_3, y_4\}$, $B = \{x_2, y_3\}$, $A = G - x_1x_2 - S - B$, then $(x_1x_2, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the W' -framework; if $x_2y_2 \in E(C)$, by letting $S = \{x_1, y_3, v\}$ such that $v \in \Gamma_G(x_3)$, $B = \{x_2, x_3\}$, $A = G - x_2y_2 - S - B$, then the separating group $(x_2y_2, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the W' -framework.

Let $E_0 = E(C)$, then A is an E_0 -edge-vertex-cut fragment of G such that it does not contain any inner vertex of H . Obviously, A contains an E_0 -edge-vertex-cut end-fragment as its subgraph, say A' . It is easy to see that A' does not contain any inner vertex of H . Finally, by an argument analogous to that used in the proof of Theorem 4.5, we can show that A' contains an inner vertex of some subgraph of G belonging to \mathfrak{R} , which contradicts to that A' does not contain any inner vertex of any subgraph of G belonging to \mathfrak{R} . The proof is now complete. \square

Finally, to end this paper we construct examples to show that the lower bounds for the numbers of removable edges of G that a cycle of G can contain in Theorems 4.5 and 4.6 are in some sense best possible, and we also construct an example to show that the conditions, i.e., the numbers of subgraphs of G belonging to \mathfrak{R} that a cycle of G can pass through in Theorems 4.5 and 4.6 are in some sense best possible.

Let F be a maximal k -bi-fan such that $V(F) = \{a, b, z_1, z_2, \dots, z_{k+3}\}$ and $E(F) = \{z_1 z_2, z_2 z_3, \dots, z_{k+2} z_{k+3}, a z_2, a z_3, \dots, a z_{k+2}, b z_2, \dots, b z_{k+2}\}$ where $k \geq 1$. Let L be a maximal l -belt such that $V(L) = \{x_1, x_2, \dots, x_{l+2}, y_1, y_2, \dots, y_{l+2}\}$ and $E(H) = E_1(H) \cup E_2(H)$ where $E_1(H) = \{x_1 x_2, x_2 x_3, \dots, x_{l+1} x_{l+2}, y_1 y_2, y_2 y_3, \dots, y_{l+1} y_{l+2}\}$ and $E_2(H) = \{y_1 x_2, x_2 y_2, y_2 x_3, \dots, y_l x_{l+1}, x_{l+1} y_{l+1}, y_{l+1} x_{l+2}\}$, in which $l \geq 1$.

Example 1. Identify the vertex a with x_1 , vertex b with y_{l+2} , vertex z_{k+3} with x_{l+2} , vertex z_1 with y_1 , respectively. Denote the resulting graph by G_1 . Let $G = G_1 + ab + y_1 x_{l+2}$. It is easy to see that G is a 4-connected graph. First, let $A = \{x_3, x_4, \dots, x_{l+1}, y_2, y_3, \dots, y_{l+1}\}$, $S = \{x_2, x_{l+2}, y_1\}$, $B = G - by_{l+1} - S - A$, then $(by_{l+1}, S; A, B)$ is a separating group of G , and so $by_{l+1} \in E_N(G)$. Since $y_1 x_{l+2} \in E([S])$, from Theorem 2.4 we have that $y_1 x_{l+2} \in E_R(G)$. Obviously, $(x_{l+2} z_{k+2}, S_1)$ is a separating pair such that $S_1 = \{a, b, z_2\}$, and $(z_2 y_1, S_2)$ is also a separating pair such that $S_2 = \{a, b, x_{l+2}\}$. It is easy to see that $z_i z_{i+1} \in E_N(G)$ where $i = 2, \dots, k+1$. Pick up the cycle $C_1 = y_1 x_{l+2} z_{k+2} z_{k+1} z_k \dots z_2 y_1$. Then, C_1 only passes through one subgraph of G belonging to \mathfrak{R} , and C_1 has only one removable edge $y_1 x_{l+2}$ of G . This shows that the result of Theorem 4.6 is in some sense best possible.

Example 2. First, delete the vertices z_1, z_{k+3} from F . Then, identify vertex z_2 with x_1 , vertex z_{k+2} with y_{l+2} , respectively. Denote the resulting graph by G_2 . Let $G = G_2 + ab + ay_1 + bx_{l+2} + y_1 x_{l+2}$. It is easy to see that G is a 4-connected graph. Let $A = \{x_3, \dots, x_{l+1}, y_2, \dots, y_{l+1}\}$, $S = \{y_1, x_{l+2}, x_2\}$, $B = G - z_{k+2} y_{l+1} - S - A$, then $(z_{k+2} y_{l+1}, S; A, B)$ is a separating group of G , and so $z_{k+2} y_{l+1} \in E_N(G)$. Since $y_1 x_{l+2} \in E([S])$, from Theorem 2.4 we have that $y_1 x_{l+2} \in E_R(G)$. Obviously, $(z_2 x_2, S_1)$ is a separating group of G such that $S_1 = \{a, b, z_{k+2}\}$, and so $z_2 x_2 \in E_N(G)$. By a similar argument, we can get that $ay_1, bx_{l+2} \in E_N(G)$. Since $ab \in E([S_1])$, we have $ab \in E_R(G)$. Pick up the cycle $C_2 = abx_{l+2} y_1 a$. Then, C_2 does not pass through any subgraph of G belonging to \mathfrak{R} , and C_2 has exactly two removable edges $ab, y_1 x_{l+2}$ of G . This shows that the result of Theorem 4.5 is in some sense best possible.

The following example shows that if a cycle C of G passes through two subgraphs of G belonging to \mathfrak{R} , then it may not contain any removable edge of G .

Example 3. First, delete the vertices z_{k+3} from F . Then, identify the vertex a with x_1 , vertex b with x_{l+2} , vertex z_{k+2} with y_{l+2} , vertex z_1 with y_1 , respectively. Denote the resulting graph by G_3 . Let $G = G_3 + ab + y_1 x_{l+2}$. It is easy to see that G is a 4-connected graph. Pick up the cycle $C_3 = y_1 y_2 \dots y_{l+2} z_{l+2} z_{l+1} \dots z_2 y_1$. Then, C_3 passes through two subgraphs of G belonging to \mathfrak{R} . It is easy to see that $E(C_3) \subset E_N(G)$, and so C_3 does not contain any removable edge of G . This in some sense shows that the conditions of Theorems 4.5 and 4.6 are best possible.

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