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# Connectivity of $k$-extendable graphs with large $k$

Dingjun Lou[a], Qinglin Yu[b]

[a] *Department of Computer Science, Zhongshan University, Guangzhou 510275, People's Republic of China*

[b] *Department of Mathematics and Statistics, University College of the Cariboo, Kamloops, B.C., Canada*

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**Abstract**

Let $G$ be a simple connected graph on $2n$ vertices with perfect matching. For a given positive integer $k$ $(0 \leqslant k \leqslant n-1)$, $G$ is *k-extendable* if any matching of size $k$ in $G$ is contained in a perfect matching of $G$. It is proved that if $G$ is a $k$-extendable graph on $2n$ vertices with $k \geqslant n/2$, then either $G$ is bipartite or the connectivity of $G$ is at least $2k$. As a corollary, we show that if $G$ is a maximal $k$-extendable graph on $2n$ vertices with $n+2 \leqslant 2k+1$, then $G$ is $K_{n,n}$ if $k+1 \leqslant \delta \leqslant n$ and $G$ is $K_{2n}$ if $2k+1 \leqslant \delta \leqslant 2n-1$. Moreover, if $G$ is a minimal $k$-extendable graph on $2n$ vertices with $n+1 \leqslant 2k+1$ and $k+1 \leqslant \delta \leqslant n$ then the minimum degree of $G$ is $k+1$. We also discuss the relationship between the $k$-extendable graphs and the Hamiltonian graphs.



*Keywords:* $k$-Extendable graph; Minimal $k$-extendable graph; Maximal $k$-extendable graph; Minimum degree; Hamiltonian graph

## 1. Introduction and terminology

All graphs considered in this paper are finite, undirected and simple. For the terminology and notation not defined in this paper, the reader is referred to [4].

Let $G$ and $H$ be two graphs. Let $kH$ denote $k$ disjoint copies of $H$ and $G+H$ denote the union of $G$ and $H$ with each vertex of $G$ joining to every vertex of $H$.

A graph $G$ is said to be *factor-critical* if $G-v$ has a perfect matching for each $v \in V(G)$. Let $G$ be a graph with a perfect matching. Then $G$ is said to be *k-extendable* for $0 \leqslant k \leqslant (v-2)/2$ if any matching in $G$ of size $k$ is contained in a perfect matching of $G$. And $G$ is said to be *maximal k-extendable* if $G$ is $k$-extendable and for each

*E-mail address:* issldj@zsu.edu.cn (D. Lou).

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$e \in E(\bar{G})$, where $\bar{G}$ is the complement of $G$, $G \cup \{e\}$ is not $k$-extendable. And $G$ is said to be *minimal $k$-extendable* if $G$ is $k$-extendable and for each $e \in E(G)$, $G - e$ is not $k$-extendable.

The concept of $k$-extendable graphs was introduced by Plummer [7] in 1980. Since then, extensive researches on this topic have been done (see [1,2,6–10]). In [2], Ananchuen and Caccetta proved the following result about the minimum degree of $k$-extendable graphs.

**Lemma 1** (Ananchuen and Caccetta [2]). *Suppose* $1 \leqslant k \leqslant (v-2)/2$ *and* $|V(G)| = v$. *Then if $G$ is $k$-extendable, then either* $k+1 \leqslant \delta \leqslant v/2$ *or* $2k+1 \leqslant \delta \leqslant v-1$.

For each value of $\delta$ given in Lemma 1, there exist $k$-extendable graphs with the minimum degree $\delta$. However, the problem that which value in these ranges is attainable for maximal $k$-extendable graphs remains open. Plummer [9] proposed the following problem.

**Problem 1.** Suppose $1 \leqslant k \leqslant (v-2)/2$ and $k+1 \leqslant j \leqslant v/2$ or $2k+1 \leqslant j \leqslant v-1$. Then which $k$-extendable graphs having minimum degree $j$ are maximal $k$-extendable?

Motivated by this problem, we study the $k$-extendable graphs with $k \geqslant v/4$, that is $v/2+1 \leqslant 2k+1$, which means the two intervals for $\delta$ in Lemma 1 are separated. We prove that if $G$ is a $k$-extendable graph with $k \geqslant v/4$, then either $G$ is bipartite or $\kappa(G) \geqslant 2k$. As corollaries, we characterize the maximal $k$-extendable graphs with $v/2+2 \leqslant 2k+1$ and we show that the minimum degree of a minimal $k$-extendable graph with $v/2+1 \leqslant 2k+1$ and with $k+1 \leqslant \delta \leqslant v/2$ is $k+1$. Also we prove that a $k$-extendable graph with $k \geqslant v/4$ is Hamiltonian, which shows the relation between $k$-extendable graphs and Hamiltonian graphs.

## 2. Main result

We start this section with a few basic lemmas on $k$-extendable graphs.

**Lemma 2** (Yu [10]). *A graph $G$ is $k$-extendable if and only if for any matching $M$ of size $r$ in* $G (1 \leqslant r \leqslant k)$, $G - V(M)$ *is* $(k-r)$*-extendable.*

**Lemma 3** (Yu [10]). *Let $G$ be a connected $k$-extendable non-bipartite graph. Then for each edge* $e \in E(\bar{G})$, $G + e$ *is* $(k-1)$*-extendable.*

**Lemma 4** (Plummer [7]). *If $G$ is $k$-extendable, then* $\kappa(G) \geqslant k+1$.

**Lemma 5.** *Let $G$ be a graph and* $S \subseteq V(G)$. *If the size of a maximum matching of* $G - S$ *is $m$, then the size of a maximum matching of $G$ is at most* $m + |S|$.

**Proof.** Obvious. □

We need the following lemma to prove our main result, this lemma itself may serve as a useful tool in other research on matching theory.

**Lemma 6.** *Let $G$ be a graph with order $v = 2r + m$. If $G$ has a matching of size $r$ and deleting any vertex from $G$, the resulting graph still has a matching of size $r$, then $G$ has a matching of size $r + 1$ unless $G$ has exactly $m$ odd components and no even components and each odd component is factor-critical.*

**Proof.** Suppose that the maximum matchings of $G$ have size $r$. Then by Berge's formula on maximum matching, there exists a set $S \subseteq V(G)$ such that $o(G-S)-|S|=m$. If $S \neq \emptyset$, let $v \in S$, $G' = G - v$ and $S' = S \setminus \{v\}$. Then $o(G' - S') - |S'| = o(G - S) - |S| + 1 = m + 1$. So the maximum matching in $G'$ has size at most $(|V(G')| - (o(G' - S') - |S'|))/2 = (2r + m - 1 - (m + 1))/2 = r - 1$, contradicting to the hypothesis that deleting any vertex from $G$ the resulting graph still has a matching of size $r$. So $S = \emptyset$ and $G$ has exactly $m$ odd components. If $G$ has an even component $C$, deleting a vertex $v$ from $C$, $G - v$ has a maximum matching of size less than $r$ since there is a vertex in each of the $m + 1$ odd components which is not covered by the maximum matching and also $v$ is not covered by the maximum matching. Hence, $G$ has no even component. But deleting any vertex $v$ from each odd component $C$ of $G$, $C - v$ must have a perfect matching, otherwise by counting the number of vertices of $G$, $G - v$ has no matching of size $r$. So each component of $G$ is factor-critical. □

Now we give the proof of our main result.

**Theorem 7.** *If $G$ is a $k$-extendable graph on $v$ vertices with $k \geqslant v/4$, then either $G$ is bipartite or $\kappa(G) \geqslant 2k$.*

**Proof.** By contradiction. Suppose that $G$ is a connected $k$-extendable graph with connectivity at most $2k - 1$ but not bipartite. Let $S$ be a minimum cutset of $G$ and let $M$ be a maximum matching in $G[S]$. Let $T = S \setminus V(M)$ and $r = |M|$. Since $|S| \leqslant 2k - 1$, $|M| \leqslant k - 1$. By Lemmas 2 and 4, $G - V(M)$ is $(k - r + 1)$-connected. Then we have

$$|T| \geqslant k - r + 1 \geqslant 2 \tag{1}$$

and we have $2k - 1 \geqslant 2r + |T| \geqslant k + r + 1$, so

$$r \leqslant k - 2. \tag{2}$$

**Claim 1.** *For every perfect matching $F$ containing $M$, $F \cap E(G - S)$ is a maximum matching in $G - S$ and $|F \cap E(G - S)| \leqslant k - 1$.*

Since $T$ is an independent set of $G$, by (1) and assumption that $|V(G)| \leqslant 4k$,

$$\begin{aligned}|F \cap E(G - S)| &= (|V(G)| - 2|M| - 2|T|)/2 \\ &= |V(G)|/2 - r - |T| \leqslant 2k - (k + 1) = k - 1.\end{aligned}$$

If $F \cap E(G-S)$ is not a maximum matching in $G-S$, then there is a matching $F_1$ in $G-S$ such that $|F_1| = |F \cap E(G-S)| + 1 \leqslant k$. But by Lemma 5, the size of a maximum matching in $G - V(F_1)$ is at most

$$|V(G-S-V(F_1))| + |M| \leqslant |V(G)|/2 - |F_1| - 1,$$

hence $G - V(F_1)$ does not have perfect matching, this contradicts the $k$-extendability of $G$. The proof of Claim 1 is complete. □

By Claim 1 and the fact that $T$ is an independent set of $G$, we easily prove the following claim.

**Claim 2.** *The size of every maximum matching in* $G-S$ *is* $|V(G)|/2 - |M| - |T|$.

By (1), there are two distinct vertices $x$ and $y$ in $T$. By Lemma 3, the graph $H = G + xy$ is $(k-1)$-extendable. By (2), $M_1 = M \cup \{xy\}$ is a matching in $H$ which has size at most $k-1$. Then $H - V(M_1)$ has a perfect matching $M^*$ and $M^*$ matches each vertex of $T \setminus \{x, y\}$ to a vertex in $V(G-S)$. Hence, $M^* \cap E(G-S)$ is a matching of size $|V(G)|/2 - |M| - |T| + 1$ in $G-S$. This contradicts Claim 2. The proof of Theorem 7 is complete. □

**Remark 1.** The lower bound on connectivity in Theorem 7 is best possible. Let $H_1 = K_{2k}$, $H_2 = K_r$ and $H_3 = K_s$ with $4 \leqslant r+s \leqslant 2k-2$ and both $r$ and $s$ being positive even integers. Then $G = H_1 + (H_2 \cup H_3)$ is $k$-extendable but with $\kappa(G) = 2k$. Also the lower bound on $k$ in Theorem 7 is best possible. The hypothesis $k \geqslant v/4$ is equivalent to $v \leqslant 4k$. Let $H_1 = \bar{K}_{k+1}$, $H_2 = K_{k+1}$ and $H_3 = K_{2k}$, where $\bar{K}_{k+1}$ is the complement of $K_{k+1}$. Then $G = H_1 + (H_2 \cup H_3)$ is a $k$-extendable graph with $v = 4k+2$ that is not bipartite but has connectivity $k+1$.

## 3. Maximal $k$-extendable graphs with large $k$

In this section, we characterize all maximal $k$-extendable graphs with $v/2+2 \leqslant 2k+1$. Then we show some maximal $k$-extendable graphs with $2k+1 \leqslant v/2+1$ and with $\delta \geqslant v/2$. Our results partially answer Problem 1.

**Lemma 8** (Ananchuen and Caccetta [1]). *If* $G \neq K_v$ *is a maximal* $k$-*extendable graph on* $v$ *vertices, then*

(a) *if* $v/2 < 2k$, *then* $\delta \leqslant v/2$, *while*
(b) *if* $v/2 \geqslant 2k$, *then* $\delta \leqslant v/2 + 2\lfloor (k-1)/2 \rfloor$.

**Lemma 9** (Plummer [8] and Yu [10]). *If* $G = (X, Y) \neq K_{n,n}$ *is a connected* $k$-*extendable bipartite graph and* $e = xy \in E(\bar{G})$, *where* $x \in X$ *and* $y \in Y$, *then* $G \cup \{e\}$ *is also* $k$-*extendable*.

**Corollary 10.** *Let $G$ be a maximal $k$-extendable graph on $v$ vertices with $v/2+2 \leqslant 2k+1$. Then*

(a) *if $k+1 \leqslant \delta \leqslant v/2$, then $G$ is $K_{v/2,v/2}$ and hence $\delta = v/2$;*
(b) *if $2k+1 \leqslant \delta \leqslant v-1$, then $G$ is $K_v$ and hence $\delta = v-1$.*

**Proof.** By Theorem 7, if $k+1 \leqslant \delta \leqslant v/2$, then $G$ is bipartite. Otherwise $\delta(G) \geqslant \kappa(G) \geqslant 2k$. When $v/2+2 \leqslant 2k+1$, $\delta(G) \neq 2k$ by Lemma 1. Hence, $\delta(G) \geqslant 2k+1 \geqslant v/2+2$ and $G$ is non-bipartite. By Lemma 9, we have conclusion (a). By Lemma 8(a), we have conclusion (b). □

**Remark 2.** Corollary 10 characterizes all maximal $k$-extendable graphs with $v < 4k$. It shows that the minimum degree of a maximal $k$-extendable graph $G$ with $v \leqslant 4k-2$ is either $v/2$ or $v-1$. But for the case of $v \geqslant 4k$, we give a family of maximal $k$-extendable graphs to show that the minimum degree of $G$ can be much more diverse.

Let $G_i = K_{r_i}$, $i = 1,2,\ldots,m$, where each $r_i$ is an odd number and $r_1 + r_2 + \cdots + r_m = 2k-2+m$. Let $H_j = K_{s_j}$, $j = 1,2,\ldots,m$, where each $s_j$ is an odd number and $s_1+s_2+\cdots+s_m = 2k-2+m$. And let $G = (G_1 \cup G_2 \cup \cdots \cup G_m) + (H_1 \cup H_2 \cup \cdots \cup H_m)$. Then it is not too difficult to verify that $G$ is maximal $k$-extendable but not $(k+1)$-extendable. When we take $m = 2$, by choosing proper $r_i$ and $s_i$ $(i = 1,2)$, we have $\delta(G) = t$ for all even numbers $t$ such that $v/2 \leqslant t \leqslant v/2 + 2\lfloor (k-1)/2 \rfloor$. When we take $m = 3$, by choosing proper $r_i$ and $s_i$ $(i = 1,2,3)$, we have $\delta(G) = t$ for all odd numbers $t$ such that $v/2 \leqslant t \leqslant v/2 + \lfloor (2k+1)/3 \rfloor - 1$.

## 4. Minimal *k*-extendable graphs with large *k*

In this section, we show that the minimum degree of a minimal $k$-extendable graph with $v \leqslant 4k$ and $k+1 \leqslant \delta \leqslant v/2$ is $k+1$. We introduce a result of Lou [6] as a lemma.

**Lemma 11** (Lou [6]). *If $G$ is a minimal $k$-extendable bipartite graph, then $\delta(G) = k+1$, and furthermore, there are at least $2k+2$ vertices of degree $k+1$ in $G$.*

**Corollary 12.** *Let $G$ be a minimal $k$-extendable graph on $v$ vertices with $v/2+1 \leqslant 2k+1$. If $k+1 \leqslant \delta(G) \leqslant v/2$, then $\delta(G) = k+1$. Furthermore, there are at least $2k+2$ vertices of degree $k+1$ in $G$.*

**Proof.** By Theorem 7, if $k+1 \leqslant \delta(G) \leqslant v/2$, then $G$ is bipartite. By Lemma 11, the result follows. □

Since a $k$-extendable graph with $k \geqslant v/4$ is rather dense, we make the following conjectures.

**Conjecture 1.** *Let $G$ be a minimal $k$-extendable graph on $v$ vertices with $v/2+1 \leqslant 2k+1$. Then $\delta(G) = k+1$, $2k$ or $2k+1$.*

In particular, for the case of $v \leqslant 4k - 2$, we have the following conjecture.

**Conjecture 2.** *Let G be a minimal k-extendable graph on v vertices with* $v/2+2 \leqslant 2k+1$. *If* $2k+1 \leqslant \delta \leqslant v-1$, *then* $\delta(G) = 2k+1$.

## 5. Hamiltonicity of *k*-extendable graphs with large *k*

In this section, we show that a $k$-extendable graph is Hamiltonian if $k$ is sufficiently large with respect to its order.

**Lemma 13** (Dirac [5]). *If* $\delta(G) \geqslant v/2$, *then G is Hamiltonian.*

**Lemma 14** (Jackson [3]). *Let* $G = (X, Y)$ *be a connected bipartite graph with* $|X| = |Y| = n$. *If* $\delta(G) \geqslant (n+1)/2$, *then G is Hamiltonian.*

**Corollary 15.** *If G is a k-extendable graph with* $k \geqslant v/4$, *then G is Hamiltonian.*

**Proof.** By Theorem 7, if $k+1 \leqslant \delta(G) \leqslant v/2$, $G = (X, Y)$ is bipartite with $|X| = |Y| = v/2 \leqslant 2k$. However, $\delta(G) \geqslant k+1 = (2k+2)/2 > (|X|+1)/2$, by Lemma 14, $G$ is Hamiltonian. Otherwise $\delta(G) \geqslant \kappa(G) \geqslant 2k \geqslant v/2$, by Lemma 13, $G$ is Hamiltonian. □

**Remark 3.** Although we did not find new Hamiltonian graphs in Corollary 15, we did show the relation between $k$-extendable graphs and Hamiltonian graphs that a $k$-extendable graph with sufficiently large $k$ with respect to the order $v(G)$ is Hamiltonian. In fact, we suspect that the lower bound on $k$ in Corollary 15 is not best possible. And hence, we give the following conjecture.

**Conjecture 3.** *If G is a k-extendable graph with* $k > (v-2)/6$, *then G is Hamiltonian.*

The lower bound on $k$ in Conjecture 3 is best possible. Let $S = \{v_1, v_2, \ldots, v_{2k}\}$ be an independent set and $H = (2k+1)K_2$ with $V(H) \cap S = \emptyset$. Then $G = S + H$ is a $k$-extendable graph but $G$ is not Hamiltonian as $G$ is not 1-tough. Here $v(G) = 6k+2$, that is $k = (v-2)/6$. The above counterexamples also show that a $k$-extendable graph with arbitrarily large $k$ (but $v$ is also sufficiently large) is not guaranteed to be Hamiltonian.

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