Computing the Rupture Degrees of Graphs *

Fengwei Li and Xueliang Li

Center for Combinatorics and LPMC
Nankai University
Tianjin 300071, P.R. China
x.li@eyou.com

Abstract

The rupture degree of a noncomplete connected graph G is defined by $r(G) = max\{\omega(G-X)-|X|-m(G-X) : X \subset V(G), \omega(G-X) \geq 2\}$, where $\omega(G-X)$ denotes the number of components in the graph $G-X$. For a complete graph K_n, we define $r(K_n) = 1-n$. This parameter can be used to measure the vulnerability of a graph. To some extent, it represents a trade-off between the amount of work done to damage the network and how badly the network is damaged. In this paper, we prove that the problem of computing the rupture degree of a graph is NP-complete. We find the rupture degree of the Cartesian product of some graphs and also give the exact values or bounds for the rupture degrees of Harary graphs.

Keyworks: rupture degree, NP-completeness, Cartesian product, Harary graph.

AMS subject classification(2000): 94A05, 94C15, 90B18, 68R10.

1. Preliminaries

The stability of a communication network, composed of processing nodes and communication links, is of prime importance to network designers. As the

*Research supported by National Science Foundation of China.

network begins losing links or nodes, eventually there is a loss in its effectiveness. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. Many graph theoretical parameters have been used in the past to describe the stability of communication networks. Most notably, the vertex-connectivity and the edge-connectivity have been frequently used. The difficulty with this parameters is that they do not take into account what remains after the graph is disconnected. Consequently, a number of other parameters have been introduced that attempt to cope with this difficulty, including toughness and edge-toughness in [7, 8], integrity and edge-integrity in [1], tenacity and edge-tenacity in [4, 5, 9], and scattering number in [10]. Unlike the connectivity measures, each of these parameters shows not only the difficulty to break down the network but also the damage that has been caused.

Before we formally define the rupture degree of a graph, we recall some parameters of [2] and [4]. Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$, let $\omega(G-S)$ and $m(G-S)$, respectively, denote the number of components and the order of a largest component in $G-S$. A set $S \subseteq V(G)$ is a cut set of G, if either $G-S$ is disconnected or $G-S$ has only one vertex. We shall use $\lceil x \rceil$ for the smallest integer not smaller than x, and $\lfloor x \rfloor$ for the largest integer not larger than x. $\alpha(G)$ denotes the independent number of the graph G. We use Bondy and Murty for terminology and notations not defined here. For comparing, the following graph parameters are listed.

The Vertex-Connectivity of G:

$$\kappa(G) = min\{|S| : S \subseteq V(G) \; is \; a \; cut \; set \; of \; G\}.$$

The Vertex-Toughness of G (Chvátal (1973) [7]):

$$t(G) = min\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G) \; is \; a \; cut \; set \; of \; G\}.$$

The Vertex-Integrity of G (Barefoot et al (1987) [1]:

$$I(G) = min\{|S| + m(G-S) : S \subseteq V(G)\}.$$

The Vertex-Tenacity of G (Cozzens et al (1995) [4]):

$$T(G) = min\{\frac{|S| + m(G-S)}{\omega(G-S)} : S \subseteq V(G) \; is \; a \; cut \; set \; of \; G\}.$$

The corresponding edge analogues of these concepts are defined similarly, see [1,8,9].

Definition 1.1 Let G be a noncomplete connected graph. Then the rupture degree $r(G)$ of G is defined by

$$r(G) = max\{\omega(G-X) - |X| - m(G-X) : X \subset V(G), \omega(G-X) \geq 2\}.$$

In particular, the rupture degree of a complete graph K_n is defined to be $1-n$.

The concept of rupture degree was first introduced in [6], where the authors determined the rupture degrees of several classes of graphs, and gave formulas for the rupture degrees of join graphs and some bounds of rupture degrees. Some Nordhaus-Goddard-type results for the rupture degree are also deduced.

In this paper, we consider the complexity for computing the rupture degrees of graphs. In Section 2, we prove that the problem of computing the rupture degree of a graph is NP-complete. In Sections 3 and 4, we give rupture degrees for the Cartesian product of some graphs and for Harary graphs. Some other results on rupture degree are given in Section 5.

2. NP-Completeness Result

From above, we know that the rupture degree can be used to measure the vulnerability of networks. So, clearly it is of prime importance to determine this parameter for a graph. A noncomplete connected graph G is said to be r-rupture if $\omega(G-X) \leq |X| + m(G-X) + r$ for all $X \subset V(G)$ with $\omega(G-X) \geq 2$. Thus, $r(G)$ is the maximum r for which G is r-rupture. In this section, we will consider the computational aspect for the rupture degree of a graph. We begin by considering the following problem.

Problem 2.1 Not r-Rupture

Instance: A noncomplete connected graph G; and an integer r.

Question: Does there exist an $X \subset V(G)$ with $\omega(G-X) \geq 2$ such that $\omega(G-X) > |X| + m(G-X) + r$?

To prove the NP-completeness, we will use the following NP-complete problem to reduce, see [7, p194].

Problem 2.2 Independent Majority

Instance: A undirected graph G.

Question: Does G contain an independent set $I \subseteq V(G)$ such that $|I| \geq$

$\frac{1}{2}|V(G)|$?

Theorem 2.1 For any integer r, the **NOT r-Rupture** problem is NP-complete.

Proof. Clearly, **NOT r-Rupture** is in the class NP. Next, let G be a noncomplete connected graph with vertex set $\{u_1, u_2, \cdots, u_n\}$. Construct another graph $G^{'}$ from G as follows: Let a and b be integers such that $a \geq 1, b \geq 1$ and $r = b - a$. Add to G a set $A = \{v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_{b+2}\}$ of independent vertices, and join vertices u_i to v_i by an edge for $i = 1, 2, \cdots, n$. Then, add another set B of $\lfloor\frac{1}{2}(n-1)+a\rfloor$ vertices, which induces a complete graph, and then join each vertex of B to every vertex of $G \cup A$. To complete the proof, it is sufficient to show that G contains an independent set I such that $|I| \geq \frac{1}{2}n$ if and only if $G^{'}$ is not r-rupture.

Suppose first that G contains an independent set I such that $|I| \geq \frac{1}{2}n$. Then, define vertex set $X^{'} \subseteq V(G^{'})$ by $X^{'} = (V(G) - I) \cup B$. It is easy to see that $\omega(G^{'} - X^{'}) = n + b + 2$. Note that $|X^{'}| = |V(G) - I| + |B| \leq n - \frac{1}{2}n + \lfloor\frac{1}{2}(n-1)\rfloor + a < n + a$ and $m(G - X) = 2$. Thus, we have $\omega(G^{'} - X^{'}) - |X^{'}| - m(G^{'} - X^{'}) > n + b + 2 - n - a - 2 = b - a = r$, i.e., $G^{'}$ is not r-rupture.

On the other hand, if $G^{'}$ is not r-rupture, then there exists an $X^{'} \subset V(G)$ with $\omega(G^{'} - X^{'}) \geq 2$ such that $\omega(G^{'} - X^{'}) > |X^{'}| + m(G^{'} - X^{'}) + r$. It is obvious that $B \subseteq X^{'}$; otherwise, $\omega(G^{'} - X^{'}) = 1$. We can assume that $X^{'} \cap A = \emptyset$, since otherwise, by setting $X^{''} = X^{'} - A$, we have

$$\begin{aligned}\omega(G^{'} - X^{''}) &= \omega(G^{'} - X^{'}) - |X^{'} \cap A| \\ &> |X^{'}| + m(G^{'} - X^{'}) - |X^{'} \cap A| + r \\ &= |X^{''}| + m(G^{'} - X^{'})| + r.\end{aligned}$$

Noticing that $m(G^{'} - X^{''}) = m(G^{'} - X^{'})$, we have $\omega(G^{'} - X^{''}) > |X^{''}| + m(G^{'} - X^{''}) + r$, and so we can use $X^{'} - A$ instead of $X^{'}$. Let $X = X^{'} \cap V(G)$. Then, $\omega(G - X) \geq 2, \omega(G^{'} - X^{'}) \geq 2$, and $|X^{'}| = |X| + \lfloor\frac{1}{2}(n-1)\rfloor + a$ and $\omega(G^{'} - X^{'}) = |X| + b + 2 + \omega(G - X) > |X^{'}| + m(G^{'} - X^{'}) + r$. So, we have

$$\omega(G - X) > |X^{'}| + m(G^{'} - X^{'}) + r - |X| - b - 2$$

$$= |X| + \lfloor\frac{1}{2}(n-1)\rfloor - |X| + a - b + r + m(G^{'} - X^{'}) - 2$$

$$\geq \lfloor\frac{1}{2}(n-1)\rfloor.$$

It is obvious that $G - X$ contains at least $\frac{1}{2}n$ components. Choosing one vertex in each components of $G - X$ yields a set of at least $\frac{1}{2}n$ independent vertices in G. The proof is now complete. ∎

Since computing the rupture degree of a graph is NP-complete in general, it becomes an interesting question to calculate the rupture degrees for some special classes of interesting or practically useful graphs. In the following two sections we will deal with this question.

3. Rupture Degree of the Cartesian Product of Two Special Graphs

The Cartesian product of two graphs G_1 and G_2, denoted by $G_1 \times G_2$, is defined as follows: $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_1v_1 \in E(G_1)$ and $u_2 = v_2$. The Cartesian product of n graphs $G_1, G_2, \cdots, G_n$, denoted by $G_1 \times G_2 \times \cdots \times G_n$, is defined inductively as the Cartesian product of $G_1 \times G_2 \times \cdots \times G_{n-1}$ and G_n. In particular, the Cartesian product of k copies of K_2, denoted by Q_k, is called a hypercube of dimensional k.

In this section, we determine the rupture degree of the Cartesian product of two special graphs.

Theorem 3.1 If $n \geq m > 1$, then $r(K_m \times K_n) = n - mn + m - \lceil \frac{n}{m} \rceil$.

To prove the theorem, we first introduce the following definition and lemmas.

Definition 3.1 Let G be a noncomplete connected graph, a set $S \subset V(G)$ is called an R-set if it satisfies that $r(G) = \omega(G - S) - |S| - m(G - S)$.

Lemma 3.1 If G is a noncomplete connected graph, $\alpha(G)$ is the independent number of G and $T(G)$ is the vertex-tenacity of G, then we have $r(G) \leq \alpha(G)(1 - T(G))$.

Proof. Suppose that S is an R-set of G. Then, by the definition of an R-set, we have $r(G) = \omega(G-S) - |S| - m(G-S)$, where $\omega(G-S) \geq 2$. So we have $\frac{r(G)}{\omega(G-S)} = 1 - \frac{|S|+m(G-S)}{\omega(G-S)}$. By the definition of vertex-tenacity, we know that $\frac{|S|+m(G-S)}{\omega(G-S)} \geq T(G)$. On the other hand, it is obvious that $\omega(G - S) \leq \alpha(G)$. So we have $r(G) \leq \alpha(G)(1 - T(G))$.

Lemma 3.2 ([5]) If $n \geq m > 1$, then $T(K_m \times K_n) = \frac{mn-n+\lceil \frac{n}{m} \rceil}{m}$.

Proof of Theorem 3.1 It is easy to see that $\alpha(K_m \times K_n) = m$. So, by Lemmas 3.1 and 3.2 we have $r(K_m \times K_n) \leq \alpha(K_m \times K_n)(1 - T(K_m \times K_n)) = m - mn + n - \lceil \frac{n}{m} \rceil$.

On the other hand, let $V(K_m) = \{1, 2, \cdots, m\}, V(K_n) = \{1, 2, \cdots, n\}$. Then $V(K_m \times K_n) = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$. Also, let $n = am + b$, where $0 \leq b < m$. So, if $b = 0$, then $a = \lceil \frac{n}{m} \rceil = \frac{n}{m}$, and otherwise, $a + 1 = \lceil \frac{n}{m} \rceil$. Now we define the sets W_i as follows:

$$W_i = \begin{cases} \{(i, ia + a + 1), \cdots, (i, ia)\} & if\ b = 0, 1 \leq i \leq m \\ \{(i, ia + i - a), \cdots, (i, ia + a)\} & if\ b \geq 1, 1 \leq i \leq b \\ \{(i, ia + b - a + 1), \cdots, (i, ia + b)\} & if\ b \geq 1, b + 1 \leq i \leq m \end{cases}$$

Let $W = \cup_{i=1}^{m} W_i$. So we know $|W| = n$. Define $S = V(K_m \times K_n) - W$, and so $|S| = mn - n$. It is easy to see that W_i $(i = 1, 2, \cdots, m)$ are the components of $K_m \times K_n - S$, and so $m(K_m \times K_n - S) = \lceil \frac{n}{m} \rceil$, $\omega(G - S) = m$. Then, by the definition of rupture degree of a graph we know that

$$\begin{aligned} r(G) &\geq \omega(K_m \times K_n - S) - |S| - m(K_m \times K_n - S) \\ &\geq m - mn + n - \lceil \frac{n}{m} \rceil. \end{aligned}$$

The proof is complete. ∎

In the following, we will give the rupture degree of the Cartesian product of two special graphs. First, we give a necessary lemma.

Lemma 3.3 If G is a bipartite, k-connected, k-regular graph on p vertices, then the rupture degree $r(G)$ of G is -1.

Proof. In [4] Cozzens et al proved that $T(G) \geq 1 + \frac{1}{\alpha(G)}$, and it is easy to see that $\alpha(G) = \frac{p}{2}$. Thus, by Theorem 3.1 we have $r(G) \leq \alpha(G)(1 - T(G)) \leq -1$.

On the other hand, let A be one of the partite set of G. Thus $|A| = \frac{p}{2}$, $m(G - A) = 1$ and $\omega(G - A) = \frac{p}{2}$. Then, by definition we have $r(G) \geq \omega(G - A) - |A| - m(G - A) \geq \frac{p}{2} - \frac{p}{2} - 1 = -1$, which completes the proof. ∎

Theorem 3.2 If G_1 is a bipartite, n-regular and n-connected graph with p_1 vertices, and G_2 is a bipartite, m-regular and m-connected graph with p_2 vertices, then $r(G_1 \times G_2) = -1$.

Proof. It is obvious that the graph $G_1 \times G_2$ is an $(m + n)$-regular and $(m + n)$-connected bipartite graph with mn vertices. Then, by Lemma 3.3 we have $r(G_1 \times G_2) = -1$.

The following two results can be verified by Lemma 3.3, directly.

Theorem 3.3 If m and n are even integers, then $r(C_n \times C_m) = -1$ and $r(C_n \times K_2) = -1$.

Theorem 3.4 The rupture degree $r(Q_k)$ of the hypercube Q_k is -1.

4. Rupture Degrees of Harary Graphs

In 1962 Harary researched a problem of reliable communication network: for given order n and a nonegative integer l ($l < n$), how to construct a simple graph of order n such that $\kappa(G) = l$, and G has as few edges as possible. For any fixed integers n and p such that $p \geq n+1$, Harary constructed the class of graphs $H_{n,p}$ which are n-connected with the minimum number of edges on p vertices. Thus Harary graphs are examples of graphs which in some sense have the maximum possible connectivity and hence are of interests as possibly having good stability properties. $H_{n,p}$ is constructed as follows.

Case 1. If n is even, let $n = 2r$, then $H_{n,p}$ has vertices $0, 1, 2, \cdots, p-1$, and two vertices i and j are adjacent if and only if $|i-j| \leq r$, where the addition is taken modulo p.

Case 2. If n is odd ($n > 1$) and p is even, let $n = 2r+1$ ($r > 0$), then $H_{n,p}$ is constructed by first drawing $H_{2r,p}$, and then adding edges joining vertex i to vertex $i + \frac{P}{2}$ for $1 \leq i \leq \frac{P}{2}$.

Case 3. If n is odd ($n > 1$) and p is odd, let $n = 2r+1 (r > 0)$, then $H_{2r+1,p}$ is constructed by first drawing $H_{2r,p}$, and then adding edges joining vertex i to vertex $i + \frac{p+1}{2}$ for $0 \leq i \leq \frac{p-1}{2}$. Note that under this definition, vertex 0 is adjacent to both vertex $\frac{p+1}{2}$ and $\frac{p-1}{2}$. Again note that all vertices of $H_{n,p}$ have degree n except vertex 0, which has degree $n+1$.

As a useful reliable network, Harary graphs have arouse interests in many network designers. Harary [1] proved that the Harary graphs $H_{n,p}$ is n-connected. Ouyang et al [15] gave the scattering number of the Harary graphs. In [14] Cozzens et al gave exact values or good bounds for the tenacity of the Harary graphs. In this section, we compute the rupture degrees of the Harary graphs. Throughout this section, we set the connectivity $n = 2r$ or $n = 2r+1$ and the number of vertices $p = k(r+1) + s$ for $0 \leq s \leq r+1$. So we can see that $p \equiv s \mod (r+1)$ and $k = \lfloor \frac{p}{r+1} \rfloor$. We assume that the graph $H_{n,p}$ is not complete, and so $n+1 < p$, which implies that $k \geq 2$.

Lemma 4.1 If S is a minimal R-set for the graph $H_{n,p}$, $n = 2r$, then S

consists of the union of sets of r consecutive vertices such that there exists at least one vertex not in S between any two sets of consecutive vertices in S.

Proof. We assume that the vertices of $H_{n,p}$ are labelled by $0, 1, 2, \cdots, n, p-1$. Let S be a minimal R-set of $H_{n,p}$ and j be the smallest integer such that $T = \{j, j+1, \cdots, j+t-1\}$ is a maximum set of consecutive vertices such that $T \subseteq S$. Relabel the vertices of $H_{n,p}$ as $v_1 = j, v_2 = j+1, \cdots, v_t = j+t-1, \cdots, v_p = j-1$. Since $S \neq V(H_{n,p})$ and $T \neq V(H_{n,p})$, v_p does not belong to S. Since S must leave at least two components of $G-S$, we have $t \neq p-1$, and so $v_{t+1} \neq v_p$. Therefore, $\{v_{t+1}, v_p) \cap S = \Phi$. Choose v_i such that $1 \leq i \leq t$, and delete v_i from S yielding a new set $S' = S - \{v_i\}$ with $|S'| = |S|-1$. Now suppose $t < r$. By the definition of $H_{n,p}$ $(n = 2r)$ we know that the edges v_iv_p and v_iv_{t+1} are in $H_{n,p} - S'$. Consider a vertex v_k adjacent to v_i in $H_{n,p} - S'$. If $k \geq t+1$, then $k < t+r$. So, v_k is also adjacent to v_{t+1} in $H_{n,p} - S'$. If $k < p$, then $k \geq p-r+1$ and v_k is also adjacent to v_p in $H_{n,p} - S'$. Since $t < k$, then v_p and v_{t+1} are adjacent in $H_{n,p} - S'$. Therefore, we can conclude that deleting vertex v_i from S does not change the number of components, and so $\omega(H_{n,p} - S') = \omega(H_{n,p} - S)$ and $m(H_{n,p} - S') \leq m(H_{n,p} - S) + 1$. Thus, we have

$$\begin{aligned} &\omega(H_{n,p} - S') - |S'| - m(H_{n,p} - S') \\ &\geq \omega(H_{n,p} - S) - |S| + 1 - m(H_{n,p} - S) - 1 \\ &= \omega(H_{n,p} - S) - |S| - m(H_{n,p} - S) = r(H_{n,p}). \end{aligned}$$

This is contrary to our choice of S. Thus we must have $t \geq r$. Now suppose $t > r$. Delete v_t from the set S yielding a new set $S_1 = S - \{v_t\}$. Since $t > r$, the edge v_tv_p is not in $H_{n,p} - S_1$. Consider a vertex v_k adjacent to v_t in $H_{n,p} - S_1$. Then, $k \geq t+1$ and $k \leq t+r$, and so v_k is also adjacent to v_{t+1} in $H_{n,p} - S_1$. Therefore, deleting v_t from S yields $\omega(H_{n,p} - S_1) = \omega(H_{n,p} - S)$ and $m(H_{n,p} - S_1)$=$m(H_{n,p} - S) + 1$. So,

$$\begin{aligned} &\omega(H_{n,p} - S_1) - |S_1| - m(H_{n,p} - S_1) \\ &\geq \omega(H_{n,p} - S) - |S| + 1 - m(H_{n,p} - S) - 1 \\ &= \omega(H_{n,p} - S) - |S| - m(H_{n,p} - S) = r(H_{n,p}), \end{aligned}$$

which is again contrary to our choice of S. Thus, $t = s$, and so S consists of the union of sets of exactly r consecutive vertices. ∎

Lemma 4.2 There is an R-set S for the graph $H_{n,p}$, $n = 2r$, such that all components of $H_{n,p} - S$ have order $m(H_{n,p} - S)$ or $m(H_{n,p} - S) - 1$.

Proof. Among all R-sets of minimum order, consider those R-sets such that the number of components with minimum order is maximum, and let s denote the order of the minimum component. Among these R-sets, let S be one with minimum number of components of order s. Suppose $s \leq m(H_{n,p}-S)-2$. Note that all of the components must be sets of consecutive vertices. Assume that C_k is a smallest component. Then $|V(C_k)|=s$, and without loss of generality, let $C_k=\{v_1,v_2,\cdots,v_s\}$. Suppose C_e is a largest component, and so $|V(C_e)|=m(H_{n,p}-S)=m$ and let $C_e=\{v_j,v_{j+1},\cdots,v_{j+m-1}\}$. Let $C_1,C_2,\cdots,C_a$ be the components with vertices between v_s of C_k and v_j of C_e, such that $|C_i|=p_i$ for $1 \leq i \leq a$, and let $C_i=\{v_{i_1},v_{i_2},\cdots,v_{i_{p_i}}\}$. Now we construct vertex set S' as follows: $S'=S-\{v_{s+1},v_{1_{p_1+1}},v_{2_{p_2+1}},\cdots,v_{a_{p_a+1}}\}\cup\{v_{1_1},v_{2_2},\cdots,v_{a_1},v_j\}$. Therefore, $|S'|=|S|$, $m(H_{n,p}-S') \leq m(H_{n,p}-S)$ and $\omega(H_{n,p}-S')=\omega(H_{n,p}-S)$. So we have

$$\omega(H_{n,p}-S')-|S'|-m(H_{n,p}-S')$$

$$\geq \omega(H_{n,p}-S)-|S|-m(H_{n,p}-S).$$

Therefore, $m(H_{n,p}-S')=m(H_{n,p}-S)$. But $H_{n,p}-S'$ has one less components of order s than $H_{n,p}-S$ does, which is a contradiction. Thus, all components of $H_{n,p}-S$ have order $m(H_{n,p}-S)$ or $m(H_{n,p}-S)-1$. So, $m(H_{n,p}-S)=\lceil\frac{p-r\omega}{\omega}\rceil$. ∎

By the above two lemmas we give the exact values of rupture degrees of the Harary graphs for $n=2r$.

Theorem 4.1 Let $H_{n,p}$ be a Harary graph with $n=2r$ and $p=k(r+1)+s$ for $0 \leq s < r+1$. Then

$$r(H_{n,p})=\begin{cases} 2-r-\lceil\frac{p}{2}\rceil & if\ p \leq 4(r-1) \\ m-(m-1)r-\lceil\frac{p}{m}\rceil & if\ p\ > 4(r-1) \end{cases}$$

where $m=\lfloor\sqrt{\frac{p}{r-1}}\rfloor$.

Proof. Let S be a minimum R-set of $H_{n,p}$. By Lemmas 4.1 and 4.2 we know that $|S|=r\omega$, and $m(H_{n,p}-S)=\lceil\frac{p-r\omega}{\omega}\rceil$. Thus, from the definition of rupture degree we have

$$r(H_{n,p}-S)=max\{\omega-r\omega-\lceil\frac{p-r\omega}{\omega}\rceil|2 \leq \omega \leq k\}.$$

Now we consider the function

$$f(\omega)=\omega-r\omega-\lceil\frac{p-r\omega}{\omega}\rceil.$$

It is easy to see that $f'(\omega)=1-r-\lceil\frac{-p}{\omega^2}\rceil=\lceil\frac{(1-r)\omega^2+p}{\omega^2}\rceil$. Since $\omega^2>0$, we have $f'(\omega) \geq 0$ if and only if $g(\omega)=(1-r)\omega^2+p \geq 0$. Since the two roots of the equation $g(\omega)=(1-r)\omega^2+p=0$ are $\omega_1=-\sqrt{\frac{p}{r-1}}$ and $\omega_2=\sqrt{\frac{p}{r-1}}$.

But $\omega_1 < 0$, and so it is deleted. Then if $0 < \omega \leq \lfloor \omega_2 \rfloor$, we have $f^{'}(\omega) \geq 0$, and so $f(\omega)$ is an increasing function; if $\omega \geq \lceil \omega(2) \rceil$, then $f^{'}(\omega) \leq 0$, and so $f(\omega)$ is a decreasing function. Thus, we have the following cases:

Case 1. If $p \leq 4(r-1)$, then $\lfloor \omega_2 \rfloor \leq 2$. Since we know that $2 \leq \omega \leq k$, we have that $f(\omega)$ is a decreasing function and the maximum value occurs at the boundary. So, $\omega = 2$ and $r(H_{n,p}) = f(2) = 2 - r - \lceil \frac{p}{2} \rceil$.

Case 2. If $p > 4(r-1)$, then $\lfloor \omega_2 \rfloor > 2$. So, we have

Subcase 2.1 If $2 \leq \omega \leq \lfloor \omega_2 \rfloor$, then $f(\omega)$ is an increasing function.

Subcase 2.2 If $\lceil \omega_2 \rceil \leq \omega \leq k$, then $f(\omega)$ is a decreasing function.

Thus the maximum value occurs when $\omega = \lfloor \omega_2 \rfloor$. Then, $r(H_{n,p}) = f(\lfloor \omega_2 \rfloor) = m - (m-1)r - \lceil \frac{p}{m} \rceil$, where $m = \lfloor \omega_2 \rfloor = \lfloor \sqrt{\frac{p}{r-1}} \rfloor$. The proof is now complete. ∎

The following two lemmas can be found in [14] or easily seen

Lemma 4.3 ([14]) Let $H_{n,p}$ be a Harary graph with $n = 2r$ and p even. Then

$$\alpha(H_{n,p}) = \begin{cases} k & if\ p \neq 0\ mod(n+1) \\ k-1 & if\ p = 0\ mod(n+1). \end{cases}$$

Lemma 4.4 If H is a spanning subgraph of a connected graph G, then $r(G) \leq r(H)$.

Lemma 4.5 Let G be a noncomplete connected graph of order n. Then $r(G) \geq 2\alpha(G) - n - 1$.

Proof. Let S be a maximum independent set of G. Then $|S| = \alpha(G)$. Let A=$V(G) - S$. Then $\omega(G-A) = \alpha(G)$, $m(G-A) = 1$ and $|A| = n - \alpha(G)$. So, by the definition of rupture degree we have $r(G) = \omega(G-A) - |A| - m(G-A) = 2\alpha - n - 1$. ∎

Theorem 4.2 Let $H_{n,p}$ be a Harary graph with p even, n odd and $n = 2r+1$, then

(1) If $p \leq 4(r-1)$,

$$2 - r - \lceil \frac{p}{2} \rceil \geq r(H_{n,p}) \geq \begin{cases} k - kr - s - 1 & if\ p \neq 0\ mod(n+1) \\ k - kr - s - 3 & if\ p = 0\ mod(n+1). \end{cases}$$

(2) If $p > 4(r-1)$,

$$m-(m-1)r-\lceil\frac{p}{m}\rceil \geq r(H_{n,p}) \geq \begin{cases} k-kr-s-1 & if\ p \neq 0\ mod(n+1) \\ k-kr-s-3 & if\ p = 0\ mod(n+1), \end{cases}$$

where $m = \lfloor\sqrt{\frac{p}{r-1}}\rfloor$.

Proof. Since $V(H_{2r+1,p}) = V(H_{2r,p})$, $E(H_{2r+1,p}) \subseteq E(H_{2r,p})$, it is obvious that $H_{2r,p}$ is a connected spanning subgraph of $H_{2r+1,p}$. So, by Lemma 4.4 we have

$$r(H_{2r+1,p}) \leq r(H_{2r,p}) = \begin{cases} 2-r-\lceil\frac{p}{2}\rceil & if\ p \leq 4(r-1) \\ m-(m-1)r-\lceil\frac{p}{m}\rceil & if\ p > 4(r-1), \end{cases}$$

where $m = \lfloor\sqrt{\frac{p}{r-1}}\rfloor$.

On the other hand, by Lemmas 4.3 and 4.5 we have

$$r(H_{n,p}) \geq \begin{cases} k-kr-s-1 & if\ p \neq 0\ mod(n+1) \\ k-kr-s-3 & if\ p = 0\ mod(n+1). \end{cases}$$

The theorem is thus proved. ∎

From above theorem, the following corollaries are easily obtained.

Corollary 4.1 If n is odd, p is even and $s \neq 0$, then

$$k-kr-s-1 \leq r(H_{n,p}) \leq \begin{cases} 2-r-\lceil\frac{p}{2}\rceil & if\ p \leq 4(r-1) \\ m-(m-1)r-\lceil\frac{p}{m}\rceil & if\ p > 4(r-1), \end{cases}$$

where $m = \lfloor\sqrt{\frac{p}{r-1}}\rfloor$.

Corollary 4.2 If n is odd, p is even, $s = 0$ and k is odd, then

$$k-kr-s-1 \leq r(H_{n,p}) \leq \begin{cases} 2-r-\lceil\frac{p}{2}\rceil & if\ p \leq 4(r-1) \\ m-(m-1)r-\lceil\frac{p}{m}\rceil & if\ p > 4(r-1), \end{cases}$$

where $m = \lfloor\sqrt{\frac{p}{r-1}}\rfloor$.

Corollary 4.3 If n is odd, p is even, $s = 0$ and k is even, then

$$k-kr-s-3 \leq r(H_{n,p}) \leq \begin{cases} 2-r-\lceil\frac{p}{2}\rceil & if\ p \leq 4(r-1) \\ m-(m-1)r-\lceil\frac{p}{m}\rceil & if\ p > 4(r-1), \end{cases}$$

where $m = \lfloor\sqrt{\frac{p}{r-1}}\rfloor$.

In the following we will give some lower and upper bounds for $r(H_{n,p})$ such that both n and p are odd.

Lemma 4.6 ([14]) Let $H_{n,p}$ be a Harary graph such that both n and p are odd, $n = 2r + 1$ and $r > 0$. Then

$$\alpha(H_{n,p}) = \begin{cases} k - kr - s - 1 & if\ p \neq 1\ mod(n+1) \\ k - kr - s - 3 & if\ p = 1\ mod(n+1). \end{cases}$$

Theorem 4.3 Let $H_{n,p}$ be a Harary graph such that both n and p are odd, $n = 2r + 1$ and $r > 0$. Then

(1) If $p \leq 4(r-1)$, then

$$2 - r - \lceil\frac{p}{2}\rceil \geq r(H_{n,p}) \geq \begin{cases} k - kr - s - 1 & if\ p \neq 1\ mod(n+1) \\ k - kr - s - 3 & if\ p = 1\ mod(n+1). \end{cases}$$

(2) If $p > 4(r-1)$, then

$$m - (m-1)r - \lceil\frac{p}{m}\rceil \geq r(H_{n,p}) \geq \begin{cases} k - kr - s - 1 & if\ p \neq 1\ mod(n+1) \\ k - kr - s - 3 & if\ p = 1\ mod(n+1), \end{cases}$$

where $m = \lfloor\sqrt{\frac{p}{r-1}}\rfloor$.

Proof. The proof is similar to that of Theorem 4.2. ∎

Lemma 4.6 ([14]) Let $H_{n,p}$ be a Harary graph such that $n = 2r + 1$, p is even, $r \geq 2$, $0 < s < r + 1$, $s < k$, and k is odd. Then there exists a cut set S with kr elements, such that $\omega(H_{n,p} - S) = k$ and $m(H_{n,p} - S) = 2$.

Theorem 4.4 Let $H_{n,p}$ be a Harary graph such that $n = 2r + 1$, p is even, $r \geq 2$, $0 < s < r + 1$, $s < k$, and k is odd. Then

$$k - kr - 2 \leq r(H_{n,p}) \leq \begin{cases} 2 - r - \lceil\frac{p}{2}\rceil & if\ p \leq 4(r-1) \\ m - (m-1)r - \lceil\frac{p}{m}\rceil & if\ p > 4(r-1), \end{cases}$$

where $m = \lfloor\sqrt{\frac{p}{r-1}}\rfloor$.

Proof. First note that if $r = 1$, then $s = 1$ and so $p = 2k + 1$, a contradiction. So, $r \geqslant 2$. By Lemma 4.6 and the definition of rupture degree we have $r(H_{n,p}) \geq \omega(H_{n,p} - S) - |S| - m(H_{n,p} - S) = k - kr - 2$.

On the other hand, by Theorem 4.2 we have

$$r(H_{n,p}) \leq \begin{cases} 2 - r - \lceil \frac{p}{2} \rceil & if\ p \leq 4(r-1) \\ m - (m-1)r - \lceil \frac{p}{m} \rceil & if\ p > 4(r-1). \end{cases}$$

The theorem is thus proved. ∎

5. Some Other Results on Rupture Degrees of Graphs

In this section, we determine the rupture degree of a special permutation graph. The concept of a permutation graph was introduced by Chartrand and Harary in [13]. Since then, many parameters on this kind of graphs have been determined, such as connectivity, chromatic number, crossing number, etc.

Definition 5.1 Let G be a graph whose vertices are labelled $v_1, v_2, \cdots, v_n$ and a permutation α in S_n, where S_n is the symmetric group on $\{1, 2, \cdots, n\}$. Then the permutation graph $P_\alpha(G)$ is obtained by taking two copies of G, say G_x with vertex set $\{x_1, x_2, \cdots, x_n\}$ and G_y with vertex set $\{y_1, y_2, \cdots, y_n\}$, along with a set of permutation edges joining x_i of G_x and $y_{\alpha(i)}$ of G_y.

Theorem 5.1 If G is a k-regular and k-connected bipartite graph with partition $[M, N]$ on n vertices, then for a permutation $\alpha \in S_n$ such that

$$\alpha : \begin{cases} M_x \rightarrow N_y \\ M_y \rightarrow N_x \end{cases}$$

the rupture degree $r(P_\alpha(G))$ of the permutation graph $P_\alpha(G)$ is -1, where $[M_x, M_y]$ is the partition of the first copy of G, and $[N_x, N_y]$ is the partition of the second copy of G.

Proof. It is easy to verify that the graph $P_\alpha(G)$ is a $(k+1)$-regular and $(k+1)$-connected bipartite graph with partition $[M_x \cup M_y, N_x \cup N_y]$. By Lemma 3.3 we know that $r(P_\alpha(G)) = -1$. ∎

Theorem 5.2 If G is a connected graph with n vertices, then for any vertex $v \in V(G)$ we have $r(G - v) \leq r(G) + 1$.

Proof. Let $G^{'} = G - v$. We distinguish the following two cases:

Case 1. If G is a complete graph K_n, then $G^{'}$ is the complete graph K_{n-1}. By the definition of rupture degree of a complete graph we have $r(G) = r(K_n) = 1 - n$ and $r(G^{'}) = r(k_{n-1}) = 2 - n$. So it is easy to see that $r(G^{'}) = r(G) + 1$.

The result is true.

Case 2. If G is a noncomplete connected graph, then, on one hand, if $G^{'} = K_{n-1}$, then by the definition of rupture degree we know that $r(G^{'})$=$2-n$. Clearly, $\alpha(G)=2$. Then, by Lemma 4.5 we have $r(G) \geq 2\alpha(G)-n-1 = 3-n$. It is easy to see that $r(G^{'}) < r(G)+1$.

On the other hand, if $G^{'} \neq K_{n-1}$, let $S^{'}$ be a R-set of $G^{'}$ such that $|S^{'}|=t$, then $r(G^{'}) = \omega(G^{'}-S^{'})-|S^{'}|-m(G^{'}-S^{'})$. Now define $S = S^{'}\cup\{v\}$. Clearly, S is a cut set of G, and so

$$r(G) \geq \omega(G-S)-|S|-m(G-S).$$

But $|S|=t+1$, $\omega(G^{'}-S^{'})$=$\omega(G-S)$ and $m(G^{'}-S^{'})$=$m(G-S)$. So,

$$r(G^{'}) = \omega(G^{'}-S^{'})-|S^{'}|-m(G^{'}-S^{'})$$

$$= \omega(G-S)-|S|+1-m(G-S) \leq r(G)+1.$$

The theorem is thus proved. ∎

For the graphs K_n, P_n (n is even) and C_n (n is odd), the above upper bound can be achieved. One can easily find more examples to show that our upper bound is best possible.

References

[1] C.A. Barefoot, R.Entringer and H. Swart, Vulnerability in graphs - A comparative survey, Congressus Numerantium 1(1987), 12-22.

[2] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New york, 1976.

[3] D. Bauer, S.L. Hakimi and E. Schmeichel, Recognizing tough graphs is NP-hard, Discrete Math. 28(1990), 191-195.

[4] M. Cozzens, D. Moazzami and S. Stueckle, The tenacity of a graph, Proc. Seventh International Conference on the Theory and Applications of Graphs, Wiley, New York, 1995, 1111-1122.

[5] S.A. Choudum and N. Priya, Tenacity of complete graph product and grids, Networks 34(1999), 192-196.

[6] Y.K. Li, S.G. Zhang and X.L. Li, Rupture degree of graphs, Preprint 2003.

[7] V. Chvátal, Tough graph and hamiltonian circuits, Discrete Math. 5(1973), 215-228.

[8] Y.H. Peng, C.C. Chen and K.M. Koh, On the edge-toughness of a graph (I), Southeast Asian Math. Bull. 12(1988), 109-122.

[9] B.L. Piazza, F.S. Roberts and S.K. Stueckle, Edge tenacious networks, Networks 25(1995), 7-17.

[10] S.G. Zhang and Z.G. Wang, Scattering number in graphs, Networks 37(2001), 102-106.

[11] G. Almasi and A. Gottlieb, Highly parallel computing, Benjamin/Cummings, Redwood City, CA, 1989.

[12] M.J. Quinn, Parallel computing: theory and practice, New York, 1994.

[13] G. Chartrand and F. Harary, Planar permutation graphs, Ann. Inst. H. Poincare Sec. B (N.S.) 3(1967), 433-438.

[14] M. Cozzens and D. Moazzami, The tenacity of the Harary graphs, J. Combin. Math. Combin. Comput. 16(1994), 33-56.

[15] K.Z. Ouyang and K.Y. Ouyang, Relative breaktivity of graphs, J. Lanzhou University (Natural Science) 29(3)(1993), 43-49.