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Applicability of the q-analogue of Zeilberger's algorithm

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Abstract

The applicability or terminating condition for the ordinary case of Zeilberger's algorithm was recently obtained by Abramov. For the q-analogue, the question of whether a bivariate q-hypergeometric term has a qZ-pair remains open. Le has found a solution to this problem when the given bivariate q-hypergeometric term is a rational function in certain powers of q. We solve the problem for the general case by giving a characterization of bivariate q-hypergeometric terms for which the q-analogue of Zeilberger's algorithm terminates. Moreover, we give an algorithm to determine whether a bivariate q-hypergeometric term has a qZ-pair.

MSC: 33F10; 68W30

Keywords: Zeilberger's algorithm; q-hypergeometric term; Z-pair; qZ-pair; Proper hypergeometric term; q-proper hypergeometric term

1. Introduction

Zeilberger's algorithm (Graham et al., 1994; Petkovšek et al., 1996; Zeilberger, 1991), also known as the method of *creative telescoping*, is devised for proving hypergeometric identities of the form

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$$\sum_{k=-\infty}^{\infty} F(n,k) = f(n),$$

where $F(n,k)$ is a bivariate hypergeometric term and $f(n)$ is a given function (for most cases a hypergeometric term plus a constant). The algorithm can be easily adapted to the q-case, which is called the q-analogue of Zeilberger's algorithm (Böing and Koepf, 1999; Koornwinder, 1993; Paule and Riese, 1997; Wilf and Zeilberger, 1992). Let N and K be the shift operators with respect to n and k respectively, defined by

$$NT(n,k) = T(n+1,k) \qquad \text{and} \qquad KT(n,k) = T(n,k+1).$$

Given a bivariate q-hypergeometric term $T(n,k)$, the q-analogue of Zeilberger's algorithm aims to find a *qZ-pair* (L, G), where L is a linear difference operator with coefficients in the ring of polynomials in q^n

$$L = a_0(q^n)N^0 + a_1(q^n)N^1 + \cdots + a_r(q^n)N^r$$

and G is a bivariate q-hypergeometric term $G(n,k)$ such that

$$LT(n,k) = (K-1)G(n,k).$$

Zeilberger's algorithm has been widely used as a powerful tool to prove hypergeometric identities. It was an open question when the algorithm terminates. This problem was solved recently by Abramov (2002, 2003). For the q-analogue of Zeilberger's algorithm, Abramov and Le (2002) found a solution to the termination problem for the case of rational functions. In this paper we provide a solution for the general q-case.

We begin with an additive decomposition of univariate q-hypergeometric terms. Using this decomposition, a univariate q-hypergeometric term $T(n)$ can be represented as

$$T(n) = (N-1)T_1(n) + T_2(n),$$

where $T_1(n)$ and $T_2(n)$ are q-hypergeometric terms, and $T_2(n)$ has the following form:

$$T_2(n) = \frac{u_1(q^n)}{u_2(q^n)} \prod_{j=n_0}^{n-1} \frac{f_1(q^j)}{f_2(q^j)},$$

where u_1, u_2, f_1, f_2 are polynomials, n_0 is a nonnegative integer, and for any integer m, $u_2(x)$ and $u_2(xq^m)$ have no common factors except for a power of x. Consequently, a bivariate q-hypergeometric term $T(n,k)$ can be decomposed as

$$T(n,k) = (K-1)T_1(n,k) + T_2(n,k) \tag{1.1}$$

such that

$$T_2(n,k) = T(n,k_0)V(q^n,q^k) \prod_{j=k_0}^{k-1} F(q^n,q^j),$$

where V, F are rational functions, n_0 is a nonnegative integer, and the denominator v_2 of V satisfies the conditions that for any integer m, $v_2(x,y)$ and $v_2(x,yq^m)$ have no common factors except for a power of y. The polynomial $v_2(x,y)$ with the above property is called $\boldsymbol{\varepsilon}_y$-free. We should note that the above decomposition does not solve the minimal additive decomposition problem and is not unique (see Abramov and Petkovšek (2002a) for a precise definition). However, for the purpose of constructing a qZ-pair, it turns out that one may choose any decomposition.

Then we consider the structure of bivariate q-hypergeometric terms. The structure of ordinary hypergeometric terms has been studied by Ore (1930), Sato et al. (1990), Gel'fand et al. (1992), Abramov and Petkovšek (2002b) and Hou (2004). To a large extent, the q-case is analogous to the ordinary case. For each bivariate q-hypergeometric term, we associate it with a normal representation (q-NR) which consists of four polynomials r, s, u, v. Based on the properties of the representation, we may give a definition of q-proper hypergeometric terms and prove that under the condition that v is $\boldsymbol{\varepsilon}_y$-free, a bivariate q-hypergeometric term has a qZ-pair if and only if it is a q-proper term. Applying the decomposition (1.1), we deduce that for any bivariate q-hypergeometric term T, it has a qZ-pair if and only if T_2 is q-proper.

We conclude with some examples.

2. $\boldsymbol{\varepsilon}$-free decomposition

Throughout the paper, we let $\mathbb{Z}, \mathbb{Z}^+$ and $\mathbb{N}$ denote the set of integers, positive integers and nonnegative integers, respectively. For integers (or polynomials) a, b, we denote by $\gcd(a, b)$ the (monic) greatest common divisor of a and b. We also write $a \perp b$ to indicate that a and b are relatively prime, i.e., $\gcd(a, b) = 1$.

Let $\mathbb{F}$ be a field of characteristic zero, $q \in \mathbb{F}$ a nonzero element which is not a root of unity, and x transcendental over $\mathbb{F}$. Denote by $\boldsymbol{\varepsilon}$ the unique automorphism of $\mathbb{F}(x)$ which fixes $\mathbb{F}$ and satisfies $\boldsymbol{\varepsilon} x = qx$. Then $\mathbb{F}(x)$ together with the *q-shift operator* $\boldsymbol{\varepsilon}$ is a difference field (Cohn, 1965). Let r and s be two polynomials. We say that r/s is *$\boldsymbol{\varepsilon}$-reduced* if $r \perp \boldsymbol{\varepsilon}^h s$ for all $h \in \mathbb{Z}$.

To be more specific, the rational functions involved in the q-hypergeometric terms (see Definition 2.4) are rational functions of q^n. However, for a rational function $R \in \mathbb{F}(x)$ and a nonnegative integer n_0, we have

$$N\,R(q^n) = R(q^{n+1}) = \boldsymbol{\varepsilon} R(q^n) \quad \text{and} \quad R(q^n) = 0\ \forall\, n \geq n_0 \Leftrightarrow R(x) = 0.$$

Therefore, there is a natural one-to-one correspondence between the set of rational functions of q^n together with the shift operator N and the field $\mathbb{F}(x)$ together with the q-shift operator $\boldsymbol{\varepsilon}$. In this paper, we adopt the notation of $\mathbb{F}(x)$ as in the work of Abramov et al. (1998).

The concept of rational normal forms introduced by Abramov and Petkovšek (2002a) can be extended to the q-case.

Definition 2.1. Let $R \in \mathbb{F}(x)$ be a rational function. If polynomials $r, s, u, v \in \mathbb{F}[x]$ satisfy

(i) $R = \frac{r}{s} \cdot \frac{\boldsymbol{\varepsilon}(u/v)}{(u/v)}$, where $u \perp v$ and u, v have no factor x,
(ii) r/s is $\boldsymbol{\varepsilon}$-reduced,

then (r, s, u, v) is called a q-rational normal form (q-RNF) of R.

Recall that a monic polynomial that has no factor x is called a q-monic polynomial by Abramov et al. (1998). The following factorization theorem was given in Abramov et al. (1998).

Theorem 2.2. *Let $R \in \mathbb{F}(x) \setminus \{0\}$. Then there exist $z \in \mathbb{F}$ and monic polynomials $a, b, c \in \mathbb{F}[x]$ such that*

$$\begin{aligned} &R(x) = z\frac{a(x)}{b(x)}\frac{c(qx)}{c(x)}, \\ &\gcd(a(x), b(q^n x)) = 1, \quad \textit{for all } n \in \mathbb{N}, \\ &\gcd(a(x), c(x)) = \gcd(b(x), c(qx)) = 1 \quad \textit{and} \quad c(0) \neq 0. \end{aligned} \tag{2.1}$$

We call (az, b, c) a *q-Gosper form* (q-GF) of R.

Theorem 2.3. *Every rational function $R \in \mathbb{F}(x)$ has a q-RNF.*

Proof. It is clear that $(0, 1, 1, 1)$ is a q-RNF of 0. For $R \neq 0$, by Theorem 2.2, there exists a q-GF (az, b, c) of R. Applying Theorem 2.2 again to $b(x)/a(x)$, we get a q-GF (r, s, d). From the construction given in Abramov et al. (1998), we have $r \mid b$ and $s \mid a$. Hence $s(x) \perp r(xq^n)$ for any $n \in \mathbb{N}$ because (az, b, c) is a q-GF. Since (r, s, d) is also a q-GF, we have $r(x) \perp s(xq^n)$ for any $n \in \mathbb{N}$. Thus s/r is $\boldsymbol{\varepsilon}$-reduced and $(zs, r, c/\gcd(c, d), d/\gcd(c, d))$ is a q-RNF of R. □

The above proof provides an algorithm to generate a q-RNF of R.

Algorithm *q*-RNF

```
if R = 0 then
    return (0, 1, 1, 1);
else
    compute 'q-GF' of R, we get (a, b, c);
    compute 'q-GF' of b/a, we get (r, s, d);
    return (s, r, c/ gcd(c, d), d/ gcd(c, d)).
```

We now come to the q-multiplicative representation of a general q-hypergeometric term. This is the starting point of the $\boldsymbol{\varepsilon}$-free decomposition algorithm.

Definition 2.4. Suppose $T(n)$ is a function from $\mathbb{N}$ to $\mathbb{F}$. If there exist a nonnegative integer n_0 and a nonzero rational function $R(x) \in \mathbb{F}(x)$ such that $T(n+1) = R(q^n)T(n)$ for all $n \geq n_0$, then we call $T(n)$ a (univariate) *q-hypergeometric term*.

Suppose (r, s, u, v) is a q-RNF of a rational function R. Then the corresponding q-hypergeometric term $T(n)$ satisfies

$$T(n) = T(n_0)\prod_{j=n_0}^{n-1} R(q^j) = \frac{T(n_0)}{u(q^{n_0})/v(q^{n_0})} \cdot \frac{u(q^n)}{v(q^n)} \prod_{j=n_0}^{n-1} \frac{r(q^j)}{s(q^j)}, \quad \forall n \geq n_0.$$

This leads to the following definition.

Definition 2.5. Let $T(n)$ be a q-hypergeometric term and D, U be two rational functions such that $D(q^n)$ has neither poles nor zeros and $U(q^n)$ has no poles for all $n \geq n_0$. Suppose that

$$T(n) = U(q^n) \prod_{j=n_0}^{n-1} D(q^j), \qquad \forall n \geq n_0.$$

Then we call (D, U, n_0) a q-multiplicative representation (q-MR) of T.

Let $\Delta = N - 1$ be the difference operator with respect to n. The following lemma can be easily verified.

Lemma 2.6. *Let T and T_1 be two q-hypergeometric terms with q-MRs (D, U, n_0) and (D, U_1, n_0), respectively. Suppose that*

$$T_2 = T - \Delta T_1 \quad \text{and} \quad U_2 = U - D \cdot \boldsymbol{\varepsilon} U_1 + U_1.$$

Then (D, U_2, n_0) is a q-MR of T_2.

For $u, v \in \mathbb{F}[x]$, let $\mathcal{R}$ be the set of all nonnegative integers h such that there exists an irreducible polynomial $p(x) \neq x$ satisfying $p(x) \mid u(x)$ and $p(x) \mid v(q^h x)$. Define qdis(u, v) to be $\max\{h \in \mathcal{R}\}$ or -1 if $\mathcal{R}$ is empty. Note that $\mathcal{R}$ is a finite set, and "qdis" is well defined. If qdis$(v, v) = 0$, we say that v is *$\boldsymbol{\varepsilon}$-free*.

Given a q-hypergeometric term T with a q-MR (D, U, n_0). Usually the denominator u of U is not $\boldsymbol{\varepsilon}$-free. However, translating the decomposition algorithm of Abramov and Petkovšek (2002a) into the q-case, we have the following $\boldsymbol{\varepsilon}$-free decomposition algorithm "q-decomp", which decomposes T into $\Delta T_1 + T_2$ such that T_2 has a q-MR (F, V, n_0) where the denominator of V is $\boldsymbol{\varepsilon}$-free.

Algorithm *q*-decomp
Input: (D, U, n_0) Output: $U_1, F, V \in \mathbb{F}(x)$

```
d_1 := numer(D); d_2 := denom(D);
U_1 := 0; U_2 := U; u_2 := denom(U);
N := qdis(u_2, u_2);
for h := N down to 1 do
    v_2 := u_2/ gcd(u_2, d_2);
    s(x) := gcd(v_2(x), v_2(q^{-h} x));
    (s̃, ũ_2) := pump(s, u_2);
    write U_2 = a/ũ_2 + b/s̃ where a, b ∈ F[x];
    U_1' := −b/s̃;
    U_1 := U_1 + U_1'; U_2 := U_2 − D · εU_1' + U_1';
    u_2 := denom(U_2);
f_1 := d_1; f_2 := d_2; v_1 := numer(U_2); v_2 := denom(U_2);
w := gcd(d_2, v_2);
v_2 := v_2/w; f_2 := εw f_2/w;
F := f_1/f_2; V := (1/w(q^{n_0})) · v_1/v_2;
return (U_1, F, V).
```

The procedure "pump" is the same as in the ordinary case.

Algorithm pump

Input: $f, g \in \mathbb{F}[x]$ Output: $\tilde{f}, \tilde{g} \in \mathbb{F}[x]$

$\tilde{f} := f$; $\tilde{g} := g/f$;
repeat
$d := \gcd(\tilde{f}, \tilde{g})$; $\tilde{f} := \tilde{f}d$; $\tilde{g} := \tilde{g}/d$;
until $\deg d = 0$;
return $(\tilde{f}, \tilde{g})$.

The following theorem shows that the $\boldsymbol{\varepsilon}$-free algorithm generates the desired decomposition.

Theorem 2.7. *Let T be a q-hypergeometric term with a q-MR (D, U, n_0) and U_1, F, V be given by the algorithm q-decomp. Then there exist q-hypergeometric terms T_1 and T_2 such that*

(1) $T = \Delta T_1 + T_2$.
(2) *T_1 has a q-MR (D, U_1, n_0) and T_2 has a q-MR (F, V, n_0).*
(3) *The denominator of V is $\boldsymbol{\varepsilon}$-free.*

Furthermore, if D is $\boldsymbol{\varepsilon}$-reduced, so is F.

Proof. Let u_0 be the denominator of U. We first use induction to show that after iterating the loop of h in the algorithm i times, the denominator u_2 of U_2 satisfies:

(a) $\text{qdis}(v_2, v_2) \le N - i$,
(b) $u_2(q^n)$ has no zeros for all $n \ge n_0$,

where $v_2 = u_2/\gcd(u_2, d_2)$, and d_2 is the denominator of D.

The case for $i = 0$ is trivial. Assume that the assertion holds for $i - 1$. Let u_2 and u_2' be the denominator of U_2 after $i - 1$ and i iterations, respectively. Set $h = N - (i - 1) > 0$ and $w_2 = \gcd(u_2, d_2)$. From the algorithm q-decomp we have

$$v_2 = u_2/w_2 \quad \text{and} \quad s = \gcd(v_2(x), v_2(q^{-h}x)).$$

Suppose the prime decomposition of s is $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $v_2 = p_1^{\beta_1} \cdots p_r^{\beta_r} v'$, $w_2 = p_1^{\gamma_1} \cdots p_r^{\gamma_r} w'$ where $v' \perp s$, $w' \perp s$. Then the algorithm "pump" enables us to decompose u_2 as $p_1^{\beta_1+\gamma_1} \cdots p_r^{\beta_r+\gamma_r} \cdot (v'w')$. That is, $\tilde{s} = p_1^{\beta_1+\gamma_1} \cdots p_r^{\beta_r+\gamma_r}$ and $\tilde{u}_2 = v'w'$. Since

$$U_2 = \frac{a}{\tilde{u}_2} + \frac{d_1}{d_2} \cdot \boldsymbol{\varepsilon}\left(\frac{b}{\tilde{s}}\right),$$

it follows that u_2' divides the least common multiple of $\tilde{u}_2$ and $d_2\boldsymbol{\varepsilon}\tilde{s}$. Hence we have that u_2' divides $v'd_2 \cdot \boldsymbol{\varepsilon}\tilde{s}$. Let $v'' = v' \cdot \boldsymbol{\varepsilon}\tilde{s}$. Assume that there exist an integer $m \ge h$ and an irreducible polynomial $p(x) \ne x$ such that $p \mid v''$ and $p \mid \boldsymbol{\varepsilon}^m v''$. We may encounter four cases:

- $p \mid v'$ and $p \mid \boldsymbol{\varepsilon}^m v'$.
  From $v' \mid v_2$ and $\text{qdis}(v_2, v_2) \le h$, it follows that $m = h$. Therefore, $\boldsymbol{\varepsilon}^{-h} p \mid \boldsymbol{\varepsilon}^{-h} v_2$ and $\boldsymbol{\varepsilon}^{-h} p \mid v_2$. Consequently, we have $\boldsymbol{\varepsilon}^{-h} p \mid s$, which contradicts $v' \perp s$.

- $p \mid v'$ and $p \mid \boldsymbol{\varepsilon}^{m+1}\tilde{s}$.
  Since s and $\tilde{s}$ have the same prime factors, we have $p \mid \boldsymbol{\varepsilon}^{m+1}s$, implying that $p \mid \boldsymbol{\varepsilon}^{m+1}v_2$. On the other hand, we have $p \mid v_2$, which contradicts $\mathrm{qdis}(v_2, v_2) \le h$.
- $p \mid \boldsymbol{\varepsilon}\tilde{s}$ and $p \mid \boldsymbol{\varepsilon}^m v'$.
  In this situation, we have $\boldsymbol{\varepsilon}^{-1}p \mid \tilde{s}$, which implies that $\boldsymbol{\varepsilon}^{-1}p \mid \boldsymbol{\varepsilon}^{-h}v_2$, or equivalently, $\boldsymbol{\varepsilon}^{h-1}p \mid v_2$. On the other hand, $\boldsymbol{\varepsilon}^{h-1}p \mid \boldsymbol{\varepsilon}^{m+h-1}v_2$. Since $\mathrm{qdis}(v_2, v_2) \le h$, we get $m+h-1 \le h$, and hence $m = 1$. Now we have $p \mid \boldsymbol{\varepsilon}s$ and $p \mid \boldsymbol{\varepsilon}v'$, which contradicts $v' \perp s$.
- $p \mid \boldsymbol{\varepsilon}\tilde{s}$ and $p \mid \boldsymbol{\varepsilon}^{m+1}\tilde{s}$.
  Similarly, we have $\boldsymbol{\varepsilon}^{-1}p \mid s$ and hence $\boldsymbol{\varepsilon}^{-1}p \mid \boldsymbol{\varepsilon}^{-h}v_2$, i.e., $\boldsymbol{\varepsilon}^{h-1}p \mid v_2$. However, we have $\boldsymbol{\varepsilon}^{h-1}p \mid \boldsymbol{\varepsilon}^{m+h}v_2$. Thus, we obtain $m+h \le h$, which is also a contradiction.

In summary, we may conclude that $\mathrm{qdis}(v'', v'') \le h-1$. Because u_2' divides $v'' \cdot d_2$, there exist $\bar{v} \mid v''$ and $\bar{w} \mid d_2$ such that $u_2' = \bar{v}\bar{w}$. Let $v_2' = u_2'/\gcd(u_2', d_2)$. From $\bar{w} \mid \gcd(u_2', d_2)$, it follows that $v_2' \mid \bar{v}$. So we get $\mathrm{qdis}(v_2', v_2') \le h-1 = N-i$. Thus, we have proved (a). Since $u_2' | u_2 \cdot \boldsymbol{\varepsilon}u_2 \cdot d_2$, (b) immediately follows from the induction hypothesis.

On the other hand, since $\tilde{s} \mid u_2$, (b) implies that $U_1(q^n)$ has no poles for all $n \ge n_0$. Let

$$T_1(n) = U_1(q^n)\prod_{j=n_0}^{n-1} D(q^j) \quad \text{and} \quad T_2(n) = U_2(q^n)\prod_{j=n_0}^{n-1} D(q^j). \tag{2.2}$$

Noting that $U_2 = U - D\boldsymbol{\varepsilon}U_1 + U_1$, by Lemma 2.6, we obtain $T = \Delta T_1 + T_2$.

Because $w \mid d_2$ and $d_2(q^n) \ne 0$ for all $n \ge n_0$, we can write $T_2(n)$ as

$$T_2(n) = \frac{1}{w(q^{n_0})}U_2(q^n)w(q^n)\prod_{j=n_0}^{n-1} D(q^j)\frac{w(q^j)}{w(q^{j+1})} = V(q^n)\prod_{j=n_0}^{n-1} F(q^j).$$

Let v be the denominator of V. Then (a) implies $\mathrm{qdis}(v, v) = 0$; that is, v is $\boldsymbol{\varepsilon}$-free.

Finally, notice that $f_1 = d_1$ and $f_2 = \boldsymbol{\varepsilon}w \cdot (d_2/w)$, where $w \mid d_2$. Therefore, F is $\boldsymbol{\varepsilon}$-reduced provided that D is $\boldsymbol{\varepsilon}$-reduced. This completes the proof. □

3. Bivariate q-hypergeometric terms

We begin this section with the definition of bivariate q-hypergeometric terms.

Definition 3.1. Suppose $T(n,k)$ is a function from $\mathbb{N}^2$ to $\mathbb{F}$. If there exist rational functions $R_1(x,y), R_2(x,y) \in \mathbb{F}(x,y)$ and $n_0 \in \mathbb{N}$ such that

$$T(n+1,k) = R_1(q^n, q^k)T(n,k) \quad \text{and} \quad T(n,k+1) = R_2(q^n, q^k)T(n,k),$$

for all $n, k \ge n_0$, then we call $T(n,k)$ a bivariate q-hypergeometric term.

Without loss of generality, from now on we may assume that $n_0 = 0$ and that $R_1(q^n, q^k)$, $R_2(q^n, q^k)$ have neither zeros nor poles for all $n, k \ge 0$.

Denote by $\boldsymbol{\varepsilon}_x$ and $\boldsymbol{\varepsilon}_y$ the shift operators on $\mathbb{F}(x,y)$ defined by $\boldsymbol{\varepsilon}_x x = qx$, $\boldsymbol{\varepsilon}_x|_{\mathbb{F}(y)} = \mathrm{id}$ (the identity map) and $\boldsymbol{\varepsilon}_y y = qy$, $\boldsymbol{\varepsilon}_y|_{\mathbb{F}(x)} = \mathrm{id}$, respectively. The idea of q-RNFs can be easily adopted to the bivariate case by taking $\mathbb{F}(y)$ as the ground field. Let $R(x,y)$ be a rational function of x and y; its q-rational normal form (q-RNF with respect to $\boldsymbol{\varepsilon}_x$) is represented by (r, s, u, v) as in the univariate case. By using the ground field $\mathbb{F}(x)$, we may find a q-RNF of $R(x, y)$ with respect to $\boldsymbol{\varepsilon}_y$.

Let $T(n, k)$ be a bivariate q-hypergeometric term. By definition, there exists a rational function R such that

$$T(n+1, k)/T(n, k) = R(q^n, q^k).$$

Suppose (r, s, u, v) is a q-RNF of R with respect to $\boldsymbol{\varepsilon}_x$. We call (r, s, u, v) a q-normal representation (q-NR) of $T(n, k)$ with respect to the shift operator N. Similarly, we can define the q-NR of $T(n, k)$ with respect to the shift operator K.

We next give a characterization of the polynomials involved in the q-NR of bivariate q-hypergeometric terms.

Theorem 3.2. *Let $T(n, k)$ be a bivariate q-hypergeometric term that has a q-NR (r, s, u, v) with respect to N. Then r and s are products of polynomials having the form*

$$(x^c y^d) \cdot \prod_{l=1}^{a} p(q^{w_l} x^a y^b),$$

where p is a Laurent polynomial of one variable, $a \in \mathbb{Z}^+, b, c, d, w_l \in \mathbb{Z}$, $a \perp b$, and $w_i \not\equiv w_j \pmod a$, $\forall\, i \neq j$.

Similarly, suppose (r, s, u, v) is a q-NR of T with respect to K. Then r and s are products of polynomials having the form

$$(x^c y^d) \cdot \prod_{l=1}^{a} p(q^{w_l} x^b y^a)$$

under the same conditions.

Sketch of the proof. The proof of the ordinary case (Hou, 2004, Theorem 3.4) can be carried over to the q-case except that we need to consider the characterization of polynomials $f(x, y)$ such that $f(q^a x, q^b y) = C f(x, y)$ for certain integers a, b and $C \in \mathbb{F}$. □

Consequently, we have

Corollary 3.3. *Let $T(n, k)$ be a bivariate q-hypergeometric term that has a q-NR (r, s, u, v) with respect to N (or K respectively). Then we have*

$$T(n, k) = C \cdot \frac{u(q^n, q^k)}{v(q^n, q^k)} \cdot \frac{\prod_{l=1}^{uu} \prod_{j=0}^{a_l n + b_l k + c_l} f_l(q^j)}{\prod_{l=1}^{vv} \prod_{j=0}^{a'_l n + b'_l k + c'_l} g_l(q^j)},$$

where $C \in \mathbb{F}$, $uu, vv \in \mathbb{N}$, $a_l, b_l, c_l, a'_l, b'_l, c'_l \in \mathbb{Z}$ and f_l, g_l are polynomials.

Corollary 3.3 enables us to give the following definition of q-proper hypergeometric terms.

Definition 3.4. A polynomial $f \in \mathbb{F}[x, y]$ is said to be q-proper if, for each of its irreducible factors $p(x, y) \in \mathbb{F}[x, y]$, there exist $a, b \in \mathbb{Z}$, not both zeros, such that $p(x, y) | p(q^a x, q^b y)$. A bivariate q-hypergeometric term T is said to be q-proper if v is a q-proper polynomial, where (r, s, u, v) is a q-NR of T with respect to N or K.

Suppose that T is a bivariate q-hypergeometric term that has a q-NR (r, s, u, v) with respect to N (or K). Theorem 3.2 guarantees that r and s are both q-proper polynomials.

As in the case of ordinary bivariate hypergeometric terms (Hou, 2004, Theorem 4.2), we have an analogous "fundamental theorem" for the q-case.

Theorem 3.5. *Let $T(n, k)$ be a bivariate q-hypergeometric term. Then T is q-proper if and only if there exist polynomials $a_{ij}(x) \in \mathbb{F}[x]$, not all zero, such that*

$$\sum_{0 \le i \le I,\ 0 \le j \le J} a_{ij}(q^n) T(n + i, k + j) = 0 \quad \forall n, k \ge 0.$$

Based on an analogous argument for the ordinary case as in Petkovšek et al. (1996, Theorem 6.2.1), we get

Corollary 3.6. *Any q-proper hypergeometric term has a qZ-pair.*

4. The existence of *qZ*-pairs

In this section, we obtain a necessary and sufficient condition for the existence of qZ-pairs for any bivariate q-hypergeometric term based on its q-NR with respect to K.

From Theorem 3.2, we have

Corollary 4.1. *Let $T(n, k)$ be a bivariate q-hypergeometric term that has a q-NR (r, s, u, v) with respect to K. Then there exist polynomials $f_i(x), g_i(x) \in \mathbb{F}[x]$ and $a_i, a_i', b_i, b_i' \in \mathbb{Z}$ such that*

$$\prod_{j=0}^{k-1} \left(\frac{r(q^{n+1}, q^j)}{r(q^n, q^j)} \cdot \frac{s(q^n, q^j)}{s(q^{n+1}, q^j)} \right) = \prod_{i=1}^{\ell} \frac{f_i(q^{a_i k + b_i n})}{g_i(q^{a_i' k + b_i' n})}.$$

We need to consider the following ratio:

$$\frac{T(n+i, k)}{T(n, k)} = \frac{T(n+i, 0)}{T(n, 0)} \prod_{j=0}^{k-1} \left\{ \frac{T(n+i, j+1)}{T(n+i, j)} \frac{T(n, j)}{T(n, j+1)} \right\},$$

which can be rewritten as

$$\frac{T(n+i, k)}{T(n, k)} = \prod_{l=0}^{i-1} \prod_{j=0}^{k-1} \left\{ \frac{r(q^{n+l+1}, q^j)}{r(q^{n+l}, q^j)} \frac{s(q^{n+l}, q^j)}{s(q^{n+l+1}, q^j)} \right\} \prod_{l=0}^{i-1} \frac{T(n+l+1, 0)}{T(n+l, 0)}$$
$$\times \frac{u(q^{n+i}, q^k)}{u(q^{n+i}, q^0)} \frac{u(q^n, q^0)}{u(q^n, q^k)} \frac{v(q^{n+i}, q^0)}{v(q^{n+i}, q^k)} \frac{v(q^n, q^k)}{v(q^n, q^0)}. \tag{4.1}$$

From Corollary 4.1 we get the following expression.

Lemma 4.2. *Let $T(n,k)$ be a bivariate q-hypergeometric term that has a q-NR (r,s,u,v) with respect to K. Then for each $i \geq 0$, there exist q-proper polynomials $w_1^{(i)}(x,y)$ and $w_2^{(i)}(x,y)$ such that*

$$\frac{T(n+i,k)}{T(n,k)} = \frac{u(q^{n+i},q^k)}{v(q^{n+i},q^k)} \cdot \frac{v(q^n,q^k)}{u(q^n,q^k)} \cdot \frac{w_1^{(i)}(q^n,q^k)}{w_2^{(i)}(q^n,q^k)}, \quad \forall n,k \geq 0. \tag{4.2}$$

An $\boldsymbol{\varepsilon}_y$-free polynomial that is not q-proper has a special factor.

Lemma 4.3. *Let $f \in \mathbb{F}[x,y]$ be a non-q-proper and $\boldsymbol{\varepsilon}_y$-free polynomial. Then there exists an irreducible factor p of f such that*

$$\begin{aligned} &p(x,y) \perp p(q^i x, q^j y), \quad \forall (i,j) \in \mathbb{Z}^2 \setminus \{(0,0)\}, \\ &p(x,y) \perp f(q^i x, q^j y), \quad \forall (i,j) \in (\mathbb{N} \times \mathbb{Z}) \setminus \{(0,0)\}. \end{aligned} \tag{4.3}$$

Proof. Since $f(x,y)$ is non-q-proper, by definition it has an irreducible factor $p_1(x,y)$ such that $p_1(x,y) \perp p_1(q^i x, q^j y), \forall (i,j) \in \mathbb{Z}^2 \setminus \{(0,0)\}$.

We may factor $f(x,y)$ as

$$f(x,y) = p_1^{\alpha_1}(q^{a_1}x, q^{b_1}y) \cdots p_1^{\alpha_r}(q^{a_r}x, q^{b_r}y) f_1(x,y),$$

where $(a_i,b_i) \in \mathbb{Z}^2$ are distinct pairs, $\alpha_i \in \mathbb{Z}^+$, and $p_1(q^i x, q^j y) \perp f_1(x,y)$ for all $i,j \in \mathbb{Z}$. Since $f(x,y)$ is $\boldsymbol{\varepsilon}_y$-free, it follows that $a_i \neq a_j$ as long as $i \neq j$. Without loss of generality, we may assume that $a_1 < a_2 < \cdots < a_r$. Thus, $p(x,y) = p_1(q^{a_1}x, q^{b_1}y)$ satisfies the condition (4.3). □

We are now ready to give a criterion for the existence of qZ-pairs.

Theorem 4.4. *Let $T(n,k)$ be a bivariate q-hypergeometric term that has a q-NR (r,s,u,v) with respect to K such that v is $\boldsymbol{\varepsilon}_y$-free. Then $T(n,k)$ has a qZ-pair if and only if v is a q-proper polynomial.*

Proof. Because of Corollary 3.6, it suffices to show that if $T(n,k)$ has a qZ-pair, then it is q-proper. To this end, we assume that $T(n,k)$ is a bivariate q-hypergeometric term. Moreover, we assume that $T(n,k)$ is not q-proper, but it has a qZ-pair. We proceed to find a contradiction.

Clearly, for a difference operator $L \in \mathbb{F}[q^n, N]$, we have

$$(N \cdot L)T(n,k) = (K-1)G(n,k) \Longleftrightarrow LT(n,k) = (K-1)G(n-1,k).$$

Therefore, we may assume that $T(n,k)$ has a qZ-pair (L,G) of the form

$$L = \sum_{i=0}^{I} a_i(q^n) N^i,$$

where $a_i(q^n)$ are polynomials in q^n and $a_0 \neq 0$. Since LT/T and $(K-1)G/G$ are both rational functions of q^n and q^k, we may assume that

$$G(n,k)=\frac{f(q^n,q^k)}{g(q^n,q^k)}T(n,k),$$

where $f,g\in\mathbb{F}[x,y]$ are two relatively prime polynomials.

By the definition of qZ-pairs, we have

$$\sum_{i=0}^{I}a_i(q^n)\frac{T(n+i,k)}{T(n,k)}=\frac{f(q^n,q^{k+1})}{g(q^n,q^{k+1})}\frac{T(n,k+1)}{T(n,k)}-\frac{f(q^n,q^k)}{g(q^n,q^k)}. \tag{4.4}$$

Substituting (4.2) into (4.4), we obtain

$$\sum_{i=0}^{I}a_i(x)\frac{u(q^ix,y)}{v(q^ix,y)}\frac{w_1^{(i)}(x,y)}{w_2^{(i)}(x,y)}=\frac{f(x,qy)}{g(x,qy)}\frac{r(x,y)}{s(x,y)}\frac{u(x,qy)}{v(x,qy)}-\frac{f(x,y)}{g(x,y)}\frac{u(x,y)}{v(x,y)}. \tag{4.5}$$

Let $u_1=u/\gcd(u,g),\ g_1=g/\gcd(u,g)$. Multiplying

$$g_1(x,qy)g_1(x,y)v(x,qy)s(x,y)\prod_{j=0}^{I}v(q^jx,y)w_2^{(j)}(x,y)$$

to both sides of (4.5), we arrive at

$$\begin{aligned}&g_1(x,qy)g_1(x,y)v(x,qy)s(x,y)\\&\qquad\times\sum_{i=0}^{I}a_i(x)u(q^ix,y)w_1^{(i)}(x,y)\prod_{j\neq i}v(q^jx,y)w_2^{(j)}(x,y)\\&\quad=f(x,qy)r(x,y)u_1(x,qy)g_1(x,y)\prod_{j=0}^{I}v(q^jx,y)w_2^{(j)}(x,y)\\&\qquad-f(x,y)u_1(x,y)g_1(x,qy)v(x,qy)s(x,y)w_2^{(0)}(x,y)\\&\qquad\times\prod_{j=1}^{I}v(q^jx,y)w_2^{(j)}(x,y).\end{aligned} \tag{4.6}$$

Since $T(n,k)$ is not q-proper, from Lemma 4.3 it follows that there exists an irreducible factor p of v satisfying the condition (4.3). Noting that $p(x,y)$ divides each term of the left-hand side of (4.6) except for the first term, we obtain that $p(x,y)$ divides

$$\begin{aligned}&g_1(x,qy)v(x,qy)s(x,y)\prod_{j=1}^{I}v(q^jx,y)w_2^{(j)}(x,y)\\&\quad\times\big(g_1(x,y)a_0(x)u(x,y)w_1^{(0)}(x,y)+f(x,y)u_1(x,y)w_2^{(0)}(x,y)\big).\end{aligned}$$

From (4.3) it follows that

$$p(x,y)\perp v(x,qy)\prod_{j=1}^{I}v(q^jx,y).$$

Since s and $w_2^{(j)}$ are q-proper, they are also relatively prime to p. This implies that $p(x, y)$ divides

$$g_1(x, qy)\big(g_1(x, y)a_0(x)u(x, y)w_1^{(0)}(x, y) + f(x, y)u_1(x, y)w_2^{(0)}(x, y)\big). \tag{4.7}$$

Similarly, since $p(x, qy)$ divides both sides of (4.6) and $u \perp v$, we have

$$p(x, qy) \mid f(x, qy)g_1(x, y). \tag{4.8}$$

Case 1. Suppose $p(x, qy) \mid f(x, qy)$. Since $p(x, y)$ divides (4.7), it follows that

$$p(x, y) \mid g_1(x, qy)g_1(x, y)a_0(x)u(x, y)w_1^{(0)}(x, y).$$

Since $f \perp g$, $u \perp v$, a_0 and $w_1^{(0)}$ are q-proper polynomials, we may deduce that $p(x, y) \mid g_1(x, qy)$, i.e., $p(x, q^{-1}y) \mid g_1(x, y)$. Let $m(> 0)$ be the greatest integer such that $p(x, q^{-m}y) \mid g_1(x, y)$. By virtue of (4.6), we have that $p(x, q^{-m}y)$ divides

$$f(x, y)u_1(x, y)g_1(x, qy)v(x, qy)s(x, y)w_2^{(0)}(x, y) \times \prod_{j=1}^{I} v(q^j x, y)w_2^{(j)}(x, y).$$

However, $f \perp g$ and $g_1 \perp u_1$ imply that $p(x, q^{-m}y) \mid g_1(x, qy)$, which contradicts the choice of m.

Case 2. Suppose $p(x, qy) \mid g_1(x, y)$. Let $M > 0$ be the greatest integer such that $p(x, q^M y) \mid g_1(x, y)$. Similarly, from (4.6) it follows that $p(x, q^{M+1}y)$ divides

$$f(x, qy)r(x, y)u_1(x, qy)g_1(x, y) \prod_{j=0}^{I} v(q^j x, y)w_2^{(j)}(x, y).$$

Hence we get $p(x, q^{M+1}y) \mid g_1(x, y)$, which is again a contradiction. □

To extend the above result to general bivariate q-hypergeometric terms, we need the concept of similar q-hypergeometric terms. Two bivariate q-hypergeometric terms T_1, T_2 are called *similar* if there exists a rational function $R \in \mathbb{F}(x, y)$ such that $T_1(n, k)/T_2(n, k) = R(q^n, q^k)$.

As in the ordinary case, the existence of qZ-pairs is preserved under the addition of similar bivariate q-hypergeometric terms.

Lemma 4.5. *Suppose there exist qZ-pairs for two similar bivariate q-hypergeometric terms $T_1(n, k)$ and $T_2(n, k)$. Then there exists a qZ-pair for $T(n, k) = T_1(n, k) + T_2(n, k)$.*

Notice that $T(n, k) = (K - 1)G(n, k)$ has a qZ-pair $(1, G)$. Combining Theorem 4.4 and Lemma 4.5, we obtain the main result of this paper.

Theorem 4.6. *Let $T(n, k)$ be a bivariate q-hypergeometric term. Let T_1, T_2 be two similar bivariate q-hypergeometric terms satisfying*

$$T(n, k) = (K - 1)T_1(n, k) + T_2(n, k)$$

and $T_2(n,k)$ have a q-NR (r,s,u,v) with respect to K such that v is $\boldsymbol{\varepsilon}_y$-free. Then $T(n,k)$ has a qZ-pair if and only if $T_2(n,k)$ is a q-proper hypergeometric term, or equivalently, if and only if $v(x,y)$ is a q-proper polynomial.

5. Algorithms

Let $T(n,k)$ be a bivariate q-hypergeometric term. By the algorithm "q-RNF", we may find a q-NR (r,s,u,v) of $T(n,k)$ with respect to K. Let

$$F(k) = \frac{u(x,q^k)}{v(x,q^k)} \prod_{j=0}^{k-1} \frac{r(x,q^j)}{s(x,q^j)}, \quad \forall k \in \mathbb{N}.$$

Then $F(k)$ is a univariate q-hypergeometric term over the field $\mathbb{F}(x)$ with a q-MR $(r/s, u/v, 0)$. On the other hand, by Eq. (4.1), we have

$$\frac{F(k)|_{x=q^{n+1}}}{F(k)|_{x=q^n}} = \frac{u(q^{n+1},q^k)v(q^n,q^k)}{u(q^n,q^k)v(q^{n+1},q^k)} \prod_{j=0}^{k-1} \frac{r(q^{n+1},q^j)s(q^n,q^j)}{r(q^n,q^j)s(q^{n+1},q^j)}$$
$$= \frac{T(n+1,k)}{T(n,k)} \cdot \frac{T(n,0)}{T(n+1,0)} \cdot \frac{u(q^{n+1},q^0)v(q^n,q^0)}{u(q^n,q^0)v(q^{n+1},q^0)},$$

which is also a rational function of q^n and q^k. Hence $\widetilde{F}(n,k) = F(k)|_{x=q^n}$ is a bivariate q-hypergeometric term.

Using the algorithm "q-decomp" given in Section 2, one may find univariate q-hypergeometric terms $F_1(k)$, $F_2(k)$ such that

$$F(k) = (K-1)F_1(k) + F_2(k)$$

and $F_2(k)$ has a q-MR $(f_1/f_2, v_1/v_2, 0)$ with v_2 being $\boldsymbol{\varepsilon}_y$-free. Since $f_1/f_2, v_1/v_2 \in \mathbb{F}(x)(y)$, we may assume that $f_1, f_2, v_1, v_2 \in \mathbb{F}[x,y]$ and $f_1 \perp f_2$, $v_1 \perp v_2$. From the fact that r/s is $\boldsymbol{\varepsilon}_y$-reduced, it follows that f_1/f_2 is also $\boldsymbol{\varepsilon}_y$-reduced.

Let

$$T_1(n,k) = T(n,0)\frac{v(q^n,q^0)}{u(q^n,q^0)} \cdot F_1(k)|_{x=q^n},$$
$$T_2(n,k) = T(n,0)\frac{v(q^n,q^0)}{u(q^n,q^0)} \cdot F_2(k)|_{x=q^n}.$$

Since Eq. (2.2) implies that

$$F_1(k) = \frac{U_1}{u/v} \cdot F(k) \quad \text{and} \quad F_2(k) = \frac{v_1/v_2}{u/v} \cdot F(k),$$

it follows that $T_1(n,k)$ and $T_2(n,k)$ are similar bivariate q-hypergeometric terms. It is easily verified that

$$T(n,k) = (K-1)T_1(n,k) + T_2(n,k)$$

and (f_1, f_2, v_1, v_2) is a q-NR of T_2 with respect to K. Therefore, Theorem 4.6 implies that $T(n,k)$ has a qZ-pair if and only if v_2 is a q-proper polynomial.

Finally, we need the algorithm given by Abramov and Le (2002) for determining whether or not a polynomial is q-proper.

We are now ready to describe the algorithm to determine whether a bivariate q-hypergeometric term $T(n,k)$ has a qZ-pair.

1. Apply the algorithm in Böing and Koepf (1999) to find a rational function $R \in \mathbb{F}(x,y)$ such that

$$\frac{T(n,k+1)}{T(n,k)} = R(q^n, q^k).$$

2. Find a q-RNF (r,s,u,v) with respect to $\boldsymbol{\varepsilon}_y$ of R.
3. For $D = r/s, U = u/v$ and $n_0 = 0$, apply the algorithm 'q-decomp' with respect to $\boldsymbol{\varepsilon}_y$ to get $V = v_1/v_2$.
4. Use the algorithm in Abramov and Le (2002) to determine whether v_2 is q-proper. If the answer is yes, then T has a qZ-pair; otherwise, T does not have any qZ-pair.

Here are two examples.

Example 1. Let

$$T(n,k) = \frac{q^k(1+q^{n+1}+q^{k+2})}{(q^n+q^k+1)(q^n+q^{k+1}+1)\prod_{j=1}^{k+1}(1-q^j)}.$$

Then

$$\frac{T(n,k+1)}{T(n,k)} = \frac{q(1+q^{n+1}+q^{k+3})(q^n+q^k+1)}{(q^n+q^{k+2}+1)(1+q^{n+1}+q^{k+2})(1-q^{k+2})},$$

and we have

$$r = q,\ s = 1-q^2y,\ u = 1+qx+q^2y,\ v = (x+y+1)(x+qy+1)$$

is a q-NR of T with respect to K. For $D = r/s, U = u/v$ and $n_0 = 0$, applying the algorithm "q-decomp", we get

$$V = v_1/v_2 = \frac{-q^2}{(-1+q^2)(x+1)}.$$

Clearly, v_2 is q-proper, so $T(n,k)$ has a qZ-pair. Indeed, we can check that

$$L = 1, \quad G = \frac{1}{(q^n+q^k+1)\prod_{j=1}^{k}(1-q^j)}$$

is a qZ-pair for $T(n,k)$.

Example 2.

$$T(n,k) = \frac{q^k(1+q^{n+1}+q^{k+2})}{(q^n+q^k+1)(q^n+q^{k+1}+1)\prod_{j=1}^{k}(1-q^j)}.$$

Then

$$\frac{T(n,k+1)}{T(n,k)} = \frac{q(1+q^{n+1}+q^{k+3})(q^n+q^k+1)}{(q^n+q^{k+2}+1)(1+q^{n+1}+q^{k+2})(1-q^{k+1})},$$

and we have

$$r=q,\ s=1-qy,\ u=1+qx+q^2y,\ v=(x+y+1)(x+qy+1)$$

is a q-NR of T with respect to K. For $D = r/s, U = u/v$ and $n_0 = 0$, applying the algorithm "q-decomp", we get

$$V = v_1/v_2 = \frac{-(x+y+1)q^2}{(q-1)(x+1)(x+qy+1)}.$$

Since $x+qy+1$ is not a q-proper polynomial, it follows that $T(n,k)$ has no qZ-pair.

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