Evaluation of Some Hankel Determinants

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Abstract

We evaluate some Hankel determinants of Meixner polynomials, associated to the series $\exp(\sum \alpha[i]z^i/i)$, where $[1], [2], \ldots$ are the q-integers.

1. Introduction

Given a nonzero real number α, let $\sum_{i\geq 0} a_i z^i$ be the Taylor expansion of

$$f(z) = \exp\left(\sum_{i\geq 1} \alpha \frac{1-q^i}{1-q} \frac{z^i}{i}\right) = 1 + z\alpha + \frac{z^2}{2}(q+\alpha+1)\alpha + \cdots . \tag{1.1}$$

By using symmetric functions, we evaluate two families of Hankel determinants:

$$H_n^1 = \det(a_{i+j-1})_{1\leq i,j\leq n} \quad \text{and} \quad H_n^2 = \det(a_{i+j})_{0\leq i,j\leq n} . \tag{1.2}$$

What is remarkable is that H_n^1 factors into linear factors of type $(\alpha - r + rq)$ and that H_n^2/H_n^1 is a polynomial in q and α, though the coefficients of the Taylor expansion of $f(z)$ are rather complicated.

The explicit forms of H_n^1 and H_n^2 are given by the following theorem, which is our main result.

Theorem 1.1 *We have*

$$H_n^1 = w_n \prod_{n>|r|} (\alpha - r + rq)^{n-|r|}\,, \tag{1.3}$$

and

$$H_n^2/H_n^1 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \prod_{i=0}^{k-1} \frac{\alpha + (1-q)(n-i)}{2n-i}\,, \tag{1.4}$$

with

$$w_n = \frac{(-1)^{\binom{n}{2}}}{1 \cdot 2^2 \cdot 3^3 \cdots n^n (n+1)^{n-1} \cdots (2n-1)^1}\,. \tag{1.5}$$

2. Symmetric Functions

Many properties of Hankel determinants are better understood when identifying Hankel determinants with Schur functions indexed by rectangular partitions [4]. We first shall recall a few facts about symmetric functions. We follow the convention of [4] rather than [5]. The basic symmetric functions will be the power sums $p_i, i = 1, 2, \ldots$, and the argument of a symmetric function will be a rational function with coefficients in $\mathbb{C}$, of two types of variables: variables inside a box and variables without a box. We define

$$p_i(\alpha) = \alpha \quad \text{and} \quad p_i\left(\boxed{x}\right) = \boxed{x}^{\,i},$$

and require that the p_i be $\mathbb{C}$-ring homomorphisms satisfying

$$\begin{cases} p_i(\alpha A + \beta B) = \alpha p_i(A) + \beta p_i(B) \\ p_i(AB) = p_i(A) p_i(B) \\ p_i(p_j(A)) = p_{ij}(A). \end{cases}$$

Notice that

$$p_i\left(3\,\boxed{2}\right) = 3\,\boxed{2}^{\,i} = p_i\left(\boxed{2}+\boxed{2}+\boxed{2}\right),$$

but that

$$p_i\left(\boxed{2}\right) = \boxed{2}^{\,i} \neq p_i\left(\boxed{1}+\boxed{1}\right) = \boxed{1}^{\,i} + \boxed{1}^{\,i}.$$

One cannot expand the content of a box, inside the argument of a symmetric function. Symmetric functions are considered here just as a tool to write algebraic expressions, in a simpler manner than having to manipulate coefficients of different generating functions. Once they are evaluated in terms of their arguments, we can, of course, erase boxes.

The usual power sums are recovered when taking as an argument a sum of letters in boxes:

$$p_i\left(\boxed{a}+\boxed{b}+\boxed{c}+\cdots\right) = \boxed{a}^{\,i} + \boxed{b}^{\,i} + \boxed{c}^{\,i} + \cdots .$$

By abuse of language, we shall still call the argument A of a symmetric function an *alphabet*.

The complete symmetric functions of A, $S^n(A)$, are defined with the help of the following generating function

$$\sigma_z(A) = \sum_{n\geq 0} z^n S^n(A) = \exp\left(\sum_{i=1}^{\infty} z^i p_i(A)/i\right).$$

This definition implies in particular that

$$S^n(A+B) = \sum_{i=0}^{n} S^i(A) S^{n-i}(B). \tag{2.1}$$

For any $n \in \mathbb{N}$ and $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, the *Schur function* $S_I(A)$ has a determinantal expression [5, Section 1.3] :

$$S_I(A) = \left|S^{i_s+s-r}(A)\right|_{1\leq r,s\leq n}.$$

If I and J are two conjugate partitions, then [4, p. 8]

$$S_I(A) = (-1)^{|J|} S_J(-A).$$

The determinantal expressions of Schur functions occur in many fields of mathematics. For example, the notation $S_{n^n}(A - \boxed{x})$ encodes the classical determinantal expressions of orthogonal polynomials in terms of moments (cf. [1, p. 273], [4, p.117]):

$$S_{3^3}\left(A - \boxed{x}\right) = \begin{vmatrix} S^3(A - \boxed{x}) & S^4(A - \boxed{x}) & S^5(A - \boxed{x}) \\ S^2(A - \boxed{x}) & S^3(A - \boxed{x}) & S^4(A - \boxed{x}) \\ S^1(A - \boxed{x}) & S^2(A - \boxed{x}) & S^3(A - \boxed{x}) \end{vmatrix}$$

$$= \begin{vmatrix} S^3(A) & S^4(A) & S^5(A) & x^3 \\ S^2(A) & S^3(A) & S^4(A) & x^2 \\ S^1(A) & S^2(A) & S^3(A) & x^1 \\ S^0(A) & S^1(A) & S^2(A) & x^0 \end{vmatrix} .$$

More generally, given a non negative integer k and a generic alphabet A (by generic, we mean that for any $n \geq 0$, $S_{(n+k-1)^n}(A) \neq 0$), let for any $n \in \mathbb{N}$,

$$P_n(x) := S_{(n+k)^n}\left(A - \boxed{x}\right) = S_{(n+k)^n}(A) - x\, S_{n+k-1,(n+k)^{n-1}}(A)$$
$$+ x^2 S_{(n+k-1)^2,(n+k)^{n-2}}(A) + \cdots + (-1)^n x^n S_{(n+k-1)^n}(A).$$

Then $\{P_0(x), P_1(x), \ldots\}$ is a family of orthogonal polynomials associated to the functional $x^n \to \int x^n = S^{n+k}(A)$.

Notice that we have letters in boxes only as arguments of symmetric functions.

3. A transformation of alphabets

Given an alphabet A, we define a new alphabet B by the following equality of generating functions:

$$\sigma_z(B) = \sigma_{z/(1+z)}(A). \tag{3.1}$$

We have the equivalence

$$\sigma_z(B) = \sigma_{z/(1+z)}(A) \Longleftrightarrow \sigma_z(-B) = \sigma_{z/(1+z)}(-A),$$

and, by expansion of the generating functions,

$$\begin{cases} S^n(B) = \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}S^{n-i}(A) = S^n(A-(n-1)), \\ S^n(A) = \sum_{i=0}^{n-1}\binom{n-1}{i}S^{n-i}(B) = S^n(B-(n-1)\boxed{-1}). \end{cases} \tag{3.2}$$

This transformation has been used by Grothendieck to define Chern classes in K-theory (cf. [4, Exercise 2.13]).

By row and column manipulations, we can easily transform the determinant $S_{n^n}(A)$ into $S_{n^n}(B)$, i.e., one has

$$S_{n^n}(A) = S_{n^n}(B) = (-1)^{n^2}S_{n^n}(-B) = (-1)^{n^2}S_{n^n}(-A). \tag{3.3}$$

Furthermore, we have

Lemma 3.1

$$S_{(n+1)^n}\left(-A-\boxed{x{+}1}\right) = S_{(n+1)^n}\left(-B-\boxed{x}\right). \tag{3.4}$$

Proof. Define a linear functional $\int$ acting on the ring of polynomials in x by

$$\int x^n = S^{n+1}(-B), \quad \forall\, n \geq 0. \tag{3.5}$$

Then

$$\int (x+1)^n = \sum_{i=0}^{n}\binom{n}{i}S^{n-i+1}(-B) = S^{n+1}(-A)\,. \tag{3.6}$$

From (3.5), we have that $S_{(n+1)^n}(-B-\boxed{x})$ are orthogonal polynomials associated to $\int$. Similarly, (3.6) implies that $S_{(n+1)^n}(-A-\boxed{x{+}1})$ are also orthogonal polynomials associated to $\int$. Since their leading coefficients coincide, (3.4) holds. ∎

Corollary 3.2 *Let A and B be alphabets such that $\sigma_z(B) = \sigma_{z/(1+z)}(A)$. Then*

$$S_{(n+1)^n}(-A) = S_{n^{n+1}}(A) = S_{n^{n+1}}\left(B+\boxed{-1}\right) = \sum_{i=0}^{n}(-1)^{n-i}S_{i,n^n}(B).$$

4. Evaluation of Hankel Determinants

Given a nonzero real number a, let $\mathbb{A}$ be the alphabet defined by the generating function

$$\sigma_z(\mathbb{A}) = \exp\left(\sum_{i\geq 1} a\frac{1-q^i}{1-q}\frac{z^i}{i}\right). \tag{4.1}$$

We shall now evaluate the determinants $S_{n^n}(\mathbb{A})$ and $S_{n^{n+1}}(\mathbb{A})$ (these are the determinants which are needed for writing $\sigma_z(\mathbb{A})$ as a continued fraction).

Noticing that

$$\sum_{i\geq 1}\frac{1-q^i}{1-q}\frac{z^i}{i} = \frac{1}{1-q}\sum_{i\geq 1}\left(\frac{z^i}{i} - \frac{(qz)^i}{i}\right) = \frac{1}{1-q}\log\left(\frac{1-qz}{1-z}\right),$$

and taking the parameter $b = a/(1-q)$ instead of a, we derive that

$$\sigma_z(\mathbb{A}) = \left(\frac{1-qz}{1-z}\right)^b = \sigma_z\left(b\left(1-\boxed{q}\right)\right).$$

Let $\mathbb{B} = -b\boxed{q-1}$. Then

$$\sigma_{z/(1+z)}(\mathbb{A}) = \big(1-(q-1)z\big)^b = \sigma_z(\mathbb{B}).$$

and for any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$

$$S_\lambda(\mathbb{B}) = (q-1)^{\sum\lambda_i}\prod_{(i,j)\in\lambda}\frac{-b+c(i,j)}{h(i,j)},$$

where $h(i,j)$ and $c(i,j)$ are respectively the hook length and the content of the box (i,j) in the diagram of λ (cf. [5, p. 28]).

Hence,

$$\begin{aligned} S_{n^n}(\mathbb{A}) = S_{n^n}(\mathbb{B}) &= (q-1)^{n^n}\prod_{(i,j)\in n^n}\frac{-b+c(i,j)}{h(i,j)} \\ &= \prod_{(i,j)\in n^n}\frac{a+(q-1)c(i,j)}{h(i,j)}. \end{aligned} \tag{4.2}$$

For example,

$$S_{3^3}(\mathbb{A}) = (q-1)^9 \boxed{\begin{matrix} -b & -b+1 & -b+2 \\ -b-1 & -b & -b+1 \\ -b-2 & -b-1 & -b \end{matrix}} \Big/ \boxed{\begin{matrix} 5 & 4 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{matrix}},$$

(taking the product of all the elements in the arrays).

To evaluate $S_{n^{n+1}}(\mathbb{A})$, we apply Corollary 3.2 and get

$$S_{n^{n+1}}(\mathbb{A}) = \sum_{k=0}^{n} (-1)^{n-k} S_{k,n^n}(\mathbb{B})$$
$$= \sum_{k=0}^{n} (-1)^{n-k} (q-1)^{n^n+k} \prod_{(i,j)\in(k,\,n^n)} \frac{-b+c(i,j)}{h(i,j)}.$$

Therefore,

$$\begin{aligned} S_{n^{n+1}}(\mathbb{A})/S_{n^n}(\mathbb{A}) &= (-1)^n \sum_{k=0}^{n} (1-q)^k \prod_{i=1}^{k} \frac{-b+(-n+i-1)}{i} \cdot \frac{n-k+i}{2n-k+i} \\ &= (-1)^n \frac{(n!)^2}{(2n)!} \sum_{k=0}^{n} (-1)^k (1-q)^k \binom{b+n}{k} \binom{2n-k}{n-k}. \end{aligned}$$

Hence,

$$\begin{aligned} S_{n^{n+1}}(\mathbb{A})/S_{n^n}(\mathbb{A}) &= (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \prod_{i=0}^{k-1} \frac{a+(1-q)(n-i)}{2n-i} && (4.3) \\ &= (-1)^n \frac{(n!)^2}{(2n)!} S^n\Big((n+1) - (b+n)\boxed{1-q}\Big). && (4.4) \end{aligned}$$

One can write other expressions for (4.4). In terms of hypergeometric function, it reads

$$(-1)^n \; {}_2F_1\left(\begin{matrix} -b-n, -n \\ -2n \end{matrix} \; ; 1-q \right). \qquad (4.5)$$

By (3.1) and (3.2), we have

$$S^n\left((n+1)-(b+n)\boxed{1-q}\right) = S^n\left((b-n)\boxed{-1}-(b+n)\boxed{-q}\right)$$
$$= (-1)^n S^n\left((b-n)-(b+n)\boxed{q}\right). \quad (4.6)$$

For example,

$$S^2\left(3-(b+2)\boxed{1-q}\right) = S^2(3) - 3(b+2)\boxed{1-q} + \binom{b+2}{2}\boxed{1-q}^2$$
$$= S^2\left((b-2)-(b+2)\boxed{q}\right) = \binom{b-1}{2} - (b-2)(b+2)\boxed{q} + \binom{b+2}{2}\boxed{q}^2,$$

and thus

$$\binom{4}{2} - 3(b+2)(1-q) + \binom{b+2}{2}(1-q)^2$$
$$= \binom{b-1}{2} - (b-2)(b+2)q + \binom{b+2}{2}q^2.$$

The referee has given a third expression

$$\sum_{k=0}^{n}(-1)^k \frac{b\cdots(b{+}n)}{(b+k)k!(n-k)!}\frac{(n+k)!}{k!}\sum_{i=0}^{n-k}\frac{\binom{n}{i}\binom{n-k}{i}q^i}{\binom{i+k}{k}}, \quad (4.7)$$

which is no other than the expression of

$$f(b) = (-1)^n n!\, S^n\left((b-n)-(b+n)\boxed{q}\right)$$

in terms of its values at $b = 0, -1, \ldots, -n$.

Recall (cf. [2], see also [1, 3]) that the Meixner polynomials $M_k(x;\beta,c)$ have generating function

$$\sum_{k=0}^{\infty}\frac{M_k(x;\beta,c)}{k!}z^k = (1-z)^{-x-\beta}(1-z/c)^x.$$

That is,

$$M_n(x;\beta,c) = n!\, S^n\left(x+\beta-x\boxed{1/c}\right). \quad (4.8)$$

The referee pointed out that our coefficients $S^n(\mathbb{A})$ are equal to

$$\frac{a}{n!}\, M_{n-1}\left(\frac{a}{1-q} - 1; 2, 1/q\right) = \frac{a}{n}\, S^{n-1}\left((b+1) - (b-1)\boxed{q}\right) \ ,$$

and that the value of $S_{n^n}(\mathbb{A})$ results from the fact that Meixner polynomials are moments associated to Jacobi polynomials (cf. [2, 3]).

But from (4.8), we recognize that (4.4) and (4.6) themselves can be seen as specialization of Meixner polynomials:

$$S^n\left((n+1) - (b+n)\boxed{1-q}\right) = \frac{1}{n!}\, M_n(n+b; 1-b, 1/(1-q)),$$
$$S^n\left((b-n) - (b+n)\boxed{q}\right) = \frac{1}{n!}\, M_n(n+b; -2n, 1/q).$$

In fact, one can also obtain (4.6) as a specialization of a Jacobi polynomial. The homogeneous version $\widehat{P}^{\alpha,\beta}(\xi_1, \xi_2)$ of a Jacobi polynomial is (with $\xi_1 = (x+1)/2$ and $\xi_2 = (x-1)/2$ we get the usual Jacobi polynomial):

$$\widehat{P}^{\alpha,\beta}(\xi_1, \xi_2) = (-1)^n S^n\left(-(n+\alpha)\boxed{\xi_1} - (n+\beta)\boxed{\xi_2}\right).$$

Therefore, (4.6) can be written as $\widehat{P}^{-b,b}(1, q)$.

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