Lattice paths and $(n-2)$-stack sortable permutations

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Abstract. We establish a bijection between the $(n-2)$-stack sortable permutations and the labeled lattice paths. Using this bijection, we directly give combinatorial proof for the log-concavity of the numbers of $(n-2)$-stack sortable permutations with k descents. Furthermore, we prove the the numbers of $(n-2)$-stack sortable permutations with k descents satisfy interlacing log-concavity. We also consider a conjecture proposed by Bóna that the sequences of the descents of t-stack sortable permutations of $[n]$ are Hilbert functions for any t and n. We prove this conjecture for $t = n-2$.

Keywords: $(n-2)$-stack sortable permutations, simplicial complex, log-concave, lattice path

AMS Classification: 05A20

1 Introduction

The objectives of this paper are to prove the interlacing log-concavity of the descent statistic of the $(n-2)$-stack sortable permutations and construct a simplicial complex $\triangle$ whose f-vector is the sequences of the descent of the $(n-2)$-stack sortable permutations.

Let S_n denote the set of permutations on $[n] := \{1, 2, \ldots, n\}$ and suppose $\pi = \pi_1\pi_2 \cdots \pi_n$ is a permutation in S_n. The stack-sorting operation s can be defined on the set of all n-permutations as follows. Let $\pi = LnR$ be an n-permutation, with L and R denoting its substring before and after the maximal entry n, respectively. Let $s(\pi) = s(L)s(R)n$, where L and R are defined recursively by the same rule. A permutation π is called t-stack sortable if $s^t(\pi)$ is the identity permutation.

Let $W_t(n)$ denote the number of t-stack sortable permutations of length n. The study of stack-sorting problem is a major area of research, it began with Knuth's analysis [22], who proved that $W_1(n)$ is the Catalan number $C_n = \binom{2n}{n}/(n+1)$. West [25] studied thoroughly this procedure and conjectured that $W_2(n)$ is $2(3n)!/((n+1)!(2n+1)!)$. This conjecture was first proved by Zeilberger [28]. Other proofs can be found in [17], [20] and [21]. Duchi, Guerrini, Rinaldi [15] and Fang [16] gave different proofs that new combinatorial objects called "fighting fish" are counted by the numbers $W_2(n+1)$. In 2017, Defant [10] introduced "valid hook configurations" to count the cardinality of preimages of permutations under the stack sorting map. This approach further allowed Defant to generalize existing theorems about the stack-sorting map and prove new results, see [10] and [11]. Defant, Engen and Miller [12] showed that valid hook configurations of length n permutations are in bijective correspondence with certain weighted set partitions. Valid hook configurations and a generalization of stack sorting are also used to prove some results about free cumulants and classical cumulants involving colored binary plane trees [13].

There is very little known about t-stack-sortable permutations for $t \geq 3$. Úlfarsson [26] characterized 3-stack-sortable permutations in terms of new "decorated patterns". Albert, Bouvel and Féray[1] showed that for every $t \geq 1$, the set of t-stack-sortable permutations can be described by a sentence in a first-order logical theory which was called ToTo. Recently, Defant [14] gave a new proof of the Zeilberger's formula for the number $W_2(n)$ and counted 2-stack-sortable permutations according to different statistics. Furthermore, Defant also obtained a recurrence relation for $W_3(n)$.

One of the most important permutation statistics is that of the number of descents. A descent of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is an index $i \in \{2, 3, \ldots, n-1\}$ such that $\pi_i > \pi_{i+1}$. Let $W_t(n, k)$ be the number of t-stack sortable permutations with k descents, and let

$$W_{t,n}(x) = \sum_{k=0}^{n-1} W_t(n, k)x^k.$$

When $t = n-1$ and $t = 1$, $W_{t,n}(x)$ reduced to the Eulerian polynomial and the Narayana polynomial, respectively. It is well known that they have only real zeros. Thus Bóna [3] raised the question if this is true for general t and proved that for any fixed n and t, the numbers $\{W_t(n, k)\}_{k=0}^{n-1}$ form a unimodal sequence. By using certain real-rootedness preserving linear operator, Brändén [8] proved the real-rootedness for $t = n-2$. Furthermore, Brändén obtained the real-rootedness of the polynomial $A_n(x) + kxA_{n-2}(x)$, where $A_n(x)$ is the Eulerain polynomials and $k > -2$ is a real number. Zhang [27] gave another proof of the above result by using the theory of s-Eulerian polynomials.

In Section 2, we first consider the log-concavity and the interlacing log-concavity of the descent statistic of the $(n-2)$-stack sortable permutations. Recall that a sequence $\{a_n\}_{n\geq 0}$ of real positive numbers is said to be log-concave if

$$a_n^2 \geq a_{n+1}a_{n-1} \tag{1.1}$$

holds for all $n \geq 1$. If a sequence $\{a_n\}_{n\geq 0}$ which has a combinatorial meaning is log-concave, then it would be ideal to provide a combinatorial proof, see [6, 7, 23] for some techniques that are used to prove the log-concavity of sequences. Chen, Wang and Xia

[9] gave the definition of interlacing log-concavity as follow. Let $\{P_m(x)\}$ be a sequence of polynomials, where

$$P_m(x) = \sum_{i=0}^{m} a_i(m)x^m$$

is a polynomial of degree m. Let

$$r_i(m) = \frac{a_i(m)}{a_{i+1}(m)}.$$

We say that the polynomials $P_m(x)$ $(m \geq 0)$ are interlacingly log-concave if the ratios $r_i(m)$ interlace the ratios $r_i(m+1)$, that is,

$$r_0(m+1) \leq r_0(m) \leq r_1(m+1) \leq r_1(m) \leq \cdots \leq r_{m-1}(m+1) \leq r_{m-1}(m) \leq r_m(m+1).$$

Note that interlacing log-concavity is stronger than log-concavity. Chen, Wang and Xia [9] proved the Boros-Moll polynomials are interlacingly log-concave and gave a criterion for the interlacing log-concavity of the polynomials whose coefficients satisfying certain three term recurrence relations. As consequences, the interlacing log-concavity of the second kind of Stirling numbers, the Narayana numbers and the Whitney numbers are immediate. In a previous paper, the authors [19] proved the interlacing log-concavity of the Brenti's derangement polynomials and the Eulerian polynomials by a directly combinatorial injection. By a similar argument, we shall establish a bijection between the $(n-2)$-stack sortable permutations and the labeled lattice paths. Applying this construction, we give a combinatorial proof of the log-concavity and the interlacing log-concavity of the sequences $\{W_{n-2}(n,k)\}_{0\leq k\leq n-1}$.

In Section 3, we shall prove that for $n \geq 1$, the sequences $\{W_{n-2}(n,k)\}_{0\leq k\leq n-1}$ are Hilbert function. Recall that a simplicial complex is a collection of sets $\triangle$ with the property that if $A \in \triangle$ and $B \subseteq A$ then $B \in \triangle$. We call the elements of $\triangle$ the faces of $\triangle$. For $S \in \triangle$, the dimension of S is $|S|-1$. The dimension of $\triangle$ is $\dim(\triangle) \overset{\text{def}}{=} \{|A|-1 : A \in \triangle\}$. Given a simplicial complex $\triangle$ of dimension $d-1$, we define

$$f_{i-1}(\triangle) \overset{\text{def}}{=} |\{A \in \triangle : |A| = i\}|,$$

for $i = 0, 1, \ldots, d$, and call $\mathbf{f}(\triangle) \overset{\text{def}}{=} (f_0(\triangle), f_1(\triangle), \ldots, f_{d-1}(\triangle),)$ the f-vector of $\triangle$. It is known [24] that if $(f_0, f_1, \ldots, f_{d-1})$ is the f-vector of a simplicial complex then $\{1, f_0, \ldots, f_{d-1}\}$ is a Hilbert function.

Denote by $[n]$ the set $\{1, 2, \ldots, n\}$. Gasharov [18] proved that there exists a simplicial complex whose $(k-1)$-dimensional faces correspond to permutations of $[n]$ with k descents. Bóna [4] constructed a simplicial complex whose $(k-1)$-dimensional faces correspond to t-stack sortable permutations with k descents for $t = 1$ and $t = 2$ and proposed a conjecture that $W_{n,t}(x)$ are Hilbert function for all t. In Section 3, we give an affirmative answer to this question for $t = n-2$.

2 Interlacing log-concavity of $W_{n-2}(n,k)$

In this section, we will construct a bijection between the set of $(n-2)$-stack sortable permutations and a set $\mathscr{P}'(n,k)$ of certain labeled lattice paths. Recall that a lattice path P in the plane $Z \times Z$ is a path using only steps $(1,0)$ and $(0,1)$. In [5], Bóna constructed a bijection $\Upsilon_{n,k}$ between the set $\mathscr{A}(n,k)$ of n-permutations with k descents and the set $\mathscr{P}(n,k)$ of labeled lattice paths with n edges, exactly k of which are vertical.

We briefly recall Bóna's bijection here. For each $\pi \in \mathscr{A}(n,k)$, to obtain a path $p \in \mathscr{P}(n,k)$ that have edges $a_1, a_2, \ldots a_n$ and that corresponding positive integers $e_1, e_2, \ldots, e_n$ as labels, for $2 \le i \le n$, restrict π to the i first entries and relabel the entries to obtain a permutation $\gamma = \gamma_1\gamma_2\cdots\gamma_i$ of $[i]$.

1. If the position $i-1$ is a descent of the permutation p (equivalently, of the permutation γ), then let the edge a_i be vertical and the label e_i be equal to γ_i.
2. If the position $i-1$ is an ascent of the permutation p, then let the edge a_i be horizontal and the label e_i be equal to $i+1-\gamma_i$. (See Figure 1 for an example.)

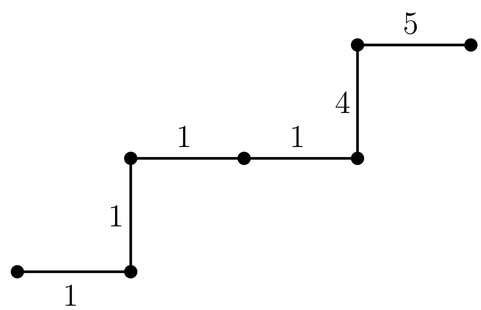

Figure 1: The path corresponding to $\pi = 314652$.

Let $\mathcal{S}(n,k)$ be the set of $(n-2)$-stack sortable permutations of length n with k descents. It is easy to check that a permutation $\pi \in \mathfrak{S}_n$ is $(n-2)$-stack sortable if and only if it is not of the form $\sigma n1$, see [2]. Therefore, the paths corresponding to $(n-2)$-stack sortable permutations of length n with k in $\mathscr{P}(n,k)$ can not have the following forms:

- a_{n-1} is horizontal and $e_{n-1} = 1$;
- a_n is vertical and $e_n = 1$.

Denote by $\mathscr{P}'(n,k)$ the subset of $\mathscr{P}(n,k)$ which is the set of labeled lattice paths corresponding to the permutations in $\mathcal{S}(n,k)$. According to the previous argument, we actually have set up a bijection between $\mathcal{S}(n,k)$ and $\mathscr{P}'(n,k)$.

Theorem 2.1 *For any positive n, the sequence $W_{n-2}(n,k)_{\{1\le k\le n-1\}}$ is log-concave, that is,*

$$W_{n-2}(n,k-1)W_{n-2}(n,k+1) \le W_{n-2}(n,k)^2. \tag{2.2}$$

Proof. According to the bijection between the set $\mathcal{S}(n,k)$ and the set $\mathscr{P}'(n,k)$, we only need to construct an injection

$$\Upsilon : \mathscr{P}'(n,k-1) \times \mathscr{P}'(n,k+1) \to \mathscr{P}'(n,k) \times \mathscr{P}'(n,k).$$

We apply the method given by Bona [5], who gave direct combinatorial proofs for the log-concavity of the Eulerian numbers, see [5]. Let $(P,Q) \in \mathscr{P}'(n,k-1) \times \mathscr{P}'(n,k+1)$. Place the initial points of P and Q at

$$u_1 = (0,0), \qquad u_2 = (1,-1),$$

respectively. Then the endpoints of P and Q are

$$v_1 = (n-k+1, k-1), \qquad v_2 = (n-k, k),$$

respectively. Thus P and Q must intersect. Let X be their first intersection point and let

$$P' = u_1 \xrightarrow{P} X \xrightarrow{Q} v_2,$$

$$Q' = u_2 \xrightarrow{Q} X \xrightarrow{P} v_1.$$

1. If P' and Q' are valid paths, that is, $(P',Q') \in \mathscr{P}'(n,k) \times \mathscr{P}'(n,k)$, then define $\Upsilon(P,Q) = (P',Q')$.
2. What remains to be done is to define $\Upsilon(P,Q)$ for those (P',Q') which are not in $\mathscr{P}'(n,k)\times\mathscr{P}'(n,k)$. In this case, X must be at their last step, and the corresponding labels are as (a) in Figure 2. Substitute (b) for (a), then it is clear that $(P',Q') \in \mathscr{P}'(n,k)\times\mathscr{P}'(n,k)$. Finally, we must show that the image of this case of the domain is disjoint from that of the previous part. This is true because in this case P' and Q' must not intersect where all the elements of the image of the previous part do not have the property.

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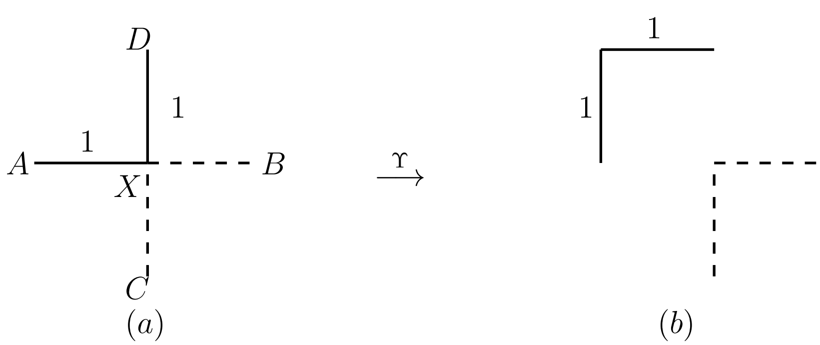

Figure 2: Labels and the changed labels around the point X.

Now let us consider the interlacing log-concavity of the sequences $\{W_{n-2}(n,k)\}_{0\le k\le n-1}$. Notice that the definition of interlacing log-concavity is equal to the following two inequalities

$$a_i(n)a_{i+1}(n+1) > a_{i+1}(n)a_i(n+1) \tag{2.3}$$

and

$$a_i(n)a_i(n+1) > a_{i-1}(n)a_{i+1}(n+1). \tag{2.4}$$

Thus we only need to prove the following theorem.

Theorem 2.2 *For $n \ge 1$ and $k \ge 0$, we have*

$$W_{n-2}(n+1,k)W_{n-2}(n,k+1) - W_{n-2}(n,k)W_{n-2}(n+1,k+1) < 0 \tag{2.5}$$

and

$$W_{n-2}(n,k)W_{n-2}(n+1,k+2) - W_{n-2}(n+1,k+1)W_{n-2}(n,k+1) < 0. \tag{2.6}$$

Proof. The proof is simliar with those in the above theorem. We only adjust the initial and final vertices. To verify (2.5), merely use initial vertex

$$u_1 = (0,0),\ \ u_2 = (1,-1),$$

and the final vertex

$$v_1 = (n-k+1,k),\ \ v_2 = (n-k,k).$$

To obtain (2.6), use initial vertex

$$u_1 = (0,0),\ \ \ u_2 = (1,-1),$$

and the final vertex

$$v_1 = (n-k,k),\ \ v_2 = (n-k,k+1).$$

∎

3 Labeled lattic path and $(n-2)$-stack sortable permutation

In this section, we will establish a bijection between labeled lattic path and $(n-2)$-stack sortable permutation. Our construction is based on Gasharov's work [18]. Given a lattice path P, we say that a horizontal edge in P is on row i if it is $i-1$ units above the initial point of P. Similarly, we say that a vertical edge is on column i, if it is $i-1$ units to the right of the initial point of P. Denote by $\mathcal{P}(n-1,k)$ the set of labeled lattice paths

P with $n-1$ edges, of which exactly k are vertical, such that a horizontal edge on row i is labeled with an integer between 1 and i, and similarly a vertical edge on column i is labeled with an integer between 1 and i. Here we do not distinguish between paths that can be obtained from each other by a translation. Let us recall Gasharov's bijection Φ between the sets $\mathscr{A}(n,k)$ and $\mathcal{P}(n-1,k)$, where $\mathscr{A}(n,k)$ denotes the set of permutations of $[n]$ with exact k descents.

For $\pi \in \mathscr{A}(n,k)$, suppose that σ (τ, respectively) is the permutation of $[j]([j+1]$, respectively) with the same order as in π. Gasharov inductively constructed a lattice path P with $n-1$ edges $a_1, a_2, \ldots, a_{n-1}$ of which exactly k are vertical and assigned a label e_i to its i-th edge ($1 \le i \le n-1$) as follow:

(1). If $\sigma \in \mathscr{A}(j,i)$ and $\tau \in \mathscr{A}(j+1,i)$, that is, the number of descents of τ is equal to that of σ, then there are exactly i positions $p_1, p_2, \ldots, p_i$ (ordered from left to right) to insert $j+1$ in σ and obtain a permutation in $\mathscr{A}(j+1,i)$. If $j+1$ has to be inserted in position p_v to obtain τ, then let a_j be horizontal, and $e_j = v$;

(2). If $\sigma \in \mathscr{A}(j,i)$ and $\tau \in \mathscr{A}(j+1,i+1)$, that is, when $j+1$ is inserted in σ, the number of descents should be increased by one. There are exactly $n-k$ positions $q_1, q_2, \ldots, q_{n-k}$ (ordered again from left to right) to insert $j+1$ in σ and obtain a permutation in $\mathscr{A}(j+1,i+1)$. If $j+1$ has to be inserted in position p_v to obtain τ, then let a_j be vertical and $e_j = v$.

See Figure 3 for an example.

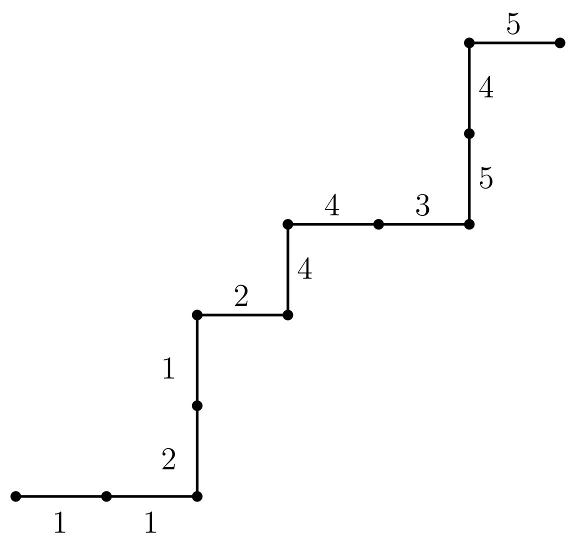

Figure 3: The path corresponding to $\pi = 5\,1\,4\,6\,2\,11\,7\,10\,9\,12\,3\,8$.

Applying this bijection, Gasharov [18] obtained a combinatorial proof of the log-concavity of the Eulerian polynomials $A_n(x)$. He also proved combinatorially that the sequence $\{A(n,k)\}_{k=1}^{n}$ of Eulerian numbers is a Hilbert function of a standard graded algebra over a field.

Now let us consider the set of $(n-2)$-stack sortable permutations with exact k descents.

Definition 3.1 *Let $\mathcal{Q}(n-1,k)$ be the set of labeled lattice paths with $n-1$ edges, exactly k of which are vertical, such that the following conditions hold:*

(1) The edge a_1 is vertical, then $e_1 = 1$,

(2) For $2 \le i \le n-2$, if a_i is a horizontal edge on row j, then $1 \le e_i \le j-1$; similarly, if a_i is a vertical edge on column j, then $1 \le e_i \le j$,

(3) The edge a_{n-1} is horizontal, and $e_{n-1} = k$.

It is obvious that $\mathcal{Q}(n-1,k)$ is a subset of $\mathcal{P}(n-1,k)$.

Theorem 3.2 *For $n \ge 1$ and $0 \le k \le n-1$, the map Φ defined above restricts a bijection between $\mathcal{Q}(n-1,k)$ and the set of permutations in $\mathfrak{S}_n$ of the form $\sigma n 1$, where σ is a permutation on $\{2,3,\ldots,n-1\}$.*

Proof. We first verify the path P corresponding to π of the form $\sigma n 1$ that satisfies the conditions in Definition 3.1 term by term.

(1) First, since 1 is in the last position, when we insert 2 as the same order in π, it must increase a descent in p_1. The permutation on $\{1, 2\}$ is 21, thus a_1 is vertical and $e_1 = 1$.

(2) Suppose we have to insert $j+1$ into the permutation γ in $[j]$ with the same order as in π. If the number of descents increases by one, then the number of positions which $j+1$ can be inserted is the same argument in Gasharov's construction. On the other hand, if the number of descents will not change after we insert $j+1$ into γ, then we cannot put $j+1$ at the end of γ since 1 must be in the last position. Thus, P satisfies the condition (2) in Definition 3.1.

(3) Suppose the last three elements of π are $tn1$, where $1 < t < n$. Note that when we insert n to obtain π, the number of descents does not change. Thus a_{n-1} is horizontal. Moreover, n must be inserted in position p_k since π has k descents, $e_{n-1} = k$.

To see that Φ is a bijection, let $P \in \mathcal{Q}(n-1,k)$ be a path satisfies the condition in Definition 3.1. Since the edge a_1 is vertical with label 1, then the permutation on $\{1,2\}$ must be 21. In general, if the ith edge a_i is a horizontal edge on row j with the condition

$1 \le e_i \le j-1$, then when we insert $i+1$ into the permutation on $[i]$, $i+1$ must be inserted before 1 since 1 is in the last position and the number of descents stays the same. Otherwise, if the ith edge a_i is a vertical edge on column j with $1 \le e_i \le j$, $i+1$ could not be inserted after 1 since the number of descents should be increased by one. Thus whenever $a_i (1 \le i \le n-2)$ is horizontal or vertical, 1 always in the last position. Finally, when we insert n into the permutation, n must be inserted right in front of 1 since a_{n-1} is a horizontal edge with label k. Therefore, the permutation corresponding P has the form of $\sigma n 1$, where σ is a permutation on $\{2, 3, \ldots, n-1\}$. ∎

Let $\mathcal{R}(n-1,k) = \mathcal{P}(n-1,k) \backslash \mathcal{Q}(n-1,k)$. The above theorem actually leads to a bijection between $\mathcal{S}(n,k)$ and $\mathcal{R}(n-1,k)$.

4 Simplicial complex

In this section, we will construct a simplicial complex $\triangle$ whose $(k-1)$-dimensional faces correspond to $(n-2)$-stack sortable permutations on $[n]$ with k descents for $1 \le k \le n$. Here we borrow the idea from Gasharov [18].

We will identify the elements of $\mathcal{S}(n,k)$ with the elements of $\mathcal{R}(n-1,k)$ via the bijection Φ. Let P be a lattice path. Denote $h(P)$ the number of horizontal edges in P. If $P \in \mathcal{R}(n-1,k)$, then for $1 \le i \le k$, we denote by $h_i(P)$ the number of horizontal edges in P whose horizontal edges are at most on row i. Let $V = \mathcal{Q}(n-1,1)$ be the vertex set of $\triangle$. Let $P \in \mathcal{Q}(n-1,k)$, we will associate to P a k-element subset $\{P_1, P_2, \ldots, P_k\}$ of $\triangle$. For a labeled path P, denote by $\overline{P}$ the path obtained from P by deleting the labels. For $1 \le i \le k$, let $\overline{P}_i$ be the unlabeled lattice path with one vertical edge, $h_1(\overline{P}_i) = h_i(P) + i - 1$, and $h(\overline{P}_i) = h(P) + k - 1$. Place the initial point of P at $(0,0)$ and for $1 \le i \le k$, place the initial point of $\overline{P}_i$ at $(-i+1, i-1)$. See Figure 4 for an example.

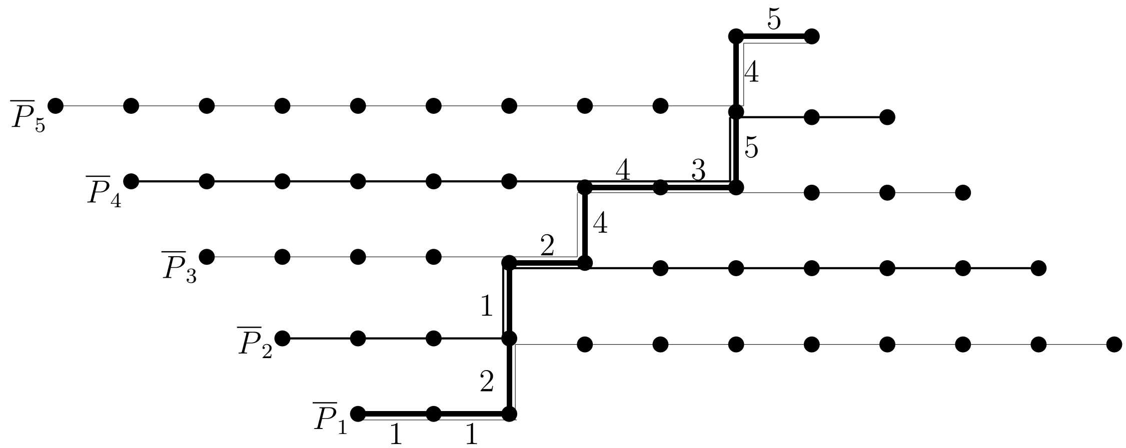

Figure 4: The subset $\{\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_5\}$ corresponding to $\pi = 5\,1\,4\,6\,2\,11\,7\,10\,9\,12\,3\,8$.

In this construction, the only vertical edge of $\overline{P}_i, 1 \leq i \leq k$, coincides with the i-th vertical edge of P and the range of labels we can assign to the vertical edge of $\overline{P}_i$ is not less than the range of labels we can assign to the i-th vertical edge of P. In fact, if the i-th vertical edge of P is t units to the right of the initial point of P, then the vertical edge of $\overline{P}_i$ is $(t+i-1)$ units to the right of the initial point of $\overline{P}_i$.

Now we proceed to label the edges of $\overline{P}_i$ to make them the vertices in V. Label the vertical edge of $\overline{P}_i, 1 \leq i \leq k$, by the label of the i-th vertical edge of P. Label the level-1 horizontal edges in $\overline{P}_1, \ldots, \overline{P}_k$ by 1. Label the level-2 horizontal edges in $\overline{P}_1$ which are also edges in P by their labels as edges in P. To label the level-2 horizontal edges in $\overline{P}_2, \ldots, \overline{P}_k$, first draw the northwest strips bounded by $\overline{P}_1$ and P. The northeast border of each such strip is either a vertical or a horizontal edge. If it is a vertical edge, label all horizontal edges in the strip 1. Now suppose the northwest border of a strip is a horizontal edge whose level and label in P are i and j, respectively. Then $j \leq i$ and the strip has exactly $i-1$ horizontal edges. Order these edges from southeast to northwest and label the first $i-j$ horizontal edges 1 and the remaining $j-1$ horizontal edges 2. In this way we obtain paths $P_1, \ldots, P_k$ in V.

Definition 4.1 *Define $\triangle_{stack}$ to be the collection of subsets $\{P_1, \ldots, P_k\}$ in V satisfy the following conditions,*

(1) $\overline{P}_i \neq \overline{P}_j$ for $1 \leq i \neq j \leq k$.

(2) Suppose $P_1, \ldots, P_k$ are ordered such that for $1 \leq i \leq k-1$, P_i has fewer level 1 horizontal edges than P_{i+1}. (This can be done in view of (1).) Then if we draw $P_1, \ldots, P_k$ such that the initial point of P_{i+1} is one unit up and to the left of the initial point of P_i for $1 \leq i \leq k-1$, the labels of their level-2 horizontal edges weakly increase in each northwest strip bounded by P_1 and P. See as Figure 5. When a northwest strip ends with a vertical edge, then all horizontal edges in it are labeled 1. If the vertical edge of P_i is t units to the right of the initial point of P_i, then the range of labels of the vertical edge of P_i is not more than $t-i+1$.

(3) The following k-element sets should not be included: The first edge of P_1 is vertical and all level-2 horizontal edges of P_1 labeled 1.

(4) The last horizontal edges of $P_2, \ldots, P_k$ are labeled 2.

Now we are in a position to prove our main theorem.

Theorem 4.2 *$\triangle_{stack}$ is a simplicial complex whose $(k-1)$-dimensional faces correspond to $(n-2)$-stack sortable permutations with k descents.*

Proof. First we aim to prove $\triangle_{stack}$ is a simplicial complex. A set of k paths from V satisfying the above conditions determines a path from $\mathcal{Q}(n,k)$. Also, given a subset of the set satisfy the above four conditions, one can see that the subset also satisfies the conditions (1), (2), (3) and(4). The above discussion shows that $\triangle$ is a simplicial complex with the desired properties.

Now we proceed to prove $(k-1)$-dimensional face of $\triangle_{stack}$ correpsond to $(n-2)$-stack sortable permutations with k descents. It is easily seen that the first two conditions correspond to permutations in $\mathfrak{S}$ with k descents. Thus we just need to verify that the last two conditions correspond the permutations on the form $\sigma n1$ with k descents. By the Definition 3.1, if the northwest border of a strip is a horizontal edge (not the last edge) whose level is i and label is j, then $j \leq i-1$. Thus the number of edges which are labeled 1 is $i-j \geq 1$, that is, whatever the northwest border of each strip is horizontal or vertical, the labels in the 2-level horizontal edges of P_1 are always 1. Moreover, since both of the level and label of the last horizontal edge is k, the number of edges which are labeled by 1 in the last strip is $i-j=0$, then all the 2-level horizontal edges of the last strip labeled 2. Thus the k-element subset $P_1,\ldots,P_k$ associated to a permutation not of the form $\sigma n1$ must satisfy the above condition. ∎

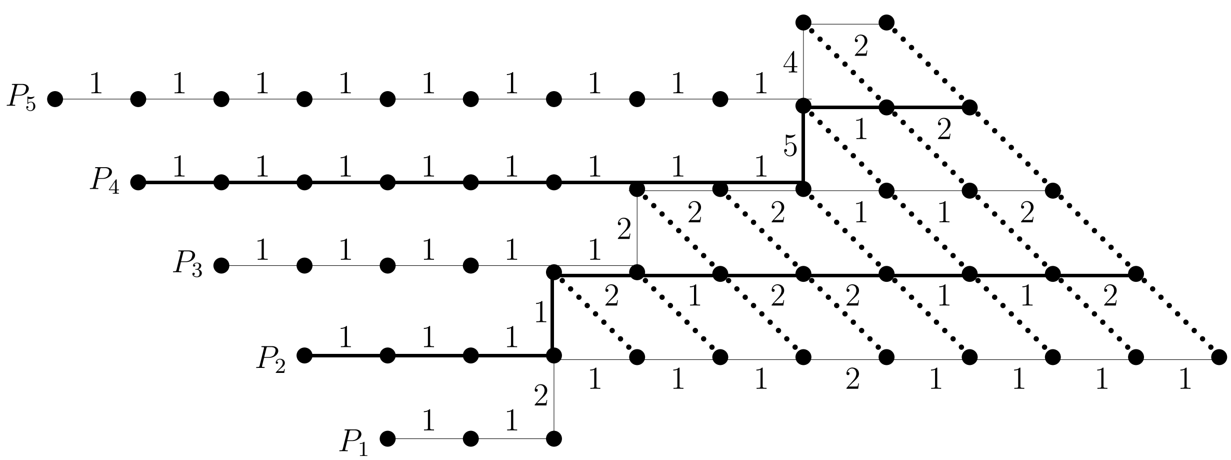

Figure 5:The simplicial complex corresponding to $\pi = 5\,1\,4\,6\,2\,11\,7\,10\,9\,12\,3\,8$.

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