CONGRUENCES FOR GENERALIZED FROBENIUS PARTITIONS WITH 6 COLORS

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ABSTRACT. In this paper, we establish the generating function for $\overline{c\phi_6}(n)$, the number of generalized Frobenius partitions of n with 6 colors whose order is 6 under cyclic permutation of the 6 colors. Furthermore, we find some congruences for $\overline{c\phi_6}(n)$ modulo 24.

1. INTRODUCTION

The concept of generalized Frobenius partitions with k colors was introduced by Andrews [1]. A generalized Frobenius partition of n with k colors is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_m \\ b_1 & b_2 & \dots & b_m \end{pmatrix},$$

where $\sum_{i=1}^{m}(a_i + b_i + 1) = n$, and where the integer entries are taken from k distinct copies of the non-negative integers distinguished by color, and the rows are ordered first by size and then by color with no two consecutive like entries in any row. The number of this kind of partitions of n is denoted by $c\phi_k(n)$. Andrews [1] showed that

$$c\phi_2(2n+1) \equiv 0 \pmod{2} \quad \text{and} \quad c\phi_2(5n+3) \equiv 0 \pmod{5}.$$

Baruah and Sarmah [3] represented the generating function of $c\phi_6(n)$ in terms of Ramanujan's theta functions and established 2, and 3-dissections of it which imply that for $n \geq 0$,

$$c\phi_6(2n+1) \equiv 0 \pmod{4},$$
$$c\phi_6(3n+1) \equiv 0 \pmod{9},$$
$$c\phi_6(3n+2) \equiv 0 \pmod{9}.$$

Xia [25] proved the following conjecture posed in [3]:

$$c\phi_6(3n+2) \equiv 0 \pmod{3^3}.$$

Later, Gu et al. [11] and Hirschhorn [12] established more congruences for $c\phi_6(n)$ modulo powers of 3. For more properties of $c\phi_k(n)$, one could see [2, 7, 9, 13–24, 26]. Then

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Kolitsch [13,14] considered the function $\overline{c\phi_k}(n)$, which denotes the number of generalized Frobenius partitions of n with k colors whose order is k under cyclic permutation of k colors. The generating function of $\overline{c\phi_k}(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{c\phi_k}(n)q^n=\frac{k\sum q^{Q(\mathbf{m})}}{(q;q)_\infty^k}, \tag{1.1}$$

where the sum of the right extends over all vectors $\mathbf{m}=(m_1,m_2,\cdots,m_k)$ with $\sum_{i=1}^k m_i=1$, and $Q(\mathbf{m})=\frac{1}{2}\sum_{i=1}^k(m_i-m_{i+1})^2$ with $m_{k+1}=m_1$. Kolitsch [13] found that for all integers $k\geq 2$,

$$\overline{c\phi_k}(n)\equiv 0 \pmod{k^2}.$$

Sellers [19,21] established that

$$\overline{c\phi_k}(kn)\equiv 0 \pmod{k^3} \quad \text{for k=2, 3, 5, 7, 11.}$$

For more results of $\overline{c\phi_k}(n)$, we refer to [22]. In [2,3], Baruah and Sarmah used the integer matrix exact covering system which was first introduced by Cao [6] to establish the generating functions of $\overline{c\phi_4}(n)$ and $c\phi_6(n)$, respectively. In this paper, using the method of Baruah and Sarmah, we focus on the generating function of $\overline{c\phi_6}(n)$. For more details of the integer matrix exact covering system, one can see [2, p. 1894–1895].

Here and in what follows, we have made use of the standard q-series notation [10]

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k), \quad |q|<1.$$

For convenience, define f_k as

$$f_k:=(q^k;q^k)_\infty.$$

Let $f(a,b)$ be Ramanujan's general theta function given by

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}}, \quad |ab|<1.$$

Jacobi's triple product identity can be stated in Ramanujan's notation as follows:

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Thus,

$$\begin{aligned}\varphi(q)&:=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2}=\frac{f_2^5}{f_1^2f_4^2},\\ \psi(q)&:=f(q,q^3)=\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}=\frac{f_2^2}{f_1},\\ f(-q)&:=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^nq^{\frac{n(3n-1)}{2}}=f_1.\end{aligned} \tag{1.2}$$

In this paper, we obtain the following congruences for $\overline{c\phi_6}(n)$.

Theorem 1.1. *We have*

$$\begin{aligned}\overline{c\phi_6}\,(2n) &\equiv 0 \pmod{24},\\ \overline{c\phi_6}\,(9n+4) &\equiv 0 \pmod{24}.\end{aligned} \tag{1.3}$$

Theorem 1.2. *For any prime $p \geq 5$,*

$$\overline{c\phi_6}\left(9p\,(pn+j)+\frac{3p^2+1}{4}\right) \equiv 0 \pmod{24},$$

where $j = 1, 2, \cdots, p-1$.

2. Main results

In this section, we first establish the generating function of $\overline{c\phi_6}(n)$ by the integer matrix exact covering system. Then we prove Theorems 1.1 and 1.2.

In order to prove the main results, the following lemmas are needed.

Lemma 2.1. *[4, Eq. (1.2.9)] For any integer n,*

$$f(a,b) = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} f(a(ab)^n, b(ab)^{-n}).$$

For any prime $p \geq 5$, define

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & p \equiv 1 \pmod{6},\\ \frac{-p-1}{6}, & p \equiv -1 \pmod{6}.\end{cases}$$

Lemma 2.2. *[8] For any prime $p \geq 5$,*

$$f(-q) = \sum_{\substack{k=-\frac{p-1}{2}\\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}).$$

Further, we claim that for $-(p-1)/2 \leq k \leq (p-1)/2$ and $k \neq (\pm p-1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

Lemma 2.3. *[8] For any odd prime p,*

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}).$$

Furthermore, we claim that for $0 \leq k \leq (p-3)/2$,

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}.$$

Setting $p = 3$ in Lemma 2.3, we arrive at

$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \tag{2.1}$$

Lemma 2.4. *We have*

$$f_1^2 \equiv f_2 \pmod{2},$$
$$f_1^4 \equiv f_2^2 \pmod{4}.$$

Set

$$a(q) := \varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6).$$

Notice that $a(q)$ is one of Borweins' cubic theta functions which were mentioned in [5].

Theorem 2.5. *We have*

$$\sum_{n=0}^{\infty} \overline{c\phi_6}(n)q^n = \frac{6S}{f_1^6},$$

where

$$\begin{aligned} S =&4q\psi^2(q)\psi(q^2)\psi(q^3)f(q^3,q^6) + 4q^2\varphi(q^2)\psi(q)\psi(q^3)\psi(q^8)f(q^{12},q^{24}) \\ &+ 4q^3\varphi(q^4)\psi(q)\psi(q^3)\psi(q^4)f(q^6,q^{30}) + 2q\varphi(q^4)\psi(q^4)a(q^2)f(q^{12},q^{24}) \\ &+ 2q^3\varphi(q^2)\psi(q^8)a(q^2)f(q^6,q^{30}). \end{aligned}$$

Proof. Setting $k = 6$ in (1.1), we have

$$\sum_{n=0}^{\infty} \overline{c\phi_6}(n)q^n = \frac{6S}{f_1^6},$$

where

$$\begin{aligned} S = \sum_{m_i=-\infty}^{\infty} & q^{3m_1^2+2m_2^2+2m_3^2+2m_4^2+3m_5^2+2m_1m_2+3m_1m_3+3m_1m_4+4m_1m_5+m_2m_3+2m_2m_4+3m_2m_5+m_3m_4} \\ & \times q^{3m_3m_5+2m_4m_5-3m_1-2m_2-2m_3-2m_4-3m_5+1}. \end{aligned}$$

We choose

$$B = 4\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Then in view of the integer matrix exact covering system in [6] and the procedures for obtaining S in [2, 3], we can write S as a linear combination of four parts as

$$\begin{aligned} S = & \sum_{n_1,n_2,n_3,n_4,n_5=-\infty}^{\infty} q^{3n_1^2+5n_2^2+3n_3^2+2n_4^2+2n_5^2+2n_1n_5+2n_2n_4+2n_3n_5-3n_1+2n_2+n_4-n_5+1} \\ & + \sum_{n_1,n_2,n_3,n_4,n_5=-\infty}^{\infty} q^{3n_1^2+5n_2^2+3n_3^2+2n_4^2+2n_5^2+2n_1n_5+2n_2n_4+2n_3n_5+3n_2+3n_3+3n_4+n_5+2} \\ & + \sum_{n_1,n_2,n_3,n_4,n_5=-\infty}^{\infty} q^{3n_1^2+5n_2^2+3n_3^2+2n_4^2+2n_5^2+2n_1n_5+2n_2n_4+2n_3n_5+n_1+7n_2+n_3+2n_4+2n_5+3} \end{aligned}$$

$$+\sum_{n_1,n_2,n_3,n_4,n_5=-\infty}^{\infty} q^{3n_1^2+5n_2^2+3n_3^2+2n_4^2+2n_5^2+2n_1n_5+2n_2n_4+2n_3n_5-2n_1+8n_2+4n_3+4n_4+2n_5+6}$$
$$= a_1(q)b_1(q)+a_2(q)b_2(q)+a_3(q)b_3(q)+a_4(q)b_4(q), \tag{2.2}$$

where

$$a_1(q) = \sum_{n_1,n_3,n_5=-\infty}^{\infty} q^{3n_1^2+3n_3^2+2n_5^2+2n_1n_5+2n_3n_5-3n_1-n_5+1},$$
$$a_2(q) = \sum_{n_1,n_3,n_5=-\infty}^{\infty} q^{3n_1^2+3n_3^2+2n_5^2+2n_1n_5+2n_3n_5+3n_3+n_5+2},$$
$$a_3(q) = \sum_{n_1,n_3,n_5=-\infty}^{\infty} q^{3n_1^2+3n_3^2+2n_5^2+2n_1n_5+2n_3n_5+n_1+n_3+2n_5+3},$$
$$a_4(q) = \sum_{n_1,n_3,n_5=-\infty}^{\infty} q^{3n_1^2+3n_3^2+2n_5^2+2n_1n_5+2n_3n_5-2n_1+4n_3+2n_5+6},$$
$$b_1(q) = \sum_{n_2,n_4=-\infty}^{\infty} q^{5n_2^2+2n_4^2+2n_2n_4+2n_2+n_4},$$
$$b_2(q) = \sum_{n_2,n_4=-\infty}^{\infty} q^{5n_2^2+2n_4^2+2n_2n_4+3n_2+3n_4},$$
$$b_3(q) = \sum_{n_2,n_4=-\infty}^{\infty} q^{5n_2^2+2n_4^2+2n_2n_4+7n_2+2n_4},$$
$$b_4(q) = \sum_{n_2,n_4=-\infty}^{\infty} q^{5n_2^2+2n_4^2+2n_2n_4+8n_2+4n_4}.$$

In addition, we find that

$$qa_1(q) = a_2(q). \tag{2.3}$$

Next, For $a_1(q)$, $a_2(q)$, and $a_3(q)$, we apply another transformation of variables by using the integer matrix exact covering system. We adopt the matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Therefore, from this integer matrix exact covering system, we can write $a_1(q), a_2(q), a_3(q)$, and $a_4(q)$ as follows.

$$\begin{aligned} a_1(q) &= q\psi(q)\psi(q^2)\psi(q^3) + q^3\psi(q)\psi(q^3)f(q^{-2},q^{10}) \\ &= 2q\psi(q)\psi(q^2)\psi(q^3), \end{aligned} \tag{2.4}$$

where we use Lemma 2.1 with $n=1$ to arrive at the last equality. Employing (2.3), we see that

$$a_2(q) = 2q^2\psi(q)\psi(q^2)\psi(q^3). \tag{2.5}$$

Similarly, we obtain

$$a_3(q) = 2q^3\varphi(q^4)\varphi(q^6)\psi(q^4) + 4q^5\varphi(q^2)\psi(q^8)\psi(q^{12}), \tag{2.6}$$

$$a_4(q) = 2q^5\varphi(q^2)\varphi(q^6)\psi(q^8) + 4q^6\varphi(q^4)\psi(q^4)\psi(q^{12}). \tag{2.7}$$

Next, we use the following integer matrix exact covering system

$$\left\{\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

to rewrite $b_1(q), b_2(q), b_3(q)$, and $b_4(q)$ as follows.

$$\begin{aligned} b_1(q) &= \sum_{x_1,x_2=-\infty}^{\infty} q^{18x_1^2+2x_2^2+3x_1+x_2} + \sum_{x_1,x_2=-\infty}^{\infty} q^{18x_1^2+2x_2^2+21x_1+3x_2+7} \\ &= f(q^{15}, q^{21})f(q, q^3) + q^7 f(q^{-3}, q^{39})f(q^{-1}, q^5) \\ &= f(q, q^3)\left(f(q^{15}, q^{21}) + q^3 f(q^3, q^{33})\right) \\ &= \psi(q)f(q^3, q^6), \end{aligned} \tag{2.8}$$

where we use Lemma 2.1 with $n = 1$ to obtain the penultimate equality, and employ the following fact to obtain the last equality.

$$\begin{aligned} f(q^3, q^6) &= \sum_{n=-\infty}^{\infty} q^{\frac{9n^2+3n}{2}} \\ &= \sum_{n=-\infty}^{\infty} q^{\frac{9\cdot 4n^2+3\cdot 2n}{2}} + \sum_{n=-\infty}^{\infty} q^{\frac{9\cdot(2n+1)^2+3\cdot(2n+1)}{2}} \\ &= f(q^{15}, q^{21}) + q^6 f(q^{-3}, q^{39}) \\ &= f(q^{15}, q^{21}) + q^3 f(q^3, q^{33}) \quad \text{(by Lemma 2.1)}. \end{aligned}$$

Similarly, we obtain

$$b_2(q) = q^{-1}\psi(q)f(q^3, q^6), \tag{2.9}$$

$$b_3(q) = 2\psi(q^4)f(q^6, q^{30}) + q^{-2}\varphi(q^2)f(q^{12}, q^{24}), \tag{2.10}$$

$$b_4(q) = q^{-2}\varphi(q^2)f(q^6, q^{30}) + 2q^{-3}\psi(q^4)f(q^{12}, q^{24}). \tag{2.11}$$

Substituting (2.4)-(2.11) into (2.2) yields that

$$\begin{aligned} S = {} & 4q\psi^2(q)\psi(q^2)\psi(q^3)f(q^3, q^6) \\ & + 4q^2\varphi(q^2)\psi(q^8)\left(\varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12})\right) f(q^{12}, q^{24}) \\ & + 4q^3\varphi(q^4)\psi(q^4)\left(\varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12})\right) f(q^6, q^{30}) \\ & + 2q\varphi(q^4)\psi(q^4)a(q^2)f(q^{12}, q^{24}) + 2q^3\varphi(q^2)\psi(q^8)a(q^2)f(q^6, q^{30}). \end{aligned} \tag{2.12}$$

In addition, Baruah and Sarmah [3, p. 368] presented that

$$\varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12}) = \psi(q)\psi(q^3).$$

Then substituting the above equation into (2.12), we arrive at the theorem. □

Proof of Theorem 1.1. To prove the theorem, we need the following properties.

$$f(q,q^2) \equiv f(-q,-q^2) = f_1 \pmod 2, \tag{2.13}$$

$$f(q,q^5) \equiv f(-q,-q^5) = \psi(q^3)\frac{f_1}{f_2} \equiv \frac{\psi(q^3)}{f_1} \pmod 2, \tag{2.14}$$

$$f_1^3 = \frac{f_1^4}{f_1} \equiv \frac{f_2^2}{f_1} = \psi(q) \pmod 2, \tag{2.15}$$

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2} \equiv \frac{f_1^{10}}{f_1^2 f_1^8} = 1 \pmod 2. \tag{2.16}$$

First, from Theorem 2.5, it can be seen that

$$\begin{aligned}
&\frac{\sum_{n=0}^{\infty} \overline{c\phi_6}(n)q^n}{6} \\
&\equiv 2q\frac{\varphi(q^2)\varphi(q^4)\varphi(q^6)\psi(q^4)f(q^{12},q^{24})}{f_1^6} + 2q^3\frac{\varphi^2(q^2)\varphi(q^6)\psi(q^8)f(q^6,q^{30})}{f_1^6} \pmod{2^2} \\
&\equiv 2q\frac{\psi(q^4)f(q^{12},q^{24})}{f_1^6} + 2q^3\frac{\psi(q^8)f(q^6,q^{30})}{f_1^6} \pmod{2^2} \\
&\equiv 2q\frac{f_8^2}{f_4}\frac{f_{12}}{f_1^6} + 2q^3\frac{f_{16}^2}{f_8}\frac{\psi(q^{18})}{f_1^6 f_6} \pmod{2^2} \\
&\equiv 2q\frac{f_4^3 f_{12}}{f_1^6} + 2q^3\frac{f_8^3\psi(q^{18})}{f_1^6 f_6} \pmod{2^2} \\
&\equiv 2q\frac{f_1^{12} f_{12}}{f_1^6} + 2q^3\frac{f_1^{24}\psi(q^{18})}{f_1^6 f_6} \pmod{2^2} \\
&= 2qf_1^6 f_{12} + 2q^3\frac{f_1^{18}\psi(q^{18})}{f_6} \\
&\equiv 2qf_2^3 f_{12} + 2q^3 f_2^9\frac{\psi(q^{18})}{f_6} \pmod{2^2} \\
&\equiv 2q\psi(q^2)f_{12} + 2q^3\psi^3(q^2)\frac{\psi(q^{18})}{f_6} \pmod{2^2} \\
&= 2q\left(f(q^6,q^{12}) + q^2\psi(q^{18})\right) f_{12} + 2q^3\left(f(q^6,q^{12}) + q^2\psi(q^{18})\right)^3 \frac{\psi(q^{18})}{f_6},
\end{aligned} \tag{2.17}$$

where we apply (1.2), (2.1), Lemma 2.4, and (2.13)-(2.16) in the above congruences. Then we find that

$$\frac{\overline{c\phi_6}\,(2n)}{6} \equiv 0 \pmod{2^2}.$$

Therefore, we prove the first congruence. Moreover, in view of (2.17), we obtain

$$\begin{aligned}
\frac{\sum_{n=0}^{\infty} \overline{c\phi_6}(3n+1)q^n}{6} &\equiv 2f(q^2,q^4)f_4 + 2q^2 f(q^2,q^4)\psi^2(q^6)\frac{\psi(q^6)}{f_2} \pmod{2^2} \\
&\equiv 2f_2 f_4 + 2q^2 f_2\psi^2(q^6)\frac{\psi(q^6)}{f_2} \pmod{2^2} \\
&\equiv 2f_2^3 + 2q^2\psi^3(q^6) \pmod{2^2}
\end{aligned}$$

$$\equiv 2\psi(q^2)+2q^2\psi^3(q^6) \pmod{2^2}$$
$$= 2\left(f(q^6,q^{12})+q^2\psi(q^{18})\right)+2q^2\psi^3(q^6), \tag{2.18}$$

where we derive the above relations by using (2.1), Lemma 2.4, (2.13), and (2.15). Since there are no terms on the right-hand side of (2.18) in which the powers of q are congruent to 1 modulo 3, we arrive at

$$\frac{\overline{c\phi_6}\left(3\left(3n+1\right)+1\right)}{6}=\frac{\overline{c\phi_6}\left(9n+4\right)}{6}\equiv 0 \pmod{2^2}.$$

Hence, we complete the proof of (1.3). □

Proof of Theorem 1.2. Based on (2.13) and (2.18), we find that

$$\frac{\sum_{n=0}^{\infty}\overline{c\phi_6}\left(3\left(3n\right)+1\right)q^n}{6}=\frac{\sum_{n=0}^{\infty}\overline{c\phi_6}\left(9n+1\right)q^n}{6}\equiv 2f(q^2,q^4)\equiv 2f_2 \pmod{2^2}.$$

Then in view of Lemma 2.2 and the above relation, we deduce that for any prime $p\geq 5$,

$$\frac{\sum_{n=0}^{\infty}\overline{c\phi_6}\left(9\left(pn+\frac{p^2-1}{12}\right)+1\right)q^n}{6}=\frac{\sum_{n=0}^{\infty}\overline{c\phi_6}\left(9pn+\frac{3p^2+1}{4}\right)q^n}{6}\equiv 2f(-q^{2p}) \pmod{2^2}.$$

Thus, it can be seen that for $j=1,2,\cdots,p-1$,

$$\overline{c\phi_6}\left(9p\left(pn+j\right)+\frac{3p^2+1}{4}\right)\equiv 0 \pmod{24}.$$

Therefore, we complete the proof. □

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