The equivariant inverse Kazhdan-Lusztig polynomials of uniform matroids

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Abstract. Motivated by the concepts of the inverse Kazhdan-Lusztig polynomial and the equivariant Kazhdan-Lusztig polynomial, Proudfoot defined the equivariant inverse Kazhdan-Lusztig polynomial for a matroid. In this paper, we show that the equivariant inverse Kazhdan-Lusztig polynomial of a matroid is very useful for determining its equivariant Kazhdan-Lusztig polynomials, and we determine the equivariant inverse Kazhdan-Lusztig polynomials for Boolean matroids and uniform matroids. As an application, we give a new proof of Gedeon, Proudfoot and Young's formula for the equivariant Kazhdan-Lusztig polynomials of uniform matroids. Inspired by Lee, Nasr and Radcliffe's combinatorial interpretation for the ordinary Kazhdan-Lusztig polynomials of uniform matroids, we further present a new formula for the corresponding equivariant Kazhdan-Lusztig polynomials.

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1 Introduction

Since the introduction of the matroid Kazhdan-Lusztig polynomials, due to Elias, Proudfoot and Wakefield [5], these polynomials have attracted much attention, for instance see [10, 11, 16, 7, 1, 24] and references therein. As noted by Proudfoot [19], the Kazhdan-Lusztig polynomials of matroids can also be considered as a special case of the Kazhdan-Lusztig-Stanley polynomials, which were first introduced by Stanley [22] and further studied by Brenti [3, 4]. Within the framework of the Kazhdan-Lusztig-Stanley theory, Gao and Xie [8] defined the inverse Kazhdan-Lusztig polynomial of a matroid. To study the properties of the matroid Kazhdan-Lusztig polynomials, Gedeon, Proudfoot, and Young

[9] introduced the concept of the equivariant Kazhdan-Lusztig polynomials of matroids. In [21] Proudfoot further developed the equivariant Kazhdan-Lusztig-Stanley theory, noted that the equivariant Kazhdan-Lusztig polynomial of a matroid can be realized as an equivariant Kazhdan-Lusztig-Stanley polynomial, and further defined the equivariant inverse Kazhdan-Lusztig polynomials of a matroid. Gao and Xie [8] showed that the inverse Kazhdan-Lusztig polynomials of uniform matroids are not only easy to compute, but also very helpful for determining their Kazhdan-Lusztig polynomials. The main objective of this paper is to develop their idea for the equivariant case of uniform matroids. To this end, we compute the equivariant inverse Kazhdan-Lusztig polynomials for Boolean matroids and uniform matroids, and further use them to give an alternative proof of Gedeon, Proudfoot and Young's formula [9] for the equivariant Kazhdan-Lusztig polynomials of uniform matroids.

Let us first follow Gedeon, Proudfoot, and Young [9] to recall some related definitions and notations on equivariant matroid Kazhdan-Lusztig polynomials. Let M be a matroid on the ground set E, and let W be a finite group acting on E and preserving M. Gedeon, Proudfoot, and Young referred to this collection of data as an equivariant matroid $W \curvearrowright M$. Let $\mathrm{VRep}(W)$ denote the ring of virtual representations of W over $\mathbb{C}$, and let $\mathrm{VRep}(W)[t]$ denote the polynomial ring in t over $\mathrm{VRep}(W)$. For the definition of virtual representations, see the end of Section 2.3 of Wachs [25]. Gedeon, Proudfoot and Young defined the equivariant characteristic polynomial of $W \curvearrowright M$ as

$$H_M^W(t) := \sum_{i=0}^{\mathrm{rk}\, M} (-1)^i t^{\mathrm{rk}\, M - i} OS_{M,i}^W \in \mathrm{VRep}(W)[t], \tag{1}$$

where $OS_{M,i}^W$ is the degree i part of the Orlik-Solomon algebra of M, a natural representation of W induced by its action on M. For the definition of the Orlik-Solomon algebra of a matroid, see [6, Definition 3.1]. Note that the graded dimension of $H_M^W(t)$ is just the usual characteristic polynomial $\chi_M(t) \in \mathbb{Z}[t]$.

Let $L(M)$ denote the lattice of flats of M. For any $F \in L(M)$, denote the localization of M at F by M^F, the matroid on the ground set F whose lattice of flats is isomorphic to $L^F := \{G \in L \mid G \leq F\}$. Dually, denote the contraction of M at F by M_F, the matroid on the ground set $E \backslash F$ with its lattice of flats isomorphic to $L_F := \{G \in L \mid G \geq F\}$. Moreover, let $W_F \subset W$ be the stablizer of F and let $\mathrm{rk}\, M$ denote the rank of M. Gedeon, Proudfoot, and Young [9, Theorem 2.8] showed that there is a unique way to assign to each equivariant matroid $W \curvearrowright M$ an element $P_M^W(t) \in \mathrm{VRep(W)}[t]$, called the equivariant Kazhdan-Lusztig polynomial, such that the following conditions are satisfied:

(1). If the ground set of M is empty, then $\deg P_M^W(t) = 0$, and $P_M^W(t)$ is the trivial representation of W.

(2). If $\mathrm{rk}\, M > 0$, then $\deg P_M^W(t) < \frac{1}{2}\,\mathrm{rk}\, M$.

(3). For every M,

$$t^{\mathrm{rk}\ M} P_M^W(t^{-1}) = \sum_{[F]\in L(M)/W} \mathrm{Ind}_{W_F}^W \Big(H_{M^F}^{W_F}(t) \otimes P_{M_F}^{W_F}(t) \Big), \tag{2}$$

where $L(M)/W$ denotes the set of orbits of the natural action of W on $L(M)$.

As noted by Proudfoot (private communication), the fourth condition in [9, Theorem 2.8] follows from the first three conditions; meanwhile, the original definition of $P_M^W(t)$ applies only to loopless matroids, see also [17, Remark 2.2].

The equivariant Kazhdan-Lusztig polynomials have been computed for uniform matroids [9], q-niform matroids [20], and thagomizer matroids [10, 26]. Gedeon, Proudfoot, and Young [9, Conjecture 2.13] conjectured that for any equivariant matroid $W \curvearrowright M$ the coefficients of $P_M^W(t)$ are isomorphism classes of honest representations of W. Recently, Braden, Huh, Matherne, Proudfoot and Wang [2] proved this conjecture by using the singular hodge theory.

As remarked by Gedeon, Proudfoot, and Young [9], their computation of the equivariant Kazhdan-Lusztig polynomials for uniform matroids relies on a guess on the generating function of these polynomials. Our proof of [9, Theorem 3.1] given here is more direct, and uses the concept of the equivariant inverse Kazhdan-Lusztig polynomials of matroids; see the proof of Theorem 3.1. As noted by Proudfoot [21, Section 4], there is a unique way to assign to each equivariant matroid $W \curvearrowright M$ an element $Q_M^W(t) \in \mathrm{VRep}(W)[t]$ such that the following conditions are satisfied:

(a). If the ground set of M is empty, then $Q_M^W(t)$ is the trivial representation in degree 0.

(b). If $\mathrm{rk}\, M > 0$, then $\deg Q_M^W(t) < \frac{1}{2}\,\mathrm{rk}\, M$.

(c). For every $W \curvearrowright M$,

$$t^{\mathrm{rk}\, M} \cdot (-1)^{\mathrm{rk}\, M} Q_M^W(t^{-1}) = \sum_{[F]\in L(M)/W} \mathrm{Ind}_{W_F}^W \Big((-1)^{\mathrm{rk}\, M^F} Q_{M^F}^{W_F}(t) \otimes t^{\mathrm{rk}\, M_F} H_{M_F}^{W_F}(t^{-1}) \Big). \tag{3}$$

We call $Q_M^W(t)$ the equivariant inverse Kazhdan-Lusztig polynomial of $W \curvearrowright M$. Substituting t for t^{-1} in (3) and then multiplying both sides by $t^{\mathrm{rk}\, M}$, we obtain an equivalent form of (3) as

$$(-1)^{\mathrm{rk}\, M} Q_M^W(t) = \sum_{[F]\in L(M)/W} \mathrm{Ind}_{W_F}^W \Big((-t)^{\mathrm{rk}\, M^F} Q_{M^F}^{W_F}(t^{-1}) \otimes H_{M_F}^{W_F}(t) \Big). \tag{4}$$

By using the equivariant Kazhdan-Lusztig-Stanley theory, one can deduce that

$$\sum_{[F]\in L(M)/W} (-1)^{\mathrm{rk}\, M^F} \mathrm{Ind}_{W_F}^W \Big(P_{M^F}^{W_F}(t) \otimes Q_{M_F}^{W_F}(t) \Big) = 0. \tag{5}$$

We would like to point out that (5) was used by Braden, Huh, Matherne, Proudfoot and Wang to define $Q_M^W(t)$. Moreover, they showed that the coefficients of $Q_M^W(t)$ are isomorphism classes of honest representations of W. For notational convenience, let

$$\hat{Q}_M^W(t) = (-1)^{\operatorname{rk} M} Q_M^W(t).$$

Then the relations (4) and (5) can be written as

$$\hat{Q}_M^W(t) = \sum_{[F]\in L(M)/W} \operatorname{Ind}_{W_F}^W \Big(t^{\operatorname{rk} F} \hat{Q}_{M^F}^{W_F}(t^{-1}) \otimes H_{M_F}^{W_F}(t) \Big), \tag{6}$$

and

$$\sum_{[F]\in L(M)/W} \operatorname{Ind}_{W_F}^W \Big(P_{M^F}^{W_F}(t) \otimes \hat{Q}_{M_F}^{W_F}(t) \Big) = 0. \tag{7}$$

This paper is organized as follows. In Section 2, we will review some basic definitions and notations of the symmetric functions and give some results which will be used later. In Section 3, we first compute the equivariant inverse Kazhdan-Lusztig polynomials for Boolean matroids, and then compute these polynomials for uniform matroids. Based on the work on inverse Kazhdan-Lusztig polynomials, we further present a new proof for Gedeon, Proudfoot, and Young's formula for the equivariant Kazhdan-Lusztig polynomial for uniform matroids, as well as a new formula of these equivariant Kazhdan-Lusztig polynomials.

2 Symmetric functions

The aim of this section is to review some basic definitions and results on symmetric functions. We refer the reader to Stanley [23], Macdonald [18] and Haglund [12] for undefined terminology from the theory of symmetric functions. We also prove some symmetric function identities which will be used in the evaluation of the (inverse) equivariant Kazhdan-Lusztig polynomials for uniform matroids.

2.1 The Frobenius characteristic map

Let Λ_n denote the $\mathbb{Z}$-module of symmetric functions of degree n in the variables $\mathbf{x} = (x_1, x_2, \ldots)$. From the theory of symmetric functions, the Frobenius characteristic map is an isomorphism

$$\operatorname{ch} : \operatorname{VRep}(S_n) \longrightarrow \Lambda_n,$$

which maps the irreducible representation V_λ of S_n to the Schur function $s_\lambda(\mathbf{x})$ for each partition λ of n. In particular, the image of the trivial representation $V_{(n)}$ is $s_{(n)}(\mathbf{x}) = h_n(\mathbf{x})$, which is called the complete symmetric function, and the image of the representation $V_{(1^n)}$ is $s_{(1^n)}(\mathbf{x}) = e_n(\mathbf{x})$, which is called the elementary symmetric function. Moreover, the image of the skew Specht module $V_{\lambda/\mu}$ under ch is the skew Schur function $s_{\lambda/\mu}(\mathbf{x})$. The definition of ch carries over directly from $\mathrm{VRep}(S_n)$ to $\mathrm{VRep}(S_n)[t]$. It has the property that, given two graded virtual representations $V_1 \in \mathrm{VRep}(S_{n_1})[t]$ and $V_2 \in \mathrm{VRep}(S_{n_2})[t]$, we have

$$\mathrm{ch}\,\mathrm{Ind}_{S_{n_1}\times S_{n_2}}^{S_{n_1+n_2}}(V_1 \otimes V_2) = \mathrm{ch}(V_1)\mathrm{ch}(V_2). \tag{8}$$

2.2 Plethystic

Here we adopt the notation of Haglund [12]. Let $E = (t_1, t_2, t_3, \ldots)$ be a formal series of rational functions in the parameters. Let $p_k[E]$ denote the plethystic substitution of E into the k-th power sum p_k, i.e.,

$$p_k[E] = E(t_1^k, t_2^k, \ldots).$$

For any symmetric function f, suppose that $f = \sum_\lambda c_\lambda p_\lambda = \sum_\lambda c_\lambda \prod_i p_{\lambda_i}$, and then define

$$f[E] = \sum_\lambda c_\lambda \prod_i p_{\lambda_i}[E].$$

Note that if $X = \sum_i x_i$, then $p_k[X] = p_k(\mathbf{x})$, from which we get $f[X] = f(\mathbf{x})$ for any symmetric function f. One can show that $p_k[-X] = -p_k(\mathbf{x}) = -p_k[X]$ and $e_m[tX] = t^m e_m(\mathbf{x})$.

The following lemma will be used for the computation of the equivariant inverse Kazhdan-Lusztig polynomials of boolean matroids.

Lemma 2.1 ([15], Section 3.3)**.** *Let $E = E(t_1, t_2, t_3, \ldots)$ and $F = F(w_1, w_2, w_3, \ldots)$ be two formal series of rational functions in the parameters $t_1, t_2, \ldots$ and $w_1, w_2, \ldots$. Then*

$$e_m[E - F] = \sum_{j=0}^{m} (-1)^{m-j} e_j[E] h_{m-j}[F].$$

2.3 The Pieri rule and symmetric function identities

In this subsection we present three symmetric function identities. These identities might be known, but we could not find them in the literature. To be self-contained here, we will use the Pieri rule to give proofs of these identities.

Let us first recall the well known Pieri rule on Schur functions; see Stanley [23, Theorem 7.15.7] and the paragraphs following it for details. For any $i \geq 1$, we have

$$s_{(i)}(\mathbf{x}) s_\lambda(\mathbf{x}) = \sum_\mu s_\mu(\mathbf{x})$$

summing over all partitions μ such that μ/λ is a horizontal strip of size i. Meanwhile, we have

$$s_{(1^i)}(\mathbf{x})s_\lambda(\mathbf{x}) = \sum_{\mu} s_\mu(\mathbf{x})$$

summing over all partitions μ such that μ/λ is a vertical strip of size i. Recall that a horizontal strip is a skew partition with no two squares in the same column of its diagram, and a vertical strip is a skew shape with no two squares in the same row.

We proceed to prove the first symmetric function identity, which is stated as follows.

Lemma 2.2. *For $m \geq 2$, $i \geq 0$ and $j - i \geq 2$, we have*

$$s_{(1^{i+1})}(\mathbf{x})s_{(m,1^{j-1})}(\mathbf{x}) - s_{(1^i)}(\mathbf{x})s_{(m,1^j)}(\mathbf{x}) = s_{(m+1,2^i,1^{j-i-1})}(\mathbf{x}) + s_{(m,2^{i+1},1^{j-i-2})}(\mathbf{x}). \tag{9}$$

Proof. For $m \geq 2$ and $i, j \geq 0$, by the Pieri rule, we have

$$s_{(1^i)}(\mathbf{x})s_{(m,1^j)}(\mathbf{x}) = \sum_{a=0}^{\min\{i-1,j\}} s_{(m+1,2^a,1^{i+j-1-2a})}(\mathbf{x}) + \sum_{a=0}^{\min\{i,j\}} s_{(m,2^a,1^{i+j-2a})}(\mathbf{x}).$$

It follows that, for $m \geq 2$, $i \geq 0$ and $j \geq 1$,

$$\begin{aligned} & s_{(1^{i+1})}(\mathbf{x})s_{(m,1^{j-1})}(\mathbf{x}) - s_{(1^i)}(\mathbf{x})s_{(m,1^j)}(\mathbf{x}) \\ &= \sum_{a=0}^{\min\{i,j-1\}} s_{(m+1,2^a,1^{i+j-1-2a})}(\mathbf{x}) + \sum_{a=0}^{\min\{i+1,j-1\}} s_{(m,2^a,1^{i+j-2a})}(\mathbf{x}) \\ &\quad - \sum_{a=0}^{\min\{i-1,j\}} s_{(m+1,2^a,1^{i+j-1-2a})}(\mathbf{x}) - \sum_{a=0}^{\min\{i,j\}} s_{(m,2^a,1^{i+j-2a})}(\mathbf{x}). \end{aligned}$$

In view of that $j - i \geq 2$, and hence $i \leq j - 2$, the above four summations reduce to the right hand side of (9). This completes the proof. □

The second symmetric function identity we are to prove is as follows.

Lemma 2.3. *For $n \geq 0$, $m \geq 2$ and $j \geq 1$, we have*

$$\sum_{i=0}^{n}(-1)^i s_{(i)}(\mathbf{x})s_{(m,2^j,1^{n-i})}(\mathbf{x}) = (-1)^n \sum_{a=0}^{\min\{m-2,n\}} s_{(m+n-a,2+a,2^{j-1})}(\mathbf{x}).$$

Proof. When $n = 0$, the statement can be verified by plugging in $n = 0$ and checking both sides. It remains to prove the identity for $n \geq 1$. Given $m \geq 2$, $j \geq 1$ and $n \geq 1$, let $\lambda^i = (m, 2^j, 1^{n-i})$ for $0 \leq i \leq n$. According to the Pieri rule, we have

$$s_{(i)}(\mathbf{x})s_{(m,2^j,1^{n-i})}(\mathbf{x}) = \sum_{\mu} s_\mu(\mathbf{x}), \tag{10}$$

where the summation ranges over all partitions $\mu \vdash m+2j+n$ such that μ/λ^i is a horizontal strip of size i. Considering the shape of λ^i, such a partition μ must be of the form $(A, B, 2^u, 1^v)$ for some $A \geq m$, $B \geq 2$, $u = j-1$ or j, and nonnegative $v = n-i-1$, $n-i$ or $n-i+1$. Assume that

$$\sum_{i=0}^{n} (-1)^i s_{(i)}(\mathbf{x}) s_{(m,2^j,1^{n-i})}(\mathbf{x}) = \sum_{\mu} c_\mu s_\mu(\mathbf{x}),$$

where $\mu = (A, B, 2^u, 1^v) \vdash m+2j+n$ for $A \geq m$, $B \geq 2$, $u = j-1$ or j, and $v \geq 0$. It remains to determine the coefficient c_μ.

Now fix a partition $\mu = (A, B, 2^u, 1^v)$. Observe that

$$c_\mu = \sum_i (-1)^i,$$

where the sum is over all i such that $s_{(i)}(\mathbf{x}) s_{(m,2^j,1^{n-i})}(\mathbf{x})$ contains $s_\mu(\mathbf{x})$ in (10). We claim that $c_\mu = 0$ for $v \geq 1$. There are two cases to consider.

(i) Suppose that $v \geq 1$ and $u = j-1$. To guarantee that $s_\mu(\mathbf{x})$ appears in the Schur expansion of $s_{(i)}(\mathbf{x}) s_{\lambda^i}(\mathbf{x})$, by the Pieri rule we need to append $A-m$ squares to the first row of λ^i and $B-2$ squares to the second row of λ^i. According to whether or not appending a square to the first column of λ^i, we will have $i = (A-m)+(B-2)+1$ or $i = (A-m)+(B-2)$. For the former case, we have $v = n-i+1$ and $\lambda^i = (m, 2^j, 1^{v-1})$; for the latter case, we have $v = n-i$ and $\lambda^i = (m, 2^j, 1^v)$. Thus, $c_\mu = 0$.

(ii) Suppose that $v \geq 1$ and $u = j$. By using similar arguments as in (i), to guarantee that $s_\mu(\mathbf{x})$ appears in the Schur expansion of $s_{(i)}(\mathbf{x}) s_{\lambda^i}(\mathbf{x})$, there are only two choices for i: $i = (A-m)+(B-2)+1$ or $i = (A-m)+(B-2)+2$. Precisely, we have $\lambda^i = (m, 2, 2^j, 1^{v+1})$ or $\lambda^i = (m, 2^j, 1^v)$. Thus, $c_\mu = 0$.

We proceed to determine the coefficient c_μ for μ being of the form $(A, B, 2^u)$. We would like to point out that for $0 \leq i \leq n-2$ the product $s_{(i)}(\mathbf{x}) s_{(m,2^j,1^{n-i})}(\mathbf{x})$ does not contribute such $s_\mu(\mathbf{x})$ due to the fact that the number of occurrences of 1 in partition $(m, 2^j, 1^{n-i})$ is $n-i \geq 2$, which implies that, for each $s_\mu(\mathbf{x})$ in the Schur expansion of the product $s_{(i)}(\mathbf{x}) s_{(m,2^j,1^{n-i})}(\mathbf{x})$, the partition μ must contain 1 as a part by the Pieri rule. It remains to consider the contribution of $s_{(i)}(\mathbf{x}) s_{(m,2^j,1^{n-i})}(\mathbf{x})$ when $i = n-1$ and $i = n$. Note that, by the Pieri rule,

$$s_{(n-1)}(\mathbf{x}) s_{(m,2^j,1)}(\mathbf{x}) = \sum_{a=0}^{\min\{m-2,n-2\}} s_{(m+n-2-a,2+a,2^j)}(\mathbf{x}) + \sum_{\rho} s_\rho(\mathbf{x})$$

and

$$s_{(n)}(\mathbf{x}) s_{(m,2^j)}(\mathbf{x}) = \sum_{a=0}^{\min\{m-2,n\}} s_{(m+n-a,2+a,2^{j-1})}(\mathbf{x}) + \sum_{a=0}^{\min\{m-2,n-2\}} s_{(m+n-2-a,2+a,2^j)}(\mathbf{x})$$

$$+\sum_{\tau} s_{\tau}(\mathbf{x}),$$

where ρ and τ are all possible partitions of size $m+n+2j$ that have 1 as a part. Hence, we have

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^i s_{(i)}(\mathbf{x}) s_{(m,2^j,1^{n-i})}(\mathbf{x}) \\
&\quad =(-1)^{n-1} \sum_{a=0}^{\min\{m-2,n-2\}} s_{(m+n-2-a,2+a,2^j)}(\mathbf{x}) \\
&\qquad + (-1)^n \Bigg(\sum_{a=0}^{\min\{m-2,n\}} s_{(m+n-a,2+a,2^{j-1})}(\mathbf{x}) + \sum_{a=0}^{\min\{m-2,n-2\}} s_{(m+n-2-a,2+a,2^j)}(\mathbf{x}) \Bigg) \\
&\quad =(-1)^n \sum_{a=0}^{\min\{m-2,n\}} s_{(m+n-a,2+a,2^{j-1})}(\mathbf{x}).
\end{aligned}$$

This completes the proof. □

The third symmetric function identity we are to prove is as follows.

Lemma 2.4. *For $n \geq 0$ and $m \geq 1$, we have*

$$\sum_{i=0}^{n}(-1)^i s_{(i)}(\mathbf{x}) s_{(m+1,1^{n-i})}(\mathbf{x}) = (-1)^n s_{(m+n+1)}(\mathbf{x}).$$

Proof. For $n=0$ the statement can be verified by plugging in $n=0$ and checking both sides. Hence we may assume from now on that $n \geq 1$. Now apply the Pieri rule to $s_{(i)}(\mathbf{x}) s_{(m+1,1^{n-i})}(\mathbf{x})$. For $i=n$, we have

$$s_{(n)}(\mathbf{x}) s_{(m+1)}(\mathbf{x}) = \sum_{a=0}^{\min\{m+1,n\}} s_{(m+1+n-a,a)}(\mathbf{x}).$$

For $1 \leq i \leq n-1$, there holds

$$s_{(i)}(\mathbf{x}) s_{(m+1,1^{n-i})}(\mathbf{x}) = \sum_{a=0}^{\min\{m,i-1\}} s_{(m+i-a,a+1,1^{n-i})}(\mathbf{x}) + \sum_{a=0}^{\min\{m,i\}} s_{(m+1+i-a,a+1,1^{n-i-1})}(\mathbf{x}),$$

and hence

$$\begin{aligned}
&\sum_{i=1}^{n-1}(-1)^i s_{(i)}(\mathbf{x}) s_{(m+1,1^{n-i})}(\mathbf{x}) \\
&\quad = \sum_{i=1}^{n-1}(-1)^i \sum_{a=0}^{\min\{m,i-1\}} s_{(m+i-a,a+1,1^{n-i})}(\mathbf{x}) + \sum_{i=1}^{n-1}(-1)^i \sum_{a=0}^{\min\{m,i\}} s_{(m+1+i-a,a+1,1^{n-i-1})}(\mathbf{x})
\end{aligned}$$

$$= -\sum_{i=0}^{n-2}(-1)^i \sum_{a=0}^{\min\{m,i\}} s_{(m+1+i-a,a+1,1^{n-i-1})}(\mathbf{x}) + \sum_{i=1}^{n-1}(-1)^i \sum_{a=0}^{\min\{m,i\}} s_{(m+1+i-a,a+1,1^{n-i-1})}(\mathbf{x})$$

$$= - s_{(m+1,1^n)}(\mathbf{x}) + (-1)^{n-1} \sum_{a=1}^{\min\{m+1,n\}} s_{(m+n+1-a,a)}(\mathbf{x}).$$

Therefore, we obtain

$$\sum_{i=0}^{n}(-1)^i s_{(i)}(\mathbf{x}) s_{(m+1,1^{n-i})}(\mathbf{x})$$

$$=(-1)^n s_{(n)}(\mathbf{x}) s_{(m+1)}(\mathbf{x}) + s_{(m+1,1^n)}(\mathbf{x}) + \sum_{i=1}^{n-1}(-1)^i s_{(i)}(\mathbf{x}) s_{(m+1,1^{n-i})}(\mathbf{x})$$

$$=(-1)^n \sum_{a=0}^{\min\{m+1,n\}} s_{(m+1+n-a,a)}(\mathbf{x}) + (-1)^{n-1} \sum_{a=1}^{\min\{m+1,n\}} s_{(m+n+1-a,a)}(\mathbf{x})$$

$$=(-1)^n s_{(m+n+1)}(\mathbf{x}),$$

as desired. This completes the proof. □

2.4 The Littlewood-Richardson rule

The Littlewood-Richardson rule gives a combinatorial interpretation of the Schur expansion of a skew Schur function. There are many ways to state the Littlewood-Richardson rule; see Stanley [23] and references therein. Here will use its Littlewood-Richardson tableaux version.

Recall that, given two partitions λ and μ with $\mu \subseteq \lambda$ (i.e., $\mu_i \le \lambda_i$ for all i), a semistandard Young tableau of shape λ/μ is defined to be an array $T = (T_{ij})$ of positive integers of shape λ/μ that is weakly increasing along every row and strictly increasing down every column. We say that T has type $\alpha = (\alpha_1, \alpha_2, \ldots)$, if T has α_i entries equal to i. The reverse reading word of T is the sequence of entries of T obtained by concatenating the rows of T from right to left, top to bottom. For example, the left tableau in Figure 1 is of type $(7, 2, 2, 2)$ and has the reverse reading word 1111 22 33 44111, and the right one is of type $(4, 5, 2, 2)$ and has the reverse reading word 2111 22 33 44221. We say that a word $a_1a_2 \cdots a_n$ is a lattice permutation if in any initial factor $a_1a_2 \cdots a_j$, the number of i's is at least as the number of $i+1$'s (for all i). A Littlewood-Richardson tableau is a semistandard Young tableau T such that its reverse reading word is a lattice permutation. One can verify that the left tableau in Figure 1 is a Littlewood-Richardson tableau, while the right one is not.

The well known Littlewood-Richardson rule is stated as follows.

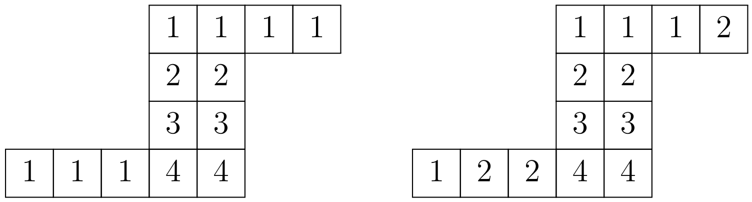

Figure 1: Two tableaux of shape $(7,5,5,5)/(3,3,3)$

Theorem 2.5 ([23, Section 7.10]). *If*

$$s_{\lambda/\mu}(\mathbf{x}) = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}(\mathbf{x}),$$

then the Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$ is equal to the number of Littlewood-Richardson tableaux of shape λ/μ and type ν.

3 Uniform matroids

Given a positive integer d and a nonnegative integer m, let $U_{m,d}$ be the uniform matroid of rank d on $m+d$ elements which admits an action of the symmetric group S_{m+d}. By using the generating functions and the Frobenius characteristic map, Gedeon, Proudfoot, and Young [9, Proposition 3.9] obtained a formula for computing the equivariant Kazhdan-Lusztig polynomial for equivariant uniform matroids, which could be stated as follows.

Theorem 3.1 ([9, Theorem 3.1]). *For any $m \geq 0$ and $d \geq 1$, we have*

$$P^{S_{m+d}}_{U_{m,d}}(t) = V_{(m+d)} + \sum_{j=1}^{\lfloor (d-1)/2 \rfloor} t^j \sum_{a=1}^{\min\{m,d-2j\}} V_{(m+d-2j-a+1,a+1,2^{j-1})}. \tag{11}$$

The main result of this section is as follows.

Theorem 3.2. *For any equivariant uniform matroid $S_{m+d} \curvearrowright U_{m,d}$ with $m \geq 0$ and $d \geq 1$, we have*

$$Q^{S_{m+d}}_{U_{m,d}}(t) = \sum_{j=0}^{\lfloor (d-1)/2 \rfloor} V_{(m+1,2^j,1^{d-2j-1})} t^j, \tag{12}$$

where $V_{(m+1,2^j,1^{d-2j-1})}$ vanishes if $(m+1,2^j,1^{d-2j-1})$ is not a valid partition.

In subsection 3.1 we first prove Theorem 3.2 for the case of $m = 0$. Then in subsection 3.2 we prove Theorem 3.2 for general m. In subsection 3.3 we use (12) to give a new proof of (11). Finally in subsection 3.4 we present a new formula for $P^{S_{m+d}}_{U_{m,d}}(t)$, which is, up to partition conjugation, a generalization of Lee, Nasr and Radcliffes combinatorial interpretation for the ordinary Kazhdan-Lusztig polynomials of uniform matroids to the equivariant setting.

3.1 Proof of Theorem 3.2 for $m = 0$

Given a positive integer n, let B_n denote the Boolean matroid of rank n, which is equipped with a natural action of the symmetric group S_n. Gedeon, Proudfoot, and Young [9] obtained a formula for computing the equivariant characteristic polynomials for equivariant Boolean matroids, which could be written as follows in terms of the plethystic notation.

Lemma 3.3 ([9, Proposition 3.9])**.** *For any equivariant Boolean matroid $S_n \curvearrowright B_n$ with $n \geq 1$, we have*

$$\mathrm{ch}\, H_{B_n}^{S_n}(t) = h_n[(t-1)X]. \tag{13}$$

Note that the equivariant Boolean matroid $S_n \curvearrowright B_n$ can be considered as the equivariant uniform matroid $S_n \curvearrowright U_{0,n}$. Thus, the case $m = 0$ of Theorem 3.2 is equivalent to the following statement.

Theorem 3.4. *For any equivariant Boolean matroid $S_n \curvearrowright B_n$ with $n \geq 1$, we have*

$$Q_{B_n}^{S_n}(t) = V_{(1^n)}.$$

Proof. Since the Frobenius characteristic map is an isomorphism between the ring of virtual representations of the symmetric group and the ring of symmetric functions, it suffices to show that, for $n \geq 1$,

$$\mathrm{ch}\, \hat{Q}_{B_n}^{S_n}(t) = (-1)^n e_n(\mathbf{x}).$$

Applying (6) to the equivariant Boolean matroid $S_n \curvearrowright B_n$, we obtain

$$\begin{aligned}\hat{Q}_{B_n}^{S_n}(t) &= \sum_{[F]\in L(B_n)/S_n} \mathrm{Ind}_{(S_n)_F}^{S_n}\left(t^{\mathrm{rk}\, F}\hat{Q}_{(B_n)^F}^{(S_n)_F}(t^{-1}) \otimes H_{(B_n)_F}^{(S_n)_F}(t)\right)\\ &= \sum_{i=0}^{n} \mathrm{Ind}_{S_i\times S_{n-i}}^{S_n}\left(t^i \hat{Q}_{B_i}^{S_i\times S_{n-i}}(t) \otimes H_{B_{n-i}}^{S_i\times S_{n-i}}(t)\right),\end{aligned}$$

which is equivalent to

$$\hat{Q}_{B_n}^{S_n}(t) = \sum_{i=0}^{n} \mathrm{Ind}_{S_i\times S_{n-i}}^{S_n}\left(t^i \hat{Q}_{B_i}^{S_i}(t) \otimes H_{B_{n-i}}^{S_{n-i}}(t)\right).$$

From (8) and (13) it follows that

$$\mathrm{ch}\, \hat{Q}_{B_n}^{S_n}(t) = \sum_{i=0}^{n} t^i \mathrm{ch}\, \hat{Q}_{B_i}^{S_i}(t^{-1}) \cdot \mathrm{ch}\, H_{B_{n-i}}^{S_{n-i}}(t) = \sum_{i=0}^{n} t^i \mathrm{ch}\, \hat{Q}_{B_i}^{S_i}(t^{-1}) \cdot h_{n-i}[(t-1)X].$$

Since $\deg \hat{Q}_{B_n}^{S_n}(t) < \frac{n}{2}$, it will be convenient for us to determine $\mathrm{ch}\, \hat{Q}_{B_n}^{S_n}(t)$ if we move the $i = n$ term of the second summation to the left hand side. If so, then the above equation is written as

$$\mathrm{ch}\, \hat{Q}_{B_n}^{S_n}(t) - t^n \mathrm{ch}\, \hat{Q}_{B_n}^{S_n}(t^{-1}) = \sum_{i=0}^{n-1} t^i \mathrm{ch}\, \hat{Q}_{B_i}^{S_i}(t^{-1}) \cdot h_{n-i}[(t-1)X]. \tag{14}$$

Now we proceed to prove that $\mathrm{ch}\,\hat{Q}_{B_n}^{S_n}(t) = (-1)^n e_n(\mathbf{x})$ by induction on the value of n. For the base case, assume $n = 1$. Now (14) turns out to be

$$\mathrm{ch}\,\hat{Q}_{B_1}^{S_1}(t) - t\mathrm{ch}\,\hat{Q}_{B_1}^{S_1}(t^{-1}) = h_1[(t-1)X] = (t-1)h_1(\mathbf{x}).$$

Since $\hat{Q}_{B_1}^{S_1}(t)$ is of degree zero, we have

$$\mathrm{ch}\,\hat{Q}_{B_1}^{S_1}(t) = -h_1(\mathbf{x}) = -e_1(\mathbf{x}).$$

Assume the assertion for $n-1$, namely

$$\mathrm{ch}\,\hat{Q}_{B_i}^{S_i}(t) = (-1)^i e_i(\mathbf{x}) = (-1)^i e_i(X) \text{ for } 0 \le i < n.$$

From (14) we have

$$\begin{aligned}\mathrm{ch}\,\hat{Q}_{B_n}^{S_n}(t) - t^n\mathrm{ch}\,\hat{Q}_{B_n}^{S_n}(t^{-1}) &= \sum_{i=0}^{n-1} t^i(-1)^i e_i(X) h_{n-i}[(t-1)X] \\ &= \sum_{i=0}^{n} t^i(-1)^i e_i(X) h_{n-i}[(t-1)X] - t^n(-1)^n e_n[X].\end{aligned}$$

Recall that $t^i e_i[X] = e_i[tX]$ for $0 \le i \le n$, which tells us that

$$\mathrm{ch}\,\hat{Q}_{B_n}^{S_n}(t) - t^n\mathrm{ch}\,\hat{Q}_{B_n}^{S_n}(t^{-1}) = (-1)^n \sum_{i=0}^{n} (-1)^{n-i} e_i[tX] h_{n-i}[(t-1)X] - t^n(-1)^n e_n[X].$$

Taking $E = tX$ and $F = (t-1)X$ in Lemma 2.1 leads to

$$\mathrm{ch}\,\hat{Q}_{B_n}^{S_n}(t) - t^n\mathrm{ch}\,\hat{Q}_{B_n}^{S_n}(t^{-1}) = (-1)^n e_n[X] - t^n(-1)^n e_n[X].$$

In view of $\deg \hat{Q}_{B_n}^{S_n}(t) < \frac{n}{2}$, we find that

$$\mathrm{ch}\,\hat{Q}_{B_n}^{S_n}(t) = (-1)^n e_n[X] = (-1)^n e_n(\mathbf{x}),$$

as desired. This completes the proof. □

3.2 Proof of Theorem 3.2 for general m

We proceed to prove Theorem 3.2 for general m.

Proof of Theorem 3.2. It suffices to show for $0 \le j \le \lfloor (d-1)/2 \rfloor$, the coefficient of t^j in $\mathrm{ch}\,\hat{Q}_{U_{m,d}}^{S_{m+d}}(t)$ is

$$(-1)^d s_{(m+1,2^j,1^{d-2j-1})}(\mathbf{x}).$$

By applying (6) to the equivariant uniform matroid $S_{m+d} \curvearrowright U_{m,d}$, we obtain

$$\hat{Q}^{S_{m+d}}_{U_{m,d}}(t) = \sum_{[F]\in L(U_{m,d})/S_{m+d}} \mathrm{Ind}^{S_{m+d}}_{(S_{m+d})_F}\Big(t^{\mathrm{rk}\,(U_{m,d})_F}\hat{Q}^{(S_{m+d})_F}_{(U_{m,d})^F}(t^{-1})\otimes H^{(S_{m+d})_F}_{(U_{m,d})_F}(t)\Big). \tag{15}$$

Observe that for any flat F of $L(U_{m,d})$ with $\mathrm{rk}\,(U_{m,d})^F = i < d$ we have

$$(U_{m,d})^F \cong B_i, \qquad (U_{m,d})_F \cong U_{m,d-i}, \qquad (S_{m+d})_F \cong S_i \times S_{m+d-i}.$$

Note that there is only one flat of rank d which is in fact the ground set E, for which there holds

$$(U_{m,d})^E = U_{m,d}, \qquad (U_{m,d})_E \cong U_{0,0}, \qquad (S_{m+d})_E \cong S_{m+d},$$

and hence $H^{(S_{m+d})_E}_{(U_{m,d})_E}(t)$ is just the trivial representation of S_{m+d}. Thus, (15) can be rewritten as

$$\hat{Q}^{S_{m+d}}_{U_{m,d}}(t) = \sum_{i=0}^{d-1}\mathrm{Ind}^{S_{m+d}}_{S_i\times S_{m+d-i}}\Big(t^i\hat{Q}^{S_i}_{B_i}(t^{-1})\otimes H^{S_{m+d-i}}_{U_{m,d-i}}(t)\Big) + t^d\hat{Q}^{S_{m+d}}_{U_{m,d}}(t^{-1}). \tag{16}$$

Applying the Frobenius characteristic map on both sides leads to

$$\mathrm{ch}\,\hat{Q}^{S_{m+d}}_{U_{m,d}}(t) - t^d\mathrm{ch}\,\hat{Q}^{S_{m+d}}_{U_{m,d}}(t^{-1}) = \sum_{i=0}^{d-1} t^i\mathrm{ch}\,\hat{Q}^{S_i}_{B_i}(t^{-1})\cdot\mathrm{ch}\,H^{S_{m+d-i}}_{U_{m,d-i}}(t).$$

By a result due to Gedeon, Proudfoot, and Young [9, Proposition 3.9], we know

$$\begin{aligned}\mathrm{ch}\,H^{S_{m+d-i}}_{U_{m,d-i}}(t) = \sum_{j=0}^{d-i-1}(-1)^j t^{d-i-j}\Big(s_{(m+d-i-j,1^j)}(\mathbf{x}) + s_{(m+d-i-j+1,1^{j-1})}(\mathbf{x})\Big)\\ + (-1)^{d-i}s_{(m+1,1^{d-i-1})}(\mathbf{x}).\end{aligned} \tag{17}$$

Meanwhile, by Theorem 3.4, we have

$$\mathrm{ch}\,\hat{Q}^{S_i}_{B_i}(t^{-1}) = (-1)^i s_{(1^i)}(\mathbf{x}). \tag{18}$$

Substituting (17) and (18) into (16), we obtain

$$\begin{aligned}\mathrm{ch}\,\hat{Q}^{S_{m+d}}_{U_{m,d}}(t) - t^d\mathrm{ch}\,\hat{Q}^{S_{m+d}}_{U_{m,d}}(t^{-1}) = \sum_{i=0}^{d-1}(-1)^d t^i s_{(1^i)}(\mathbf{x})s_{(m+1,1^{d-i-1})}(\mathbf{x})\\ + \sum_{i=0}^{d-1}\sum_{j=0}^{d-i-1}(-1)^{i+j}t^{d-j}s_{(1^i)}(\mathbf{x})\Big(s_{(m+d-i-j,1^j)}(\mathbf{x}) + s_{(m+d-i-j+1,1^{j-1})}(\mathbf{x})\Big).\end{aligned}$$

Substituting $d-j$ for j in the second summation and then interchanging the order of the summation, we obtain

$$\begin{aligned}\mathrm{ch}\,\hat{Q}^{S_{m+d}}_{U_{m,d}}(t) - t^d\mathrm{ch}\,\hat{Q}^{S_{m+d}}_{U_{m,d}}(t^{-1}) &= \sum_{i=0}^{d-1}(-1)^d t^i s_{(1^i)}(\mathbf{x})s_{(m+1,1^{d-i-1})}(\mathbf{x}) \\ &+ \sum_{i=0}^{d-1}\sum_{j=i+1}^{d}(-1)^{i+d-j}t^j s_{(1^i)}(\mathbf{x})\Big(s_{(m-i+j,1^{d-j})}(\mathbf{x}) + s_{(m-i+j+1,1^{d-j-1})}(\mathbf{x})\Big) \\ &= \sum_{j=0}^{d-1}(-1)^d t^j s_{(1^j)}(\mathbf{x})s_{(m+1,1^{d-j-1})}(\mathbf{x}) \\ &+ \sum_{j=1}^{d}\sum_{i=0}^{j-1}(-1)^{i+d-j}t^j s_{(1^i)}(\mathbf{x})\Big(s_{(m-i+j,1^{d-j})}(\mathbf{x}) + s_{(m-i+j+1,1^{d-j-1})}(\mathbf{x})\Big).\end{aligned}$$

Note that the degree of $\mathrm{ch}\,\hat{Q}^{S_{m+d}}_{U_{m,d}}(t)$ is strictly less than $\frac{d}{2}$ and hence the degree of lowest term in $t^d\mathrm{ch}\,\hat{Q}^{S_{m+d}}_{U_{m,d}}(t^{-1})$ is strictly greater than $\frac{d}{2}$.

Thus,

$$[t^0]\mathrm{ch}\,\hat{Q}^{S_{m+d}}_{U_{m,d}}(t) = (-1)^d s_{(m+1,1^{d-1})}(\mathbf{x}). \tag{19}$$

Also, for $1 \le j < d/2$, the coefficient of t^j in $\mathrm{ch}\,\hat{Q}^{S_{m+d}}_{U_{m,d}}(t)$ is

$$(-1)^d s_{(1^j)}(\mathbf{x})s_{(m+1,1^{d-j-1})}(\mathbf{x}) + A(j), \tag{20}$$

where

$$\begin{aligned}A(j) &= \sum_{i=0}^{j-1}(-1)^{i+d-j}s_{(1^i)}(\mathbf{x})\Big(s_{(m-i+j,1^{d-j})}(\mathbf{x}) + s_{(m-i+j+1,1^{d-j-1})}(\mathbf{x})\Big) \\ &= \sum_{i=0}^{j-1}(-1)^{i+d-j}s_{(1^i)}(\mathbf{x})s_{(m-i+j,1^{d-j})}(\mathbf{x}) + \sum_{i=0}^{j-1}(-1)^{i+d-j}s_{(1^i)}(\mathbf{x})s_{(m-i+j+1,1^{d-j-1})}(\mathbf{x}) \\ &= \sum_{i=0}^{j-1}(-1)^{i+d-j}s_{(1^i)}(\mathbf{x})s_{(m-i+j,1^{d-j})}(\mathbf{x}) - \sum_{i=-1}^{j-2}(-1)^{i+d-j}s_{(1^{i+1})}(\mathbf{x})s_{(m-i+j,1^{d-j-1})}(\mathbf{x}) \\ &= \sum_{i=0}^{j-2}(-1)^{i+d-j}\Big(s_{(1^i)}(\mathbf{x})s_{(m-i+j,1^{d-j})}(\mathbf{x}) - s_{(1^{i+1})}(\mathbf{x})s_{(m-i+j,1^{d-j-1})}(\mathbf{x})\Big) \\ &\qquad + (-1)^{d-1}s_{(1^{j-1})}(\mathbf{x})s_{(m+1,1^{d-j})}(\mathbf{x}) + (-1)^{d-j}s_{(m+1+j,1^{d-j-1})}(\mathbf{x}).\end{aligned}$$

For $m \ge 1$, $1 \le j < \frac{d}{2}$ and $0 \le i \le j-2$, one can verify that $m-i+j \ge 2$ and $(d-j)-i > 2$. By applying Lemma 2.2, we obtain

$$s_{(1^i)}(\mathbf{x})s_{(m-i+j,1^{d-j})}(\mathbf{x}) - s_{(1^{i+1})}(\mathbf{x})s_{(m-i+j,1^{d-j-1})}(\mathbf{x})$$

$$= -s_{(m-i+j+1,2^i,1^{d-j-i-1})}(\mathbf{x}) - s_{(m-i+j,2^{i+1},1^{d-j-i-2})}(\mathbf{x}),$$

from which it follows that

$$\begin{aligned}
A(j) =& \sum_{i=0}^{j-2}(-1)^{i+d-j}\Big(- s_{(m-i+j+1,2^i,1^{d-j-i-1})}(\mathbf{x}) - s_{(m-i+j,2^{i+1},1^{d-j-i-2})}(\mathbf{x})\Big)\\
&+ (-1)^{d-1}s_{(1^{j-1})}(\mathbf{x})s_{(m+1,1^{d-j})}(\mathbf{x}) + (-1)^{d-j}s_{(m+1+j,1^{d-j-1})}(\mathbf{x})\\
=& - \sum_{i=0}^{j-2}(-1)^{i+d-j}s_{(m-i+j+1,2^i,1^{d-j-i-1})}(\mathbf{x}) + \sum_{i=1}^{j-1}(-1)^{i+d-j}s_{(m-i+j+1,2^i,1^{d-j-i-1})}(\mathbf{x})\\
&+ (-1)^{d-1}s_{(1^{j-1})}s_{(m+1,1^{d-j})}(\mathbf{x}) + (-1)^{d-j}s_{(m+1+j,1^{d-j-1})}(\mathbf{x})\\
=& - (-1)^{d-j}s_{(m+j+1,1^{d-j-1})}(\mathbf{x}) + (-1)^{d-1}s_{(m+2,2^{j-1},1^{d-2j})}(\mathbf{x})\\
&+ (-1)^{d-1}s_{(1^{j-1})}(\mathbf{x})s_{(m+1,1^{d-j})}(\mathbf{x}) + (-1)^{d-j}s_{(m+1+j,1^{d-j-1})}(\mathbf{x})\\
=&(-1)^{d-1}s_{(m+2,2^{j-1},1^{d-2j})}(\mathbf{x}) + (-1)^{d-1}s_{(1^{j-1})}(\mathbf{x})s_{(m+1,1^{d-j})}(\mathbf{x}).
\end{aligned}$$

Therefore, in view of (20), for $1 \le j < d/2$ the coefficient of t^j in $\mathrm{ch}\,\hat{Q}_{U_{m,d}}^{S_{m+d}}(t)$ is

$$\begin{aligned}
&(-1)^{d-1}s_{(m+2,2^{j-1},1^{d-2j})}(\mathbf{x}) + (-1)^{d-1}s_{(1^{j-1})}(\mathbf{x})s_{(m+1,1^{d-j})}(\mathbf{x}) + (-1)^{d}s_{(1^j)}(\mathbf{x})s_{(m+1,1^{d-j-1})}(\mathbf{x})\\
&= (-1)^{d-1}s_{(m+2,2^{j-1},1^{d-2j})}(\mathbf{x}) + (-1)^{d-1}\Big(s_{(1^{j-1})}(\mathbf{x})s_{(m+1,1^{d-j})}(\mathbf{x}) - s_{(1^j)}(\mathbf{x})s_{(m+1,1^{d-j-1})}(\mathbf{x})\Big)\\
&= (-1)^{d-1}s_{(m+2,2^{j-1},1^{d-2j})}(\mathbf{x}) + (-1)^{d-1}\Big(- s_{(m+2,2^{j-1},1^{d-2j})}(\mathbf{x}) - s_{(m+1,2^j,1^{d-2j-1})}(\mathbf{x})\Big)\\
&= (-1)^{d}s_{(m+1,2^j,1^{d-2j-1})}(\mathbf{x}),
\end{aligned} \tag{21}$$

where the second last equality is obtained by applying Lemma 2.2 with the fact that $m+1 \ge 2$ and $(d-j)-(j-1) \ge 2$. Combining (19) and (21), we obtain

$$[t^j]\mathrm{ch}\,\hat{Q}_{U_{m,d}}^{S_{m+d}}(t) = (-1)^{d}s_{(m+1,2^j,1^{d-2j-1})}(\mathbf{x}), \text{ for } 0 \le j < d/2,$$

as desired. This completes the proof. □

3.3 The equivariant Kazhdan-Lusztig polynomials

Note that the proof of Theorem 3.2 only relies on the evaluation of the equivariant characteristic polynomials for uniform matroids and the inverse Kazhdan-Lusztig polynomials for Boolean matroids. In fact, Theorem 3.1 can be proved in the same manner, assuming that the following result has been proved.

Lemma 3.5. *For any equivariant Boolean matroid $S_n \curvearrowright B_n$ with $n \ge 0$, we have*

$$P_{B_n}^{S_n}(t) = V_{(n)}.$$

We would like to point out that Lemma 3.5 is a special case of Theorem 3.1. Since this lemma can be proved following the lines of the proof of Theorem 3.4, we omit its proof here. We proceed to prove Theorem 3.1.

Proof of Theorem 3.1. Applying (7) to the equivariant uniform matroid $S_{m+d} \curvearrowright U_{m,d}$, we obtain

$$\sum_{[F]\in L(U_{m,d})/S_{m+d}} \mathrm{Ind}_{W_F}^{S_{m+d}} \left(P_{(U_{m,d})^F}^{(S_{m+d})_F}(t) \otimes \hat{Q}_{(U_{m,d})_F}^{(S_{m+d})_F}(t) \right) = 0.$$

Recalling the previous arguments on the equivalence of flats in the proof of Theorem 3.2 and applying the Frobenius characteristic map lead to

$$\sum_{i=0}^{d-1} \mathrm{ch}\, P_{B_i}^{S_i}(t) \cdot \mathrm{ch}\, \hat{Q}_{U_{m,d-i}}^{S_{m+d-i}}(t) + \mathrm{ch}\, P_{U_{m,d}}^{S_{m+d}}(t) = 0. \tag{22}$$

Lemma 3.5 tells us that, for $0 \le i \le d-1$,

$$\mathrm{ch}\, P_{B_i}^{S_i}(t) = h_i(\mathbf{x}) = s_{(i)}(\mathbf{x}).$$

Meanwhile, from Theorem 3.2 it follows that

$$\mathrm{ch}\, \hat{Q}_{U_{m,d-i}}^{S_{m+d-i}}(t) = (-1)^{d-i} \sum_{j=0}^{\lfloor (d-i-1)/2 \rfloor} s_{(m+1,2^j,1^{d-i-2j-1})}(\mathbf{x}) t^j.$$

In view of (22), we get

$$\begin{aligned}
\mathrm{ch}\, P_{U_{m,d}}^{S_{m+d}}(t) &= -\sum_{i=0}^{d-1} s_{(i)}(\mathbf{x}) \cdot (-1)^{d-i} \sum_{j=0}^{\lfloor (d-i-1)/2 \rfloor} s_{(m+1,2^j,1^{d-i-2j-1})}(\mathbf{x}) t^j \\
&= -\sum_{i=0}^{d-1} \sum_{j=0}^{\lfloor (d-i-1)/2 \rfloor} (-1)^{d-i} s_{(i)}(\mathbf{x}) s_{(m+1,2^j,1^{d-i-2j-1})}(\mathbf{x}) t^j \\
&= -\sum_{j=0}^{\lfloor (d-1)/2 \rfloor} \sum_{i=0}^{d-1-2j} (-1)^{d-i} s_{(i)}(\mathbf{x}) s_{(m+1,2^j,1^{d-i-2j-1})}(\mathbf{x}) t^j,
\end{aligned}$$

where the last equality is obtained by interchanging the order of the summation.

Hence, for $0 \le j \le \lfloor (d-1)/2 \rfloor$ the coefficient of t^j in $\mathrm{ch}\, P_{U_{m,d}}^{S_{m+d}}(t)$ is

$$[t^j]\mathrm{ch}\, P_{U_{m+d}}^{S_{m,d}}(t) = (-1)^{d+1} \sum_{i=0}^{d-1-2j} (-1)^i s_{(i)}(\mathbf{x}) s_{(m+1,2^j,1^{d-i-2j-1})}(\mathbf{x}). \tag{23}$$

To simplify the summation on the right hand side of (23), we first consider the constant term of $\mathrm{ch}\, P_{U_{m,d}}^{S_{m+d}}(t)$, for which we have

$$[t^0]\mathrm{ch}\, P_{U_{m,d}}^{S_{m+d}}(t) = (-1)^{d+1} \sum_{i=0}^{d-1} (-1)^i s_{(i)}(\mathbf{x}) s_{(m+1,1^{d-i-1})}(\mathbf{x})$$

$$= (-1)^{d+1} \cdot (-1)^{d-1} s_{(m+d)}(\mathbf{x})$$
$$= s_{(m+d)}(\mathbf{x}),$$

where the second last equality is obtained by letting $n = d - 1$ in Lemma 2.4. For $1 \le j \le \lfloor (d-1)/2 \rfloor$, letting $n = d - 1 - 2j$ and $m = m + 1$ in Lemma 2.3, we find that

$$\begin{aligned}
[t^j]\mathrm{ch}\, P_{U_{m,d}}^{S_{m+d}}(t) &= (-1)^{d+1} \cdot (-1)^{d-1-2j} \sum_{a=0}^{\min\{m-1,d-1-2j\}} s_{(m+1+d-1-2j-a,2+a,2^{j-1})}(\mathbf{x}) \\
&= \sum_{a=0}^{\min\{m-1,d-1-2j\}} s_{(m+d-2j-a,2+a,2^{j-1})}(\mathbf{x}) \\
&= \sum_{a=1}^{\min\{m,d-2j\}} s_{(m+d-2j-a+1,a+1,2^{j-1})}(\mathbf{x}).
\end{aligned}$$

To summarize, we get the desired (11). The proof is complete. □

3.4 A generalization of Lee, Nasr and Radcliffe's formula

Lee, Nasr and Radcliffe [13, 14] considered the combinatorial interpretation for the Kazhdan-Lusztig polynomials of ρ-removed uniform matroids and sparse paving matroids. In particular, they gave the following combinatorial interpretation for the Kazhdan-Lusztig polynomials $P_{U_{m,d}}(t)$ for uniform matroids $U_{m,d}$.

Theorem 3.6 ([13, Theorem 2])**.** *For any $m, d \ge 1$, suppose*

$$P_{U_{m,d}}(t) = \sum_{j=0}^{\lfloor (d-1)/2 \rfloor} c_j t^j.$$

Then for each $0 \le j \le \lfloor (d-1)/2 \rfloor$ the coefficient c_j equals the number of standard Young tableaux of shape $\big(m + d - 2j, (d - 2j + 1)^j\big)/\big((d - 2j - 1)^j\big)$.

We would like to point out that Lee, Nasr and Radcliffe actually used the conjugate partition of $\big(m + d - 2j, (d - 2j + 1)^j\big)/\big((d - 2j - 1)^j\big)$ to express c_j. Next we will show that the equivariant Kazhdan-Lusztig polynomial $P_{U_{m,d}}^{S_{m+d}}(t)$ admits a more compact form than the formula given in Theorem 3.1, which implies that the above presentation is more natural in some sense.

Theorem 3.7. *For any $m, d \ge 1$, we have*

$$P_{U_{m,d}}^{S_{m+d}}(t) = \sum_{j=0}^{\lfloor (d-1)/2 \rfloor} V_{\big(m+d-2j,(d-2j+1)^j\big)/\big((d-2j-1)^j\big)} t^j.$$

Proof. By Theorem 3.1, it suffices to show that, for any $m, d, j \geq 1$, we have

$$s_{\left(m+d-2j,(d-2j+1)^j\right)/\left((d-2j-1)^j\right)}(\mathbf{x}) = \sum_{a=1}^{\min\{m,d-2j\}} s_{(m+d-2j-a+1,a+1,2^{j-1})}(\mathbf{x}).$$

Note that, for $\lambda = \left(m+d-2j, (d-2j+1)^j\right)$ and $\mu = \left((d-2j-1)^j\right)$, since the number of cells in the i-th row of λ/μ is

$$\lambda_i - \mu_i = \begin{cases} m+1, & \text{for } i = 1; \\ 2, & \text{for } 2 \leq i \leq j; \\ d-2j+1, & \text{for } i = j+1, \end{cases}$$

there is a subdiagram of a straight shape $\nu = (m+1, 2^j)$ such that all other cells of λ/μ are to the left of ν and form a single row partition ρ. For example, for $d = 10$ and $m = j = 3$, the partition ν is composed of the cells occupied by the bold numbers in Figure 2, and the partition ρ is composed of other left cells.

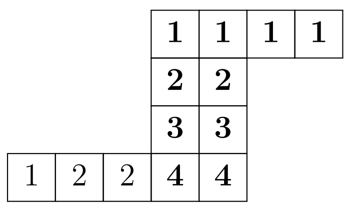

Figure 2: The Young diagrams of λ/μ, ν and ρ

In order to get a Littlewood-Richardson tableau T of shape λ/μ, there is only one way to fill the cells of ν with positive integers, namely, the i-th row of ν is filled with i's. Moreover, the cells of ρ can only be filled with 1's and 2's. Suppose that the number of 2's filled in ρ is $a-1$, and hence the number of 1's is $d-2j-a$. According to the lattice permutation condition of T, we have $1 \leq a \leq d-2j$ and $m+1 \geq a-1+2 = a+1$, from which it follows that $1 \leq a \leq \min\{m, d-2j\}$. This means that the type τ_a of T can only be $(m+d-2j-a+1, a+1, 2^{j-1})$ for some $1 \leq a \leq \min\{m, d-2j\}$, and for such a fixed a there is only one Littlewood-Richardson tableau of shape λ/μ and type τ_a. This completes the proof. □

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