Set-valued Rothe Tableaux and Grothendieck Polynomials

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Abstract

The notion of set-valued Young tableaux was introduced by Buch in his study of the Littlewood-Richardson rule for stable Grothendieck polynomials. Knutson, Miller and Yong showed that the double Grothendieck polynomials of 2143-avoiding permutations can be generated by flagged set-valued Young tableaux. In this paper, we introduce the structure of set-valued Rothe tableaux of permutations. Given the Rothe diagram $D(w)$ of a permutation w, a set-valued Rothe tableau of shape $D(w)$ is a filling of finite nonempty subsets of positive integers into the squares of $D(w)$ such that the rows are weakly decreasing and the columns are strictly increasing. We show that the double Grothendieck polynomials of 1432-avoiding permutations can be generated by flagged set-valued Rothe tableaux. When restricted to 321-avoiding permutations, our formula specializes to the tableau formula for double Grothendieck polynomials due to Matsumura. Employing the properties of tableau complexes given by Knutson, Miller and Yong, we obtain two alternative tableau formulas for the double Grothendieck polynomials of 1432-avoiding permutations.

Keywords: Grothendieck polynomial, Schubert polynomial, set-valued tableau, tableau complex

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1 Introduction

Let S_n denote the set of permutations of $\{1, 2, \ldots, n\}$. The double Grothendieck polynomials $\mathfrak{G}_w(\mathbf{x}, \mathbf{y})$ indexed by permutations $w \in S_n$ were introduced by Lascoux and Schützenberger [22] as polynomial representatives of the equivariant K-theory classes of structure sheaves of Schubert varieties in the flag manifold. For combinatorial constructions of Grothendieck polynomials, see for example [8, 12, 16, 17, 23, 33].

On the other hand, tableau formulas for $\mathfrak{G}_w(\mathbf{x},\mathbf{y})$ have been found for specific families of permutations. Based on the algebraic geometry of matrix Schubert varieties, Knutson, Miller and Yong [19] showed that the double Grothendieck polynomials for 2143-avoiding permutations (also called vexillary permutations) can be generated by flagged set-valued Young tableaux. A permutation $w = w_1w_2\cdots w_n \in S_n$ is 2143-avoiding if there do not exist indices $1 \le i_1 < i_2 < i_3 < i_4 \le n$ such that $w_{i_2} < w_{i_1} < w_{i_4} < w_{i_3}$. Set-valued Young tableaux were introduced by Buch [7] in his study of the Littlewood-Richardson rule for stable Grothendieck polynomials. As a consequence of the Knutson-Miller-Yong formula, the stable limit of the double Grothendieck polynomial for a 2143-avoiding permutation is the factorial Grothendieck polynomial studied by McNamara [28]. Moreover, when restricted to semistandard Young tableaux (namely, set-valued Young tableaux with each set containing a single integer), the Knutson-Miller-Yong formula specializes to the tableau formula for the Schubert polynomial $\mathfrak{S}_w(\mathbf{x})$ of 2143-avoiding permutations due to Wachs [32].

Knutson, Miller and Yong [18] also introduced the structure of tableau complexes, and used the tool of commutative algebra to deduce another two tableau formulas for double Grothendieck polynomials of 2143-avoiding permutations in terms of semistandard Young tableaux and limit set-valued Young tableaux, respectively.

Recently, Matsumura [25] gave a tableau formula of $\mathfrak{G}_w(\mathbf{x},\mathbf{y})$ for 321-avoiding permutations. A permutation $w = w_1w_2\cdots w_n$ is 321-avoiding if there do not exist indices $i_1 < i_2 < i_3$ such that $w_{i_1} > w_{i_2} > w_{i_3}$. To a 321-avoiding permutation w, one can associate a skew Young diagram, denoted $\sigma(w)$, see, for example, Billey, Jockusch and Stanley [6]. Matsumura [25] showed that for a 321-avoiding permutation w, $\mathfrak{G}_w(\mathbf{x},\mathbf{y})$ can be generated by flagged set-valued tableaux of shape $\sigma(w)$. This formula generalizes the tableau formula for the single Schubert polynomial $\mathfrak{S}_w(\mathbf{x})$ of a 321-avoiding permutation due to Billey, Jockusch and Stanley [6], as well as the tableau formula for the single Grothendieck polynomial $\mathfrak{G}_w(\mathbf{x})$ of a 321-avoiding permutation given by Anderson, Chen and Tarasca [1]. When restricted to semistandard Young tableaux, it specializes to the formula for the double Schubert polynomial $\mathfrak{S}_w(\mathbf{x},\mathbf{y})$ of a 321-avoiding permutation obtained by Chen, Yan and Yang [9].

In this paper, we introduce the structure of set-valued Rothe tableaux. Let $D(w)$ be the Rothe diagram of a permutation w. A set-valued Rothe tableau of shape $D(w)$ is a filling of finite nonempty subsets of positive integers into the squares of $D(w)$ such that the sets in each row are *weakly decreasing* and the sets in each column are *strictly increasing*. As defined by Buch [7], for two finite nonempty sets A and B of positive integers, $A < B$ if $\max A < \min B$, and $A \le B$ if $\max A \le \min B$. When w is a 321-avoiding permutation, $D(w)$ is a skew Young diagram after a reflection about a vertical line [6], and in this case, a set-valued Rothe tableau reduces to a set-valued Young tableau after a reflection about a vertical line.

Our main aim is to establish set-valued Rothe tableau formulas of $\mathfrak{G}_w(\mathbf{x},\mathbf{y})$ for a new family of permutations, namely, 1432-avoiding permutations. A permutation $w =$

$w_1w_2\cdots w_n$ is 1432-avoiding if there do not exist indices $i_1 < i_2 < i_3 < i_4$ such that $w_{i_1} < w_{i_4} < w_{i_3} < w_{i_2}$. Note that Stankova [30] proved that 1432-avoiding permutations in S_n have the same number as 2143-avoiding permutations in S_n. Note also that Gao [15] recently showed that any two RC-graphs of a permutation w are connected by simple ladder moves if and only if w is 1432-avoiding.

The Rothe diagram $D(w)$ of a permutation $w \in S_n$ can be viewed as a geometric configuration of the inversions of w. Consider an $n \times n$ square grid, where we use (i, j) to denote the square in row i and column j. Here the rows are numbered from top to bottom, and the columns are numbered from left to right. For $1 \le i \le n$, put a dot in the square (i, w_i). The Rothe diagram $D(w)$ is obtained by deleting the squares with dots, as well as the squares to the right or below each dot. For example, Figure 1.1(a) is the Rothe diagram of $w = 426315$.

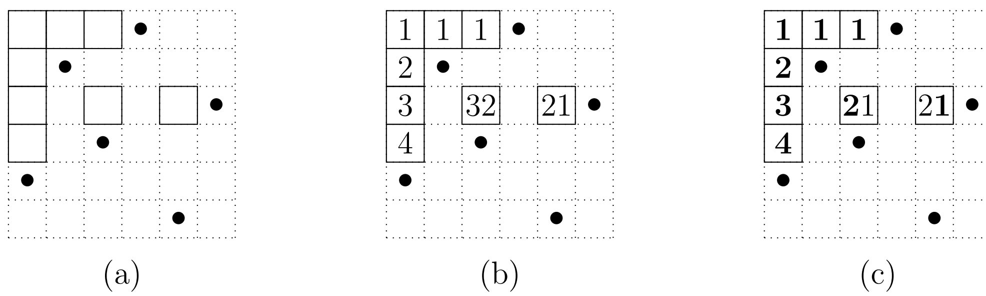

Figure 1.1: (a) The Rothe diagram $D(w)$, (b) a set-valued Rothe tableau, (c) a limit set-valued Rothe tableau.

As aforementioned, a set-valued Rothe tableau of shape $D(w)$ is a filling of finite nonempty subsets of positive integers into the squares of $D(w)$ such that the rows are weakly decreasing and the columns are strictly increasing. For example, Figure 1.1(b) depicts a set-valued Rothe tableau for $w = 426315$. We say that a set-valued Rothe tableau is flagged by a vector $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ of nonnegative integers if every integer in row i does not exceed f_i. Let $\mathrm{SVRT}(w, \mathbf{f})$ denote the set of set-valued Rothe tableaux of shape $D(w)$ flagged by $\mathbf{f}$.

For a set-valued Rothe tableau T and a square $B = (i, j)$ of T, we use $T(B)$ or $T(i, j)$ to denote the set filled in B. Write

$$|T| = \sum_{B \in D(w)} |T(B)|.$$

Let $\ell(w)$ denote the length of w, which equals the number of inversions of w:

$$\ell(w) = |\{(w_i, w_j) \mid 1 \le i < j \le n, w_i > w_j\}|.$$

It is easy to check that $\ell(w) = |D(w)|$. For a square (i, j) of $D(w)$, define

$$m_{ij}(w) = |\{(i, k) \in D(w) \mid k \le j\}|. \tag{1.1}$$

Throughout this paper, we use the following specific flag

$$\mathbf{f}_0 = (1, 2, \ldots, n).$$

Our main result is a tableau formula of $\mathfrak{G}_w(\mathbf{x}, \mathbf{y})$ for 1432-avoiding permutations. In fact, we obtain a formula for the β version $\mathfrak{G}_w^{(\beta)}(\mathbf{x}, \mathbf{y})$ of $\mathfrak{G}_w(\mathbf{x}, \mathbf{y})$, which specifies to $\mathfrak{G}_w(\mathbf{x}, \mathbf{y})$ by setting $\beta = -1$.

Theorem 1.1 *For a permutation $w \in S_n$, we have the following equivalent statements.*

(1) w is a 1432-avoiding permutation.

(2) $\mathfrak{G}_w^{(\beta)}(\mathbf{x}, \mathbf{y})$ has the following set-valued Rothe tableau formula:

$$\mathfrak{G}_w^{(\beta)}(\mathbf{x}, \mathbf{y}) = \sum_{T \in \mathrm{SVRT}(w, \mathbf{f}_0)} \beta^{|T| - \ell(w)} \prod_{(i,j) \in D(w)} \prod_{t \in T(i,j)} (x_t \oplus y_{m_{ij}(w)+i-t}), \tag{1.2}$$

where, for two variables x and y,

$$x \oplus y = x + y + \beta xy.$$

Setting $\beta = -1$ in (1.2), we are led to a tableau formula for ordinary double Grothendieck polynomials of 1432-avoiding permutations.

Corollary 1.2 *Let $w \in S_n$ be a 1432-avoiding permutation. Then*

$$\mathfrak{G}_w(\mathbf{x}, \mathbf{y}) = \sum_{T \in \mathrm{SVRT}(w, \mathbf{f}_0)} (-1)^{|T| - \ell(w)} \prod_{(i,j) \in D(w)} \prod_{t \in T(i,j)} (x_t + y_{m_{ij}(w)+i-t} - x_t y_{m_{ij}(w)+i-t}).$$

The double Schubert polynomial $\mathfrak{S}_w(\mathbf{x}, \mathbf{y})$ [21, 24] can be obtained from $\mathfrak{G}_w(\mathbf{x}, \mathbf{y})$ by extracting the monomials of the lowest degree and then replacing each y_i by $-y_i$, or equivalently, by setting $\beta = 0$ in $\mathfrak{G}_w^{(\beta)}(\mathbf{x}, \mathbf{y})$ and then replacing each y_i by $-y_i$. The single Schubert $\mathfrak{S}_w(\mathbf{x})$ is obtained from $\mathfrak{S}_w(\mathbf{x}, \mathbf{y})$ by setting $y_i = 0$ for all i. For combinatorial models of Schubert polynomials, see for example [2, 3, 4, 5, 6, 11, 13, 14, 20, 34, 35].

A single-valued Rothe tableau is a set-valued Rothe tableau with each set containing a single integer. Let $\mathrm{SRT}(w, \mathbf{f})$ denote the set of single-valued Rothe tableaux of shape $D(w)$ flagged by $\mathbf{f}$.

Corollary 1.3 *Let $w \in S_n$ be a 1432-avoiding permutation. Then*

$$\mathfrak{S}_w(\mathbf{x}, \mathbf{y}) = \sum_{T \in \mathrm{SRT}(w, \mathbf{f}_0)} \prod_{(i,j) \in D(w)} \prod_{t \in T(i,j)} (x_t - y_{m_{ij}(w)+i-t}),$$

$$\mathfrak{S}_w(\mathbf{x}) = \sum_{T \in \mathrm{SRT}(w, \mathbf{f}_0)} \prod_{(i,j) \in D(w)} \prod_{t \in T(i,j)} x_t.$$

We further investigate the structure of Rothe tableau complexes. Employing the properties of tableau complexes given by Knutson, Miller and Yong [18], we find two alternative tableau formulas of $\mathfrak{G}_w^{(\beta)}(\mathbf{x},\mathbf{y})$ for 1432-avoiding permutations, which are given in terms of single-valued Rothe tableaux and limit set-valued Rothe tableaux, respectively. A limit set-valued Rothe tableau is an assignment of finite nonempty subsets of positive integers into the squares of a Rothe diagram such that one can pick out an integer from each square to form a single-valued Rothe tableau. Figure 1.1(c) illustrates a limit set-valued Rothe tableau, where the integers in boldface form a single-valued Rothe tableau.

Let $\mathrm{LSVRT}(w,\mathbf{f})$ denote the set of limit set-valued Rothe tableaux of shape $D(w)$ flagged by $\mathbf{f}$.

Theorem 1.4 *Let $w \in S_n$ be a 1432-avoiding permutation.*

(1) For each square $B=(i,j)\in D(w)$, set

$$E_B = \bigcup_{T\in \mathrm{SRT}(w,\mathbf{f}_0)} T(i,j).$$

Then

$$\mathfrak{G}_w^{(\beta)}(\mathbf{x},\mathbf{y}) = \sum_{T\in \mathrm{LSVRT}(w,\mathbf{f}_0)} (-\beta)^{|T|-\ell(w)} \prod_{B=(i,j)\in D(w)} \prod_{t\in T(i,j)} (x_t \oplus y_{m_{ij}(w)+i-t}) \cdot \prod_{t\in E_B\setminus T(i,j)} (1+\beta x_t)(1+\beta y_{m_{ij}(w)+i-t}). \tag{1.3}$$

(2) Given $T \in \mathrm{SRT}(w,\mathbf{f}_0)$ and a square $B\in D(w)$, let $Y_{T,B}$ be the set of positive integers m such that m is larger than the (unique) integer in $T(B)$ and replacing the integer in $T(B)$ by m still yields a Rothe tableau in $\mathrm{SRT}(w,\mathbf{f}_0)$. Then

$$\mathfrak{G}_w^{(\beta)}(\mathbf{x},\mathbf{y}) = \sum_{T\in \mathrm{SRT}(w,\mathbf{f}_0)} \prod_{B=(i,j)\in D(w)} \prod_{t\in T(i,j)} (x_t \oplus y_{m_{ij}(w)+i-t}) \cdot \prod_{t\in Y_{T,B}} (1+\beta x_t)(1+\beta y_{m_{ij}(w)+i-t}). \tag{1.4}$$

The remaining of this paper is organized as follows. We prove Theorem 1.1 in Section 2. Section 3 is devoted to a proof of Theorem 1.4.

2 Proof of Theorem 1.1

In this section, we aim to prove Theorem 1.1. Let

$$G_w^{(\beta)}(\mathbf{x},\mathbf{y}) = \sum_{T\in \mathrm{SVRT}(w,\mathbf{f}_0)} \beta^{|T|-\ell(w)} \prod_{(i,j)\in D(w)} \prod_{t\in T(i,j)} (x_t \oplus y_{m_{ij}(w)+i-t})$$

$$= \beta^{-\ell(w)} \sum_{T \in \mathrm{SVRT}(w,\mathbf{f}_0)} \prod_{(i,j)\in T} \prod_{t \in T(i,j)} \beta(x_t \oplus y_{m_{ij}(w)+i-t}) \tag{2.1}$$

denote the right-hand side of (1.2). We finish the proof of Theorem 1.1 by separately proving the following two statements.

Proposition 2.1 *If w is a 1432-avoiding permutation, then $\mathfrak{G}_w^{(\beta)}(\mathbf{x},\mathbf{y}) = G_w^{(\beta)}(\mathbf{x},\mathbf{y})$.*

Proposition 2.2 *If w contains a 1432 pattern, then $\mathfrak{G}_w^{(\beta)}(\mathbf{x},\mathbf{y}) \neq G_w^{(\beta)}(\mathbf{x},\mathbf{y})$.*

We use the opportunity here to explain that when w is a 321-avoiding permutation, Proposition 2.1 recovers the tableau formula for $\mathfrak{G}_w^{(\beta)}(\mathbf{x},\mathbf{y})$ due to Matsumura [25]. Let $f(w) = (f_1, f_2, \ldots, f_k)$ (respectively, $f^c(w) = (f_1^c, f_2^c, \ldots, f_{n-k}^c)$) be the increasing arrangement of the positions i such that $w_i > i$ (respectively, $w_i \leq i$). Moreover, let $h(w) = (w_{f_1}, w_{f_2}, \ldots, w_{f_k})$ and $h^c(w) = (w_{f_1^c}, w_{f_2^c}, \ldots, w_{f_{n-k}^c})$. It can be shown that w is 321-avoiding if and only if the sequences $h(w)$ and $h^c(w)$ are both increasing [10]. One may associate a skew shape $\sigma(w) = \lambda/\mu$ to w by letting

$$\lambda_i = w_{f_k} - k - (f_i - i), \qquad \mu_i = w_{f_k} - k - (w_{f_i} - i), \tag{2.2}$$

where $1 \leq i \leq k$. For a square α of $\sigma(w)$, let $r(\alpha)$ and $c(\alpha)$ denote the row index and the column index of α, respectively.

Corollary 2.3 (Matsumura [25, Theorem 3.1]) *Let $w \in S_n$ be a 321-avoiding permutation. Then*

$$\mathfrak{G}_w^{(\beta)}(\mathbf{x},\mathbf{y}) = \sum_{T} \beta^{|T|-\ell(w)} \prod_{\alpha\in\sigma(w)} \prod_{t\in T(\alpha)} (x_t \oplus y_{\lambda_{r(\alpha)}+f_{r(\alpha)}-c(\alpha)-t+1}), \tag{2.3}$$

where T ranges over set-valued Young tableaux of shape $\sigma(w)$ flagged by $f(w)$.

Proof. We show that for a 321-avoiding permutation w, the right-hand side of (2.3) equals $G_w^{(\beta)}(\mathbf{x},\mathbf{y})$ as defined in (2.1). As observed in [6], after deleting the empty rows indexed by $f^c(w)$ and the empty columns indexed by $h(w)$ and then reflecting the resulting diagram about a vertical line, $D(w)$ coincides with the skew shape $\sigma(w)$. For example, for $w = 312465$, we see that $f(w) = (1, 5)$, $f^c(w) = (2, 3, 4, 6)$ and $h(w) = (w_{f_1}, w_{f_2}) = (3, 6)$, and so the corresponding skew diagram $\sigma(w)$ is as illustrated in Figure 2.1.

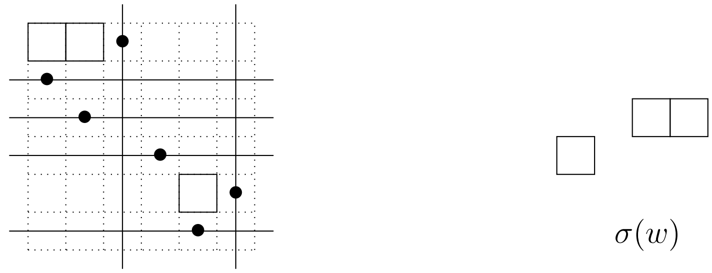

Figure 2.1: $D(w)$ and the corresponding skew shape $\sigma(w)$ for $w = 312465$.

Therefore, each set-valued Rothe tableau $T \in \mathrm{SVRT}(w, \mathbf{f}_0)$ can be viewed as a set-valued Young tableau of skew shape $\sigma(w)$ flagged by $f(w)$. For a square $(i, j) \in D(w)$, assume that α is the corresponding square of $\sigma(w)$. We need to show that

$$\lambda_{r(\alpha)} + f_{r(\alpha)} - c(\alpha) + 1 = m_{ij}(w) + i. \tag{2.4}$$

It is not hard to check that

$$r(\alpha) = i - |\{t \,|\, w_t \leq t < i\}|$$

and

$$c(\alpha) = w_{f_k} - j - |\{t \,|\, w_t > t, w_t > j\}| + 1.$$

Then, by (2.2), we have

$$\begin{aligned} \lambda_{r(\alpha)} + f_{r(\alpha)} - c(\alpha) + 1 &= w_{f_k} - k + r(\alpha) - c(\alpha) + 1 \\ &= j - k + |\{t \,|\, w_t > t, w_t > j\}| + i - |\{t \,|\, w_t \leq t < i\}| \\ &= j - |\{t \,|\, t < w_t \leq j\}| - |\{t \,|\, w_t \leq t < i\}| + i, \end{aligned} \tag{2.5}$$

where, in the last step, we used the relation

$$\begin{aligned} k - |\{t \,|\, w_t > t, w_t > j\}| &= |\{t \,|\, w_t > t\}| - |\{t \,|\, w_t > t, w_t > j\}| \\ &= |\{t \,|\, t < w_t \leq j\}|. \end{aligned}$$

Since w is 321-avoiding, it is easy to check that if there exists some integer t such that $t < w_t \leq j$, then $t < i$. Moreover, if $w_t \leq t < i$, then $w_t \leq j$. Thus we have

$$\begin{aligned} & j - |\{t \,|\, t < w_t \leq j\}| - |\{t \,|\, w_t \leq t < i\}| \\ &\quad = j - (|\{t \,|\, t < w_t \leq j, t < i\}| + |\{t \,|\, w_t \leq t < i, w_t \leq j\}|) \\ &\quad = j - |\{t \,|\, t < i, w_t \leq j\}| \\ &\quad = |\{t \,|\, t \geq i, w_t \leq j\}| \\ &\quad = |\{(i, k) \in D(w) \,|\, k \leq j\}| \\ &\quad = m_{ij}(w). \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6) yields (2.4). This completes the proof. ∎

2.1 Isobaric divided difference operator

Let s_i denote the simple transposition interchanging i and $i+1$. Notice that ws_i is obtained from w by swapping w_i and w_{i+1}. The divided difference operator ∂_i on the ring $\mathbb{Z}[\mathbf{x}] = \mathbb{Z}[x_1, x_2, \ldots, x_n]$ of polynomials with integer coefficients is defined by

$$\partial_i f(\mathbf{x}) = \frac{f(\mathbf{x}) - s_i f(\mathbf{x})}{x_i - x_{i+1}},$$

where $f(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$ and $s_i f(\mathbf{x})$ is obtained from $f(\mathbf{x})$ by interchanging x_i and x_{i+1}. Define the isobaric divided difference operator as

$$\pi_i\, f(\mathbf{x}) = \partial_i (1 + \beta x_{i+1}) f(\mathbf{x}).$$

The double Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}(\mathbf{x}, \mathbf{y})$ for $w \in S_n$ can be defined as follows. For the longest permutation $w_0 = n\,(n-1)\cdots 1$, set

$$\mathfrak{G}_{w_0}^{(\beta)}(\mathbf{x}, \mathbf{y}) = \prod_{i+j \le n} (x_i + y_j + \beta x_i y_j).$$

For $w \neq w_0$, choose a position i such that $w_i < w_{i+1}$. Notice that $\ell(ws_i) = \ell(w) + 1$. Let

$$\mathfrak{G}_w^{(\beta)}(\mathbf{x}, \mathbf{y}) = \pi_i\, \mathfrak{G}_{ws_i}^{(\beta)}(\mathbf{x}, \mathbf{y}), \tag{2.7}$$

where the operator π_i acts only on the $\mathbf{x}$-variables. The above definition is independent of the choice of the position i, since the operator π_i satisfies the Coxeter relations $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$ and $\pi_i \pi_j = \pi_j \pi_i$ for $|i - j| > 1$.

2.2 Proof of Proposition 2.1

The key in the proof of Proposition 2.1 is to show that, when w is 1432-avoiding, $G_w^{(\beta)}(\mathbf{x}, \mathbf{y})$ is compatible with the isobaric divided difference operator, see Proposition 2.5, which allows us to finish the proof of Proposition 2.1 by induction on the length of w. Such an idea was first used by Wachs [32] to establish the tableau formula for Schubert polynomials of 2143-avoiding permutations. Matsumura [26] and Matsumura and Sugimoto [27] extended this idea to reprove the Knutson-Miller-Yong set-valued tableau formula for Grothendieck polynomials of 2143-avoiding permutations. Matsumura [25] also used this technique to prove formula (2.3) for 321-avoiding permutations.

Let us first check that $\mathfrak{G}_{w_0}^{(\beta)}(\mathbf{x}, \mathbf{y}) = G_{w_0}^{(\beta)}(\mathbf{x}, \mathbf{y})$, where $w_0 = n \cdots 21$ is the longest 1432-avoiding permutation. Since $D(w_0)$ is a staircase Young diagram with $n - i$ squares in row i, there is exactly one tableau T_0 of shape $D(w_0)$ flagged by $\mathbf{f}_0$, that is, every square in the i-th row of T_0 is filled with $\{i\}$. Note that for each square (i, j) of $D(w_0)$, $m_{ij}(w_0) = j$. Thus,

$$G_{w_0}^{(\beta)}(\mathbf{x}, \mathbf{y}) = \beta^{|T_0| - \ell(w_0)} \prod_{i+j \le n} (x_i \oplus y_j) = \prod_{i+j \le n} (x_i \oplus y_j), \tag{2.8}$$

which agrees with $\mathfrak{G}_{w_0}^{(\beta)}(\mathbf{x},\mathbf{y})$.

The following observation allows us to make induction on the length $\ell(w)$. This fact does not appear in the literature as far as we are aware, but is likely known to experts.

Lemma 2.4 *Let $w \neq w_0$ be a 1432-avoiding permutation, and r be the first ascent of w. Then ws_r is a 1432-avoiding permutation.*

Proof. Write $w' = ws_r = w'_1 w'_2 \cdots w'_n$. Suppose otherwise that w' has a subsequence that is order isomorphic to 1432. Since w is 1432-avoiding and r is the first ascent, any subsequence of w' that is order isomorphic to 1432 must be of the form $w'_i w'_r w'_{r+1} w'_j$, where $i < r$ and $j > r+1$. Since w'_i is the smallest element in this subsequence, we have $w'_i < w'_{r+1}$. Noticing that $w'_i = w_i$ and $w'_{r+1} = w_r$, we see that $w_i < w_r$. However, since r is the first ascent, we must have $w_i > w_r$, leading to a contradiction. This completes the proof. ∎

In view of (2.7), (2.8) together with Lemma 2.4, to conclude Proposition 2.1, it suffices to prove the following recurrence relation.

Proposition 2.5 *Let $w \neq w_0$ be a 1432-avoiding permutation, and r be the first ascent of w. Then*

$$G_w^{(\beta)}(\mathbf{x},\mathbf{y}) = \pi_r\, G_{ws_r}^{(\beta)}(\mathbf{x},\mathbf{y}). \tag{2.9}$$

The rest of this subsection is devoted to a proof of Proposition 2.5. The proof is outlined as follows. We first define equivalence relations on the set $\mathrm{SVRT}(ws_r, \mathbf{f}_0)$ and the set $\mathrm{SVRT}(w, \mathbf{f}_0)$, see Definition 2.6. For each given equivalence class C of $\mathrm{SVRT}(ws_r, \mathbf{f}_0)$, let

$$G_{ws_r}^{(\beta)}(C;\mathbf{x},\mathbf{y}) = \sum_{T \in C} \beta^{|T|-\ell(ws_r)} \prod_{(i,j)\in D(ws_r)} \prod_{t \in T(i,j)} (x_t \oplus y_{m_{ij}(ws_r)+i-t}) \tag{2.10}$$

denote the polynomial generated by the Rothe tableaux in C. Then we have

$$G_{ws_r}^{(\beta)}(\mathbf{x},\mathbf{y}) = \sum_{C} G_{ws_r}^{(\beta)}(C;\mathbf{x},\mathbf{y}),$$

where the sum runs over the equivalence classes of $\mathrm{SVRT}(ws_r, \mathbf{f}_0)$. Similarly, for each given equivalence class C' of $\mathrm{SVRT}(w, \mathbf{f}_0)$, let

$$G_{w}^{(\beta)}(C';\mathbf{x},\mathbf{y}) = \sum_{T' \in C'} \beta^{|T'|-\ell(w)} \prod_{(i,j)\in D(w)} \prod_{t \in T'(i,j)} (x_t \oplus y_{m_{ij}(w)+i-t}).$$

Then

$$G_{w}^{(\beta)}(\mathbf{x},\mathbf{y}) = \sum_{C'} G_{w}^{(\beta)}(C';\mathbf{x},\mathbf{y}),$$

where the sum takes over the equivalence classes of $\mathrm{SVRT}(w, \mathbf{f}_0)$.

We then construct a bijection Φ between the set of equivalence classes of $\mathrm{SVRT}(ws_r, \mathbf{f}_0)$ and the set of equivalence classes of $\mathrm{SVRT}(w, \mathbf{f}_0)$. Using the bijection Φ, we complete the proof of Proposition 2.5 by showing that

$$\pi_r\, G^{(\beta)}_{ws_r}(C; \mathbf{x}, \mathbf{y}) = G^{(\beta)}_{w}(\Phi(C); \mathbf{x}, \mathbf{y}), \tag{2.11}$$

see Proposition 2.11.

We now define equivalence relations on the sets $\mathrm{SVRT}(ws_r, \mathbf{f}_0)$ and $\mathrm{SVRT}(w, \mathbf{f}_0)$. For a set-valued tableau T in $\mathrm{SVRT}(ws_r, \mathbf{f}_0)$ or $\mathrm{SVRT}(w, \mathbf{f}_0)$, let $E(T)$ be the subset of squares of T containing either r or $r+1$ (possibly both).

Definition 2.6 *Given two Rothe tableaux $T, T' \in \mathrm{SVRT}(ws_r, \mathbf{f}_0)$, we say that T is equivalent to T', denoted $T \sim T'$, if $E(T) = E(T')$ and for every square $B \in D(ws_r)$,*

$$T(B) \setminus \{r, r+1\} = T'(B) \setminus \{r, r+1\}.$$

In other words, $T \sim T'$ if and only if all entries except $r, r+1$ are the same in T, T', and the boxes with either an r or $r+1$ (possibly both) are the same. The equivalence relation on the set $\mathrm{SVRT}(w, \mathbf{f}_0)$ is defined in the same manner.

Let

$$\mathrm{SVRT}(ws_r, \mathbf{f}_0)/\!\sim \quad \text{and} \quad \mathrm{SVRT}(w, \mathbf{f}_0)/\!\sim$$

denote the sets of equivalence classes of $\mathrm{SVRT}(ws_r, \mathbf{f}_0)$ and $\mathrm{SVRT}(w, \mathbf{f}_0)$, respectively. Given a Rothe tableau T in $\mathrm{SVRT}(ws_r, \mathbf{f}_0)$ or $\mathrm{SVRT}(w, \mathbf{f}_0)$, since the columns of T are strictly increasing, each column of T contains at most two squares in $E(T)$. Let

$$P(T) = \{(i, j) \in E(T) \,|\, \text{the } j\text{-th column of } T \text{ contains exactly one square of } E(T)\}.$$

Let $Q(T) = E(T) \setminus P(T)$, namely,

$$Q(T) = \{(i, j) \in E(T) \,|\, \text{the } j\text{-th column of } T \text{ contains two squares of } E(T)\}.$$

In fact, if $T \sim T'$, then T and T' can be possibly different only in the squares of $P(T)$. For example, let $w = 426173958$. The first ascent of w is $r = 2$. For the tableau $T \in \mathrm{SVRT}(w, \mathbf{f}_0)$ displayed in Figure 2.2, we have

$$P(T) = \{(3, 3), (7, 8)\}, \qquad Q(T) = \{(2, 1), (3, 1), (3, 5), (5, 5)\}.$$

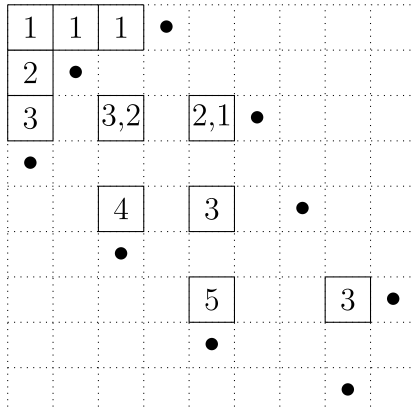

Figure 2.2: A tableau $T \in \mathrm{SVRT}(w, \mathbf{f}_0)$ for $w = 426173958$.

Let C be an equivalence class of $\mathrm{SVRT}(ws_r, \mathbf{f}_0)$ or $\mathrm{SVRT}(w, \mathbf{f}_0)$, and let T be any given Rothe tableau in C. For $i \geq 1$, let $P(T, i)$ be the set of squares of $P(T)$ in row i. Clearly, $P(T, i)$ is empty unless $i \geq r$. Denote $b_i(T) = |P(T, i)|$. Assuming that (i, p_i) is the leftmost square in $P(T, i)$, let

$$\ell_i(T) = |\{(i, k) \in T \mid k \leq p_i\}| + i - r - 1. \tag{2.12}$$

With the above notation, define a polynomial $h(C, i; \mathbf{x}, \mathbf{y})$ as follows. Set $h(C, i; \mathbf{x}, \mathbf{y}) = 1$ if $b_i(T) = 0$, and for $b_i(T) \geq 1$, let

$$\begin{aligned} h(C, i; \mathbf{x}, \mathbf{y}) =& \beta^{b_i(T)} \sum_{k=0}^{b_i(T)} \prod_{j=1}^{k} (x_{r+1} \oplus y_{\ell_i(T)+j-1}) \prod_{j=k+1}^{b_i(T)} (x_r \oplus y_{\ell_i(T)+j}) \\ &+ \beta^{b_i(T)+1} \sum_{k=1}^{b_i(T)} \prod_{j=1}^{k} (x_{r+1} \oplus y_{\ell_i(T)+j-1}) \prod_{j=k}^{b_i(T)} (x_r \oplus y_{\ell_i(T)+j}). \end{aligned} \tag{2.13}$$

Note that $h(C, i; \mathbf{x}, \mathbf{y})$ is independent of the choice of the tableau T in C.

The following lemma gives a reformulation of $G^{(\beta)}_{ws_r}(C; \mathbf{x}, \mathbf{y})$ as defined in (2.10).

Lemma 2.7 *Let $w \neq w_0$ be a 1432-avoiding permutation, and r be the first ascent of w. Assume that $C \in \mathrm{SVRT}(ws_r, \mathbf{f}_0)/\!\sim$ and T is any given Rothe tableau in C. Then,*

$$\begin{aligned} G^{(\beta)}_{ws_r}(C; \mathbf{x}, \mathbf{y}) =& \beta^{-\ell(ws_r)} \left(\prod_{(i,j) \in D(ws_r)} \prod_{\substack{t \in T(i,j) \\ t \neq r, r+1}} \beta(x_t \oplus y_{m_{ij}(ws_r)+i-t}) \right) \\ &\cdot \left(\prod_{j=1}^{b_r(T)} \beta(x_r \oplus y_{\ell_r(T)+j}) \right) \cdot H_C(\mathbf{x}, \mathbf{y}) \cdot J_C(\mathbf{x}, \mathbf{y}), \end{aligned} \tag{2.14}$$

where $H_C(\mathbf{x},\mathbf{y})$ and $J_C(\mathbf{x},\mathbf{y})$ are defined as follows:

$$H_C(\mathbf{x},\mathbf{y}) = \prod_{i>r+1} h(C,i;\mathbf{x},\mathbf{y}),$$

$$J_C(\mathbf{x},\mathbf{y}) = \prod_{(i,j)\in Q^+(T)} \beta^2(x_r \oplus y_{m_{ij}(ws_r)+i-r})(x_{r+1} \oplus y_{m_{ij}(ws_r)+i-r}),$$

where $Q^+(T)$ denotes the subset of $Q(T)$ consisting of the squares containing r.

It will be clear from the proof of Lemma 2.7 that each factor in (2.14) appears naturally. Roughly speaking, the first factor is the contribution of the integers other than r and $r+1$, the second factor is the contribution of r in $P(T,r)$, $H_C(\mathbf{x},\mathbf{y})$ is the contribution of r and $r+1$ in $P(T,i)$ with $i \geq r+1$, and $J_C(\mathbf{x},\mathbf{y})$ is the contribution of r and $r+1$ in $Q(T)$.

To prove Lemma 2.7, we need two lemmas concerning the configuration of the squares in the sets $P(T)$ and $Q(T)$.

Lemma 2.8 *Let $w \neq w_0$ be a permutation in S_n. Assume that T is a Rothe tableau in* $\mathrm{SVRT}(w,\mathbf{f}_0)$ *or* $\mathrm{SVRT}(ws_r,\mathbf{f}_0)$*, and $(i,j)\in P(T)$. Then there do not exist two squares $(i,k),(h,k)\in Q(T)$ such that $k>j$ and $h<i$.*

Proof. We only give a proof for the case when $T\in \mathrm{SVRT}(w,\mathbf{f}_0)$. The same argument applies to the case when $T\in\mathrm{SVRT}(ws_r,\mathbf{f}_0)$. Suppose to the contrary that there exist two squares $(i,k),(h,k)\in Q(T)$ such that $k>j$ and $h<i$, see Figure 2.3 for an illustration.

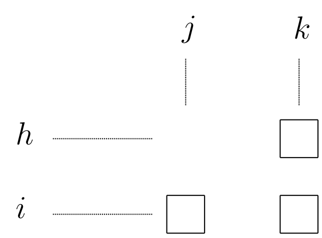

Figure 2.3: An illustration for the proof of Lemma 2.8.

Since $(i,j),(h,k)\in D(w)$, it follows that $w_h>j$ and j appears after w_h in w. This implies that $(h,j)\in D(w)$. Keep in mind that each of the sets $T(i,j)$, $T(i,k)$ and $T(h,k)$ contains at least one of the integers r and $r+1$. Since the rows of T are weakly decreasing and the columns of T are strictly increasing, we see that $r\in T(h,k)$, $r+1\in T(i,j)$. This forces that $T(h,j)=\{r\}$, and hence $(i,j)\in Q(T)$, which contradicts the assumption that $(i,j)\in P(T)$. ∎

Lemma 2.9 *Let $w \neq w_0$ be a 1432-avoiding permutation. Assume that T is a Rothe tableau in* $\mathrm{SVRT}(w, \mathbf{f}_0)$ *or* $\mathrm{SVRT}(ws_r, \mathbf{f}_0)$*, and $(i, j) \in P(T)$. If $i > r$, then there do not exist two squares $(i, k), (h, k) \in Q(T)$ such that $h > i$ and $k < j$.*

Proof. We only give a proof for $T \in \mathrm{SVRT}(w, \mathbf{f}_0)$, and the arguments for $T \in \mathrm{SVRT}(ws_r, \mathbf{f}_0)$ can be carried out in the same manner. Suppose otherwise that there exist two squares (i, k) and (h, k) in $Q(T)$ where $i < h$ and $j > k$, as illustrated in Figure 2.4. Notice that

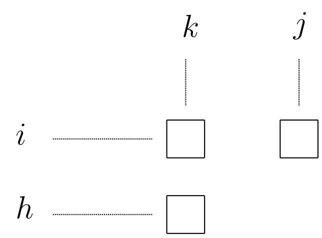

Figure 2.4: An illustration for the proof of Lemma 2.9.

both the sets $T(i, j)$ and $T(i, k)$ contain r, while the set $T(h, k)$ contains $r+1$. We have the following two claims.

Claim 1: $w_s < k$ for any $i < s < h$. Suppose otherwise that there exists some $i < s < h$ such that $w_s > k$. Then the square (s, k) belongs to $D(w)$. Since $r \in T(i, k)$ and $r + 1 \in T(h, k)$, it follows that $r < \min T(s, k) < r + 1$, leading to a contradiction.

Claim 2: $k < w_h \leq j$. Since (h, k) is a square in $D(w)$, it is clear that $k < w_h$. Suppose otherwise that $w_h > j$. It follows from Claim 1 that j must appear in w after the position h. This implies that $(h, j) \in D(w)$. Since $r \in T(i, j)$ and $r + 1 \in T(h, k)$, we must have $T(h, j) = \{r + 1\}$. This implies that $(i, j) \in Q(T)$, contradicting the assumption that $(i, j) \in P(T)$.

By Claim 2 and the fact that $w_i > j$, we see that $w_i w_h k$ forms a decreasing subsequence of w. Since w is 1432-avoiding, we have $w_t > k$ for any $1 \leq t < i$. Thus, for any $1 \leq t < i$, the square (t, k) belongs to $D(w)$. Keep in mind that each integer in row i of T cannot exceed i and the columns of T are strictly increasing. So we have $T(t, k) = \{t\}$ for $1 \leq t \leq i$. In particular, we have $T(i, k) = \{i\}$. Since $r \in T(i, k)$, we must have $i = r$, contradicting the assumption that $i > r$. This completes the proof. ∎

Based on Lemmas 2.8 and 2.9, we can now give a proof of Lemma 2.7.

Proof of Lemma 2.7. Assume that $T' \in \mathrm{SVRT}(ws_r, \mathbf{f}_0)$ is a Rothe tableau in the equivalence class C. Then T' differs from T only possibly in the squares of $P(T)$. Note that if $P(T, i)$ is nonempty, then we must have $i \geq r$. Moreover, since the integers appearing in r-th row of T' cannot exceed r, it follows that for any square $B \in P(T, r)$, $T'(B)$ does not contain $r+1$. Thus, for $B \in P(T, r)$, $r \in T(B) = T'(B)$ and $r + 1 \notin T(B) = T'(B)$.

Before we proceed, we give an illustration of the configuration of the squares in the first $r+1$ rows of $D(w)$ and $D(ws_r)$, which will be helpful to analyze the contributions of the integer r in the squares of $P(T,r)$. Notice that $D(w)$ is obtained from $D(ws_r)$ by deleting the square (r,w_r) and then moving each square in row r, that lies to the right of (r,w_r), down to row $r+1$. Since r is the first ascent of w, the first $r+1$ rows of $D(w)$ and $D(ws_r)$ are as depicted in Figure 2.5, where the square (r,w_r) of $D(ws_r)$ is signified by a symbol $\heartsuit$.

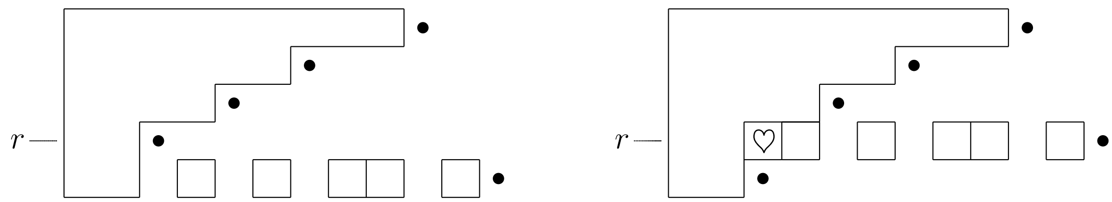

Figure 2.5: The first $r+1$ rows of $D(w)$ and $D(ws_r)$.

Obviously, the first w_r-1 squares in the r-th row (respectively, $(r+1)$-th row) of T are filled with the set $\{r\}$ (respectively, $\{r+1\}$). This implies that each square in the $(r+1)$-th row of $D(ws_r)$ belongs to $Q(T)$ and the set $P(T,r+1)$ is empty. Therefore, the contribution of the r's in squares of $P(T,r)$ to $G^{(\beta)}_{ws_r}(C;\mathbf{x},\mathbf{y})$ is

$$\prod_{(r,j)\in P(T,r)} \beta(x_r\oplus y_{m_{rj}(ws_r)}). \tag{2.15}$$

On the other hand, the contribution of the r's and $(r+1)$'s in squares of $P(T,i)$ for $i>r+1$ to $G^{(\beta)}_{ws_r}(C;\mathbf{x},\mathbf{y})$ is

$$\begin{aligned} F_C(\mathbf{x},\mathbf{y}) = \sum_{T'\in C} \prod_{i>r+1} \prod_{(i,j)\in P^+(T',i)} &\beta(x_r\oplus y_{m_{ij}(ws_r)+i-r}) \\ \cdot \prod_{(i,j)\in P^-(T',i)} &\beta(x_{r+1}\oplus y_{m_{ij}(ws_r)+i-r-1}), \end{aligned} \tag{2.16}$$

where $P^+(T',i)$ (respectively, $P^-(T',i)$) denotes the subset of $P(T',i)$ consisting of squares containing r (respectively, $r+1$). Moreover, the contribution of the r's and $(r+1)$'s in squares of $Q(T)$ to $G^{(\beta)}_{ws_r}(C;\mathbf{x},\mathbf{y})$ is

$$R_C(\mathbf{x},\mathbf{y}) = \prod_{(i,j)\in Q^+(T)} \beta(x_r\oplus y_{m_{ij}(ws_r)+i-r}) \prod_{(i,j)\in Q^-(T)} \beta(x_{r+1}\oplus y_{m_{ij}(ws_r)+i-r-1}), \tag{2.17}$$

where $Q^+(T)$ (respectively, $Q^-(T)$) denotes the subset of $Q(T)$ consisting of the squares containing r (respectively, $r+1$). Consequently, we obtain that

$$G^{(\beta)}_{ws_r}(C;\mathbf{x},\mathbf{y}) = \beta^{-\ell(ws_r)} \left(\prod_{(i,j)\in D(ws_r)} \prod_{\substack{t\in T(i,j)\\ t\neq r,r+1}} \beta(x_t\oplus y_{m_{ij}(ws_r)+i-t}) \right)$$

$$\cdot \prod_{(r,j)\in P(T,r)} \beta(x_r \oplus y_{m_{rj}(ws_r)}) \cdot F_C(\mathbf{x},\mathbf{y}) \cdot R_C(\mathbf{x},\mathbf{y}). \tag{2.18}$$

Comparing (2.18) with (2.14), in order to complete the proof of Lemma 2.7, we need to show that

$$\prod_{(r,j)\in P(T,r)} \beta(x_r \oplus y_{m_{rj}(ws_r)}) = \prod_{j=1}^{b_r(T)} \beta(x_r \oplus y_{\ell_r(T)+j}), \tag{2.19}$$

$$F_C(\mathbf{x},\mathbf{y}) = \prod_{i>r+1} h(C,i;\mathbf{x},\mathbf{y}) = H_C(\mathbf{x},\mathbf{y}), \tag{2.20}$$

$$R_C(\mathbf{x},\mathbf{y}) = J_C(\mathbf{x},\mathbf{y}). \tag{2.21}$$

Let us first prove (2.19). To this end, we show that if there are two squares (r, j_1) and (r, j_2) in $P(T, r)$ with $j_1 < j_2$ and there exists a square $(r, j) \in D(ws_r)$ for some $j_1 < j < j_2$, then $(r, j) \in P(T, r)$. It suffices to prove the following claim.

Claim 1. For $(r, j) \in P(T, r)$, there do not exist squares $(r, k), (h, k) \in Q(T)$ such that $h > r + 1$ and $k < j$.

To verify Claim 1, we construct a Rothe tableau $\overline{T}$ from T such that $\overline{T} \in \mathrm{SVRT}(w, \mathbf{f}_0)$. Let R be the set of squares of $D(ws_r)$ in row r that are strictly to the right of (r, w_r). Define $\overline{T}$ to be the tableau obtained from T by deleting the square (r, w_r) together with $T(r, w_r)$, and then moving each square B in R, together with $T(B)$, down to row $r + 1$. By construction, it is easy to check that $\overline{T} \in \mathrm{SVRT}(w, \mathbf{f}_0)$. Note that $(r, j) \in P(T)$ if and only if $(r+1, j) \in P(\overline{T})$. Applying Lemma 2.9 to $\overline{T}$, we see that if $(r+1, j) \in P(\overline{T})$, then there do not exist squares $(r, k), (h, k) \in Q(\overline{T})$ with $h > r + 1$ and $k < j$. Since $Q(T) = Q(\overline{T})$, we conclude Claim 1.

By Claim 1, the configuration of the squares of $P(T)$ and $Q(T)$ in the r-th row of $D(ws_r)$ is as illustrated in Figure 2.6, where the squares in $Q^+(T)$ (respectively, $Q^-(T)$) are marked with a $*$ (respectively, $\star$). In view of the definition $m_{i,j}(ws_r)$ in (1.1) as well as the definition $\ell_r(T)$ in (2.12), we see that (2.19) holds.

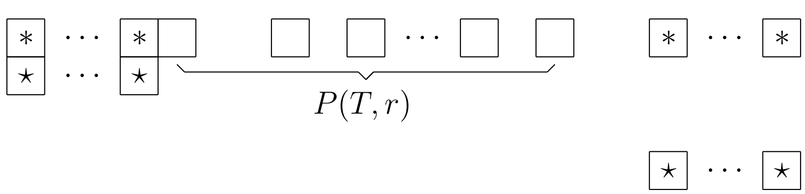

Figure 2.6: An illustration of the squares in $P(T, r)$.

We next prove (2.20). For $i > r$, by Lemma 2.8 and Lemma 2.9, the configuration of the squares of $P(T)$ and $Q(T)$ must be as illustrated as in Figure 2.7. In particular, all the squares in row i of $D(ws_r)$ that lie between the leftmost square and rightmost square of $P(T,i)$ must belong to $P(T,i)$. We have the following two cases.

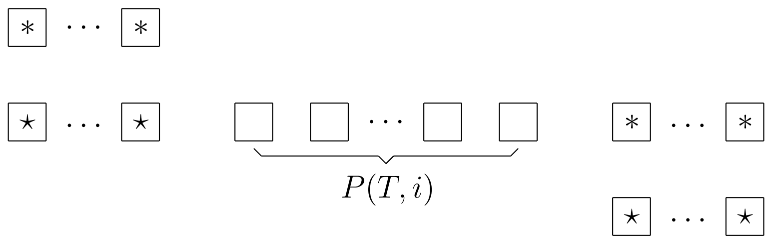

Figure 2.7: The configuration of the squares in $P(T,i)$ with $i > r+1$.

Case 1: The first k $(0 \le k \le b_i(T))$ squares in $P(T,i)$ contain $r+1$, and the remaining $b_i(T) - k$ squares in $P(T,i)$ contain r. Summing over all the Rothe tableaux in C, we have

$$\sum_{k=0}^{b_i(T)} \prod_{j=1}^{k} \beta(x_{r+1} \oplus y_{\ell_i(T)+j-1}) \prod_{j=k+1}^{b_i(T)} \beta(x_r \oplus y_{\ell_i(T)+j}). \tag{2.22}$$

Case 2: The first $k-1$ $(1 \le k \le b_i(T))$ squares in $P(T,i)$ contain $r+1$, the k-th square contains both r and $r+1$, and the remaining $b_i(T) - k$ squares in $P(T,i)$ contain r. Summing over all the Rothe tableaux in C, we obtain

$$\sum_{k=1}^{b_i(T)} \prod_{j=1}^{k} \beta(x_{r+1} \oplus y_{\ell_i(T)+j-1}) \prod_{j=k}^{b_i(T)} \beta(x_r \oplus y_{\ell_i(T)+j}). \tag{2.23}$$

Combining (2.22) and (2.23), we see that Case 1 and Case 2 together contribute the factor $h(C,i;\mathbf{x},\mathbf{y})$ as defined in (2.13) to the summation $F_C(\mathbf{x},\mathbf{y})$ in (2.16). Running over the row indices i with $i > r+1$ yields (2.20).

Finally, we verify (2.21). For each $(i,j) \in Q^+(T)$, we use (i',j) to denote the square in $Q^-(T)$ that lie in the same column as (i,j). Then we have

$$R_C(\mathbf{x},\mathbf{y}) = \prod_{(i,j)\in Q^+(T)} \beta^2(x_r \oplus y_{m_{ij}(ws_r)+i-r})(x_{r+1} \oplus y_{m_{i'j}(ws_r)+i'-r-1}). \tag{2.24}$$

Write $w' = ws_r$. We assert that $w'_t < j$ for $i < t < i'$. Suppose otherwise that $w'_t > j$. Since the square $(i',j) \in D(ws_r)$, we see that $(t,j) \in D(ws_r)$. Thus we have $r < \min T(t,j) \le \max T(t,j) < r+1$, leading to a contradiction. This verifies the assertion. By the definition of $m_{ij}(w)$ in (1.1), it is easy to see that

$$m_{ij}(w') = |\{(i,k) \in D(w') \mid k \le j\}| = |\{t > i \mid w'_t \le j\}|.$$

Therefore, by the above assertion, we obtain

$$m_{ij}(ws_r) = m_{i'j}(ws_r) + i' - i - 1,$$

and so that

$$m_{i'j}(ws_r) + i' - r - 1 = m_{ij}(ws_r) + i - r. \tag{2.25}$$

Putting (2.25) into (2.24), we arrive at the equality in (2.21). So the proof of Lemma 2.7 is complete. ∎

The following lemma provides a reformulation of the polynomial $G_w^{(\beta)}(C'; \mathbf{x}, \mathbf{y})$ for an equivalence class $C' \in \mathrm{SVRT}(w, \mathbf{f}_0)/\sim$.

Lemma 2.10 *Let $w \neq w_0$ be a 1432-avoiding permutation, and r be the first ascent of w. Assume that $C' \in \mathrm{SVRT}(w, \mathbf{f}_0)/\sim$ and T' is any given Rothe tableau in C'. Then,*

$$\begin{aligned} G_w^{(\beta)}(C'; \mathbf{x}, \mathbf{y}) =& \beta^{-\ell(w)} \left(\prod_{(i,j)\in D(w)} \prod_{\substack{t\in T'(i,j)\\ t\neq r, r+1}} \beta(x_t \oplus y_{m_{ij}(w)+i-t}) \right) \\ & \cdot h(C', r+1; \mathbf{x}, \mathbf{y}) \cdot H_{C'}(\mathbf{x}, \mathbf{y}) \cdot J_{C'}(\mathbf{x}, \mathbf{y}), \end{aligned} \tag{2.26}$$

where

$$H_{C'}(\mathbf{x}, \mathbf{y}) = \prod_{i>r+1} h(C', i; \mathbf{x}, \mathbf{y})$$

and

$$J_{C'}(\mathbf{x}, \mathbf{y}) = \prod_{(i,j)\in Q^+(T')} \beta^2 (x_r \oplus y_{m_{ij}(w)+i-r})(x_{r+1} \oplus y_{m_{ij}(w)+i-r}).$$

Sketch of the proof. The proof is nearly the same as the arguments for Lemma 2.7. The only difference is to notice that the set $P(T', r)$ is empty, and that the squares in $P(T', r+1)$ contributes the factor $h(C', r+1; \mathbf{x}, \mathbf{y})$. ∎

To finish the proof of Proposition 2.5, we still need a bijection Φ between the two sets of equivalence classes:

$$\Phi: \ \mathrm{SVRT}(ws_r, \mathbf{f}_0)/\sim \ \longrightarrow \ \mathrm{SVRT}(w, \mathbf{f}_0)/\sim .$$

Construction of the bijection Φ: Assume that $C \in \mathrm{SVRT}(ws_r, \mathbf{f}_0)/\sim$ and $T \in C$. Let T' be the Rothe tableau obtained from T by deleting the square (r, w_r) (together with the set $T(r, w_r)$), and then moving each square B in R, together with $T(B)$, down to row $r+1$, where R is the set of squares of $D(ws_r)$ in row r that are strictly to the right of (r, w_r). Evidently, T' is a set-valued Rothe tableau in $\mathrm{SVRT}(w, \mathbf{f}_0)$. Let C' be the equivalence class in $\mathrm{SVRT}(w, \mathbf{f}_0)/\sim$ containing T'. It is easily seen that C' is independent of the choice of T. Set $\Phi(C) = C'$.

The inverse of Φ can be described as follows. Let $C' \in \mathrm{SVRT}(w, \mathbf{f}_0)/\!\sim$ and $T' \in C'$. Let T'' be the Rothe tableau defined by setting $T''(B) = T'(B)$ if $B \in D(w) \backslash P(T', r+1)$, and setting

$$T''(B) = (T'(B) \setminus \{r, r+1\}) \cup \{r\}$$

if $B \in P(T', r+1)$. Notice that $T'' \in \mathrm{SVRT}(w, \mathbf{f}_0)$. We define T as the Rothe tableau obtained from T'' by adding the square (r, w_r) filled with the set $\{r\}$, and then moving each square B of T'' (together with the set $T''(B)$), which is to the right of the square $(r+1, w_r)$, up to row r. By construction, it is easily checked that $T \in \mathrm{SVRT}(ws_r, \mathbf{f}_0)$. Let C be the equivalence class in $\mathrm{SVRT}(ws_r, \mathbf{f}_0)/\!\sim$ containing T. Set $\Phi^{-1}(C') = C$.

For example, let $w = 426315$. The first ascent of w is $r = 2$. Figure 2.8 gives an illustration of the maps Φ and Φ^{-1}.

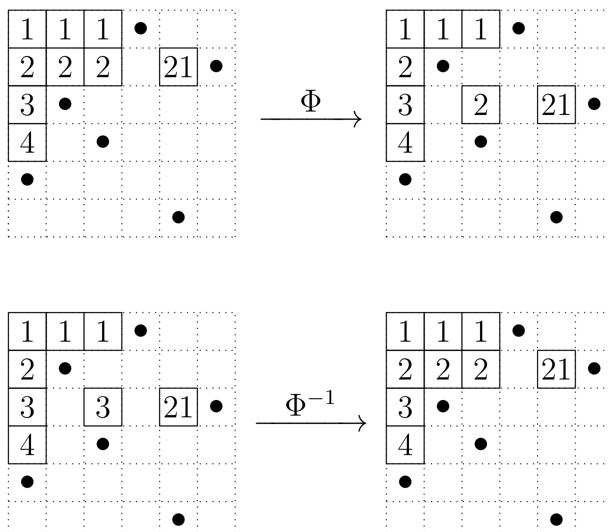

Figure 2.8: An illustration of Φ and Φ^{-1} for $w = 426315$.

Based on Lemma 2.7 and Lemma 2.10, we finally establish the following relation.

Proposition 2.11 *Let $w \neq w_0$ be a 1432-avoiding permutation, and r be the first ascent of w. For each equivalence class $C \in \mathrm{SVRT}(ws_r, \mathbf{f}_0)/\!\sim$, we have*

$$\pi_r\, G^{(\beta)}_{ws_r}(C; \mathbf{x}, \mathbf{y}) = G^{(\beta)}_{w}(\Phi(C); \mathbf{x}, \mathbf{y}). \tag{2.27}$$

The proof of Proposition 2.11 requires the following property of the operator π_r due to Matsumura [25].

Lemma 2.12 (Matsumura [25, Lemma 4.1]) *For an arbitrary sequence* $(a_1, a_2, \ldots, a_m)$ *of positive integers,*

$$\pi_r\left((x_r \oplus y_{a_1}) \cdots (x_r \oplus y_{a_m})\right) = \sum_{k=1}^{m} \prod_{j=1}^{k-1}(x_r \oplus y_{a_j}) \prod_{j=k+1}^{m}(x_{r+1} \oplus y_{a_j}) + \beta \sum_{k=1}^{m-1} \prod_{j=1}^{k}(x_r \oplus y_{a_j}) \prod_{j=k+1}^{m}(x_{r+1} \oplus y_{a_j}). \tag{2.28}$$

Furthermore, the expression in (2.28) *is symmetric in* x_r *and* x_{r+1}*.*

Proof of Proposition 2.11. Assume that T is any given Rothe tableau in C. The polynomial $h(C, i; \mathbf{x}, \mathbf{y})$ defined in (2.13) has the following reformulation:

$$h(C, i; \mathbf{x}, \mathbf{y}) = \beta^{b_i(T)} \sum_{k=1}^{b_i(T)+1} \prod_{j=1}^{k-1}(x_{r+1} \oplus y_{\ell_i(T)+j-1}) \prod_{j=k+1}^{b_i(T)+1}(x_r \oplus y_{\ell_i(T)+j-1}) + \beta^{b_i(T)+1} \sum_{k=1}^{b_i(T)} \prod_{j=1}^{k}(x_{r+1} \oplus y_{\ell_i(T)+j-1}) \prod_{j=k+1}^{b_i(T)+1}(x_r \oplus y_{\ell_i(T)+j-1}).$$

Hence $\beta^{b_i(T)} h(C, i; \mathbf{x}, \mathbf{y})$ coincides with the right-hand side of (2.28) by setting $m = b_i(T) + 1$ and setting $a_j = \ell_i(T) + j - 1$ for $1 \le j \le m$, and then exchanging the variables x_r and x_{r+1}. It follows from Lemma 2.12 that

$$h(C, i; \mathbf{x}, \mathbf{y}) = \beta^{b_i(T)}\, \pi_r \left(\prod_{j=1}^{b_i(T)+1} (x_r \oplus y_{\ell_i(T)+j-1}) \right) = \beta^{-1}\, \pi_r \left(\prod_{j=1}^{b_i(T)+1} \beta(x_r \oplus y_{\ell_i(T)+j-1}) \right), \tag{2.29}$$

which is a symmetric polynomial in x_r and x_{r+1}.

On the other hand, if a polynomial $f(\mathbf{x})$ is symmetric in x_r and x_{r+1}, then for any polynomial $g(\mathbf{x})$, it can be easily checked that

$$\pi_r(f(\mathbf{x})g(\mathbf{x})) = f(\mathbf{x})\, \pi_r g(\mathbf{x}).$$

Therefore, applying π_r to the formula of $G^{(\beta)}_{ws_r}(C; \mathbf{x}, \mathbf{y})$ in Lemma 2.7, we obtain that

$$\pi_r\, G_{ws_r}(C; \mathbf{x}, \mathbf{y}) = \beta^{-\ell(ws_r)} \left(\prod_{(i,j) \in D(ws_r)} \prod_{\substack{t \in T(i,j) \\ t \neq r, r+1}} \beta(x_t \oplus y_{m_{ij}(ws_r)+i-t}) \right.$$

$$\cdot\, H_C(\mathbf{x},\mathbf{y}) \cdot J_C(\mathbf{x},\mathbf{y}) \cdot \pi_r \left(\prod_{j=1}^{b_r(T)} \beta(x_r \oplus y_{\ell_r(T)+j}) \right). \tag{2.30}$$

Let $T' \in \Phi(C)$ be any given Rothe tableau in the equivalent class of $\Phi(C)$. By the construction of Φ, it is easy to see that

$$\prod_{(i,j)\in D(w)} \prod_{\substack{t\in T(i,j)\\ t\neq r,r+1}} \beta(x_t \oplus y_{m_{ij}(ws_r)+i-t}) = \prod_{(i,j)\in D(w)} \prod_{\substack{t\in T'(i,j)\\ t\neq r,r+1}} \beta(x_t \oplus y_{m_{ij}(w)+i-t}). \tag{2.31}$$

Again, by the construction of Φ, it is also clear that for $i > r+1$,

$$b_i(T) = b_i(T') \quad \text{and} \quad \ell_i(T) = \ell_i(T'),$$

which imply that

$$H_C(\mathbf{x},\mathbf{y}) = H_{C'}(\mathbf{x},\mathbf{y}). \tag{2.32}$$

Moreover, since $Q(T) = Q(T')$ and $m_{ij}(ws_r) = m_{ij}(w)$ for any $(i,j) \in Q^+(T)$, one has

$$J_C(\mathbf{x},\mathbf{y}) = J_{C'}(\mathbf{x},\mathbf{y}). \tag{2.33}$$

Still, by the construction of Φ, we see that

$$b_r(T) = b_{r+1}(T') + 1 \quad \text{and} \quad \ell_r(T) = \ell_{r+1}(T') - 1.$$

So, by (2.29), we have

$$\begin{aligned} \pi_r \left(\prod_{j=1}^{b_r(T)} \beta(x_r \oplus y_{\ell_r(T)+j}) \right) &= \pi_r \left(\prod_{j=1}^{b_{r+1}(T')+1} \beta(x_r \oplus y_{\ell_{r+1}(T')+j-1}) \right) \\ &= \beta\, h(C', r+1; \mathbf{x},\mathbf{y}). \end{aligned} \tag{2.34}$$

Substituting (2.31)–(2.34) into (2.30), we see that

$$\pi_r\, G^{(\beta)}_{ws_r}(C;\mathbf{x},\mathbf{y}) = G^{(\beta)}_{w}(\Phi(C);\mathbf{x},\mathbf{y}).$$

This completes the proof of Proposition 2.11. ∎

By Proposition 2.11 and the bijection Φ, we arrive at a proof of Proposition 2.5. Using induction on the length of w, we reach a proof of Proposition 2.1.

2.3 Proof of Proposition 2.2

In this subsection, we confirm Proposition 2.2 by proving the following statement.

Proposition 2.13 *If w contains a 1432 pattern, then*

$$\mathfrak{S}_w(\mathbf{x}) \neq \sum_{T\in \mathrm{SRT}(w,\mathbf{f}_0)} \prod_{(i,j)\in D(w)} \prod_{t\in T(i,j)} x_t. \tag{2.35}$$

Indeed, Proposition 2.13 implies Proposition 2.2. This can be seen as follows. Let w be a permutation containing a 1432 pattern. Suppose otherwise that $\mathfrak{G}_w^{(\beta)}(\mathbf{x},\mathbf{y}) = G_w^{(\beta)}(\mathbf{x},\mathbf{y})$. Taking $\beta = 0$ and $y_i = 0$ for all i on both sides, we are given

$$\mathfrak{S}_w(\mathbf{x}) = \sum_{T\in \mathrm{SRT}(w,\mathbf{f}_0)} \prod_{(i,j)\in D(w)} \prod_{t\in T(i,j)} x_t,$$

which is contrary to (2.35), and thus Proposition 2.2 holds.

To finish the proof of Proposition 2.13, we recall the balanced labeling model of Schubert polynomials due to Fomin, Greene, Reiner and Shimozono [11]. To a square (i,j) in the Rothe diagram $D(w)$, the associated hook $H_{i,j}(w)$ is the collection of squares (i',j') of $D(w)$ such that either $i'=i$ and $j'\geq j$, or $i'\geq i$ and $j'=j$.

A labeling L of $D(w)$ is an assignment of positive integers into the squares of $D(w)$ such that each square receives exactly one integer. We use $L(i,j)$ to denote the label in the square $(i,j)\in D(w)$. A labeling L is called *balanced* if for every square $(i,j)\in D(w)$, the label $L(i,j)$ remains unchanged after rearranging the labels in the hook $H_{i,j}(w)$ so that they are weakly increasing from right to left and from top to bottom [11].

A balanced labeling of $D(w)$ is said to be *column strict* if no column contains two equal labels. Let $\mathrm{CSBL}(w,\mathbf{f}_0)$ denote the set of column strict balanced labelings of $D(w)$ such that $L(i,j)\leq i$ for each square $(i,j)\in D(w)$. Figure 2.9 illustrates all the flagged column strict balanced labelings in $\mathrm{CSBL}(w,\mathbf{f}_0)$ for $w = 25143$.

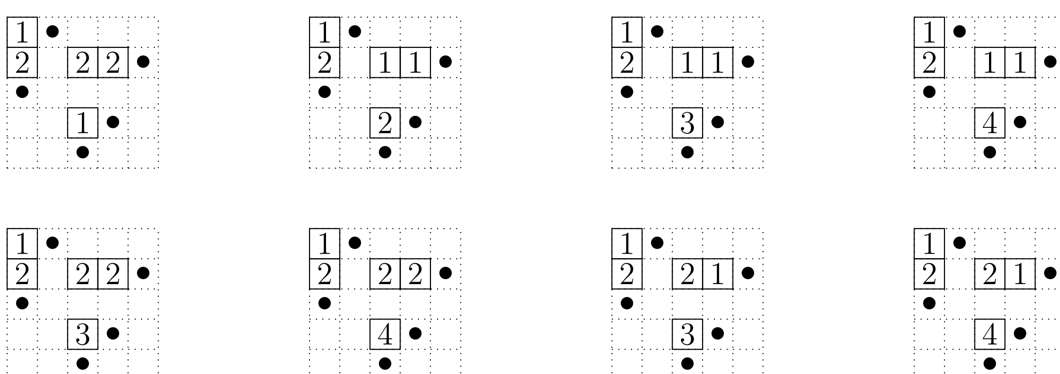

Figure 2.9: Balanced labelings in $\mathrm{CSBL}(w,\mathbf{f}_0)$ for $w = 25143$.

We are now in a position to give a proof of Proposition 2.13.

Proof of Proposition 2.13. Assume that w is a permutation that contains a pattern 1432. Recall that $\mathrm{SRT}(w, \mathbf{f}_0)$ is the set of single-valued Rothe tableaux of shape $D(w)$ flagged by $\mathbf{f}_0$. By definition, it is clear that $\mathrm{SRT}(w, \mathbf{f}_0) \subseteq \mathrm{CSBL}(w, \mathbf{f}_0)$. On the other hand, Fomin, Greene, Reiner and Shimozono [11] showed that

$$\mathfrak{S}_w(\mathbf{x}) = \sum_{L \in \mathrm{CSBL}(w, \mathbf{f}_0)} \prod_{(i,j) \in D(w)} x_{L(i,j)}. \tag{2.36}$$

In view of (2.36), to prove (2.35), it suffices to show that $\mathrm{SRT}(w, \mathbf{f}_0) \subsetneq \mathrm{CSBL}(w, \mathbf{f}_0)$. We aim to construct a balanced labeling $L \in \mathrm{CSBL}(w, \mathbf{f}_0) \backslash \mathrm{SRT}(w, \mathbf{f}_0)$.

Suppose that the subsequence $w_{i_1} w_{i_2} w_{i_3} w_{i_4}$ of w has the same relative order as the pattern 1432, that is, $w_{i_1} < w_{i_4} < w_{i_3} < w_{i_2}$. Since $w_{i_3} > w_{i_4}$, there exists at least one square in the i_3-th row of $D(w)$. Let (i_3, j) be the rightmost square in this row. Let

$$S = \{(i, j) \,|\, (i, j) \in D(w), i_1 \leq i \leq i_3\}$$

be the subset of $D(w)$ consisting of the squares in column j lying between row i_1 and row i_3. We classify S into two subsets according to whether a square $(i, j) \in S$ is the rightmost square in the row or not. Let $S_1 \subseteq S$ consists of square $(i, j) \in S$ such that (i, j) is the rightmost square in row i. Clearly, S_1 is nonempty since it contains the square (i_3, j). Let $S_2 = S \backslash S_1$ be the complement. Since $w_{i_2} > w_{i_3}$, we see that the two squares $(i_2, j), (i_2, w_{i_3})$ belong to $D(w)$. Hence $(i_2, j) \in S_2$, and so S_2 is also nonempty.

Let us use an example in Figure 2.10 to illustrate the sets S_1 and S_2. In this example, $w = 1\,4\,5\,9\,6\,10\,7\,8\,2\,3$ and the subsequence $w_1 w_6 w_7 w_9$ forms a 1432-pattern. The rightmost square of $D(w)$ in the i_3-th row is the square $(7, 3)$, and so we have

$$S = \{(i, 3) \,|\, i = 2, 3, 4, 5, 6, 7\}.$$

Moreover, the squares belonging to S_1 and S_2 are signified with $\spadesuit$ and $\clubsuit$ in Figure 2.10(a), respectively.

Let i_0 be the smallest row index such that: (1) the square $(i_0, j) \in S_1$; (2) there exists a square in S_2 lying above (i_0, j). Such an row index exists since the i_3-th row satisfies the above conditions. Let $S' = \{(i, j) \in S_1 \,|\, i \leq i_0\}$ be the subset of S_1 including the squares above (i_0, j). In the example in Figure 2.10, we see that $i_0 = 5$ and the squares of S' are signified with the symbol $\diamondsuit$.

Assume that $|S'| = k$ and $(r_1, j), \ldots, (r_k, j)$ are the squares of S', where $r_1 < \cdots < r_k = i_0$. Note that $r_1 > i_1$. This is because w_{i_1} is the smallest element of $\{w_{i_1}, w_{i_2}, w_{i_3}, w_{i_4}\}$ and thus the square $(i_1, j) \notin D(w)$.

We now construct a balanced labeling L of $D(w)$ as follows. If a square (s, t) of $D(w)$ is not contained in S', then we set $L(s, t) = s$. For the squares $(r_1, j), \ldots, (r_k, j)$ of S', we set $L(r_1, j) = i_1$ and $L(r_p, j) = r_{p-1}$ for $p = 2, \ldots, k$. For the permutation in Figure 2.10, the labeling L is given in Figure 2.11, where the integers in S' are written in boldface.

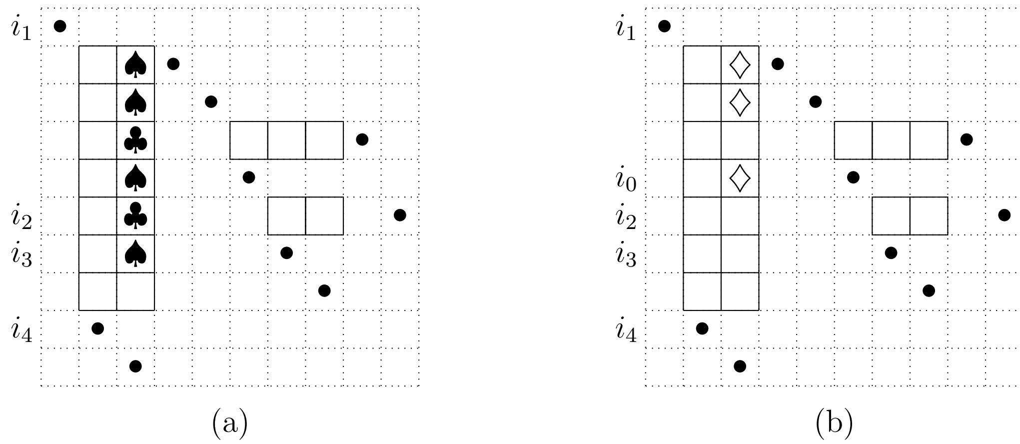

Figure 2.10: An example for the proof of Proposition 2.13.

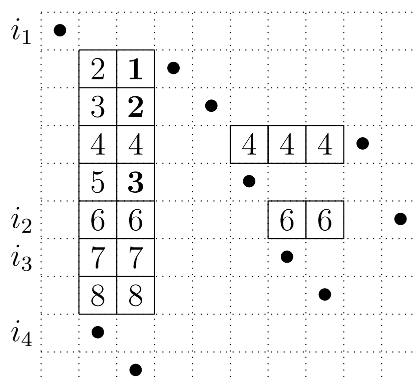

Figure 2.11: A balanced labeling in $\mathrm{CSBL}(w, \mathbf{f}_0)$, but not in $\mathrm{SRT}(w, \mathbf{f}_0)$.

By the construction of L, it is not hard to check that L is a column strict balanced labeling in $\mathrm{CSBL}(w, \mathbf{f}_0)$. Moreover, the entries in the j-th column of L are not increasing. So L does not belong to $\mathrm{SRT}(w, \mathbf{f}_0)$. This completes the proof. ∎

3 Rothe tableau complexes

In this section, we prove the tableau formulas in Theorem 1.4. To do this, we investigate the structure of Rothe tableau complexes, which is a specific family of the tableau complexes as introduced by Knutson, Miller and Yong [18]. Using Theorem 1.1 and the properties of tableau complexes established in [18], we obtain two alternative tableau formulas for the Grothendieck polynomials of 1432-avoiding permutations, as given in Theorem 1.4.

Let us proceed with a brief review of the Hilbert series of the Stanley-Reisner ring of a simplicial complex, see [29, 31] for more detailed information. An (abstract) simplicial complex Δ on a finite vertex set V is a collection of subsets of V such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. Each subset $\sigma \in \Delta$ is called a face of Δ. A face σ is called a facet of Δ if σ is not a subset of any other faces. Clearly, Δ is determined by its facets.

Let $\Bbbk[\mathbf{t}]$ be the ring of polynomials over a field $\Bbbk$ in the variables t_v where $v \in V$. The Stanley-Reisner ideal I_Δ is the ideal generated by the monomials corresponding to the subsets of V that are not faces of Δ, namely,

$$I_\Delta = \left\langle \prod_{v\in\tau} t_v \,|\, \tau \subseteq V, \text{ but } \tau \notin \Delta \right\rangle.$$

The Stanley-Reisner ring of Δ, denoted $\Bbbk[\Delta]$, is the quotient ring $\Bbbk[\mathbf{t}]/I_\Delta$. The Hilbert series $H(\Bbbk[\Delta]; \mathbf{t})$ of $\Bbbk[\mathbf{t}]/I_\Delta$ is equal to the sum of monomials not belonging to I_Δ. It is well known [29, 31] that $H(\Bbbk[\Delta]; \mathbf{t})$ has the following formula:

$$H(\Bbbk[\Delta]; \mathbf{t}) = \frac{K(\Bbbk[\Delta]; \mathbf{t})}{\prod_{v\in V}(1 - t_v)},$$

where

$$K(\Bbbk[\Delta]; \mathbf{t}) = \sum_{\sigma\in\Delta} \prod_{v\in\sigma} t_v \prod_{v\notin\sigma} (1 - t_v).$$

The numerator $K(\Bbbk[\Delta]; \mathbf{t})$ is called the K-polynomial of $\Bbbk[\Delta]$.

We now restrict attention to the K-polynomials of tableau complexes introduced in [18]. Let X and Y be two finite sets. A map f from X to Y is called a tableau, which can be viewed as an assignment of elements of Y to elements of X such that each $x \in X$ receives exactly one element of Y. A tableau f can also be identified with the following set

$$\{(x \mapsto y) \,|\, x \in X \text{ and } f(x) = y\} \subseteq X \times Y$$

of ordered pairs. Let U be a subset of tableaux from X to Y, and let $E \subseteq X \times Y$ be a set of ordered pairs such that $f \subseteq E$ for each $f \in U$. The tableau complex corresponding to U and E, denoted $\Delta_E(X \xrightarrow{U} Y)$, can be defined as follows. Let us first define a simplex Δ_E. For each pair $(x \mapsto a) \in E$, write $(x \not\mapsto y) = E \setminus \{(x \mapsto y)\}$ for the complement of $\{(x \mapsto y)\}$, and let

$$V = \{(x \not\mapsto y) \,|\, (x \mapsto y) \in E\}.$$

Denote by Δ_E the simplex with vertex set V, that is, Δ_E is the collection of all of the subsets of V.

Let $F \subseteq V$ be a face of Δ_E. Assume that F has k vertices $(x_1 \not\mapsto y_1), \ldots, (x_k \not\mapsto y_k)$. Then F can be identified with the following subset of E:

$$E \setminus \{(x_i \mapsto y_i) \,|\, 1 \leq i \leq k\}.$$

On the other hand, each subset of E can be viewed as a set-valued tableau from X to Y, that is, a map that assigns each element of X with a subset of Y. To be more specific, for a subset A of E, the corresponding set-valued tableau is defined by assigning $x \in X$ with the subset $\{y \in Y \mid (x \mapsto y) \in A\}$. So the face F of Δ_E can also be identified with a set-valued tableau such that for $x \in X$,

$$F(x) = \{y \in Y \mid (x \mapsto y) \in E, (x \mapsto y) \neq (x_i \mapsto y_i) \text{ for } 1 \leq i \leq k\}.$$

From now on, a face F of Δ_E can be identified either with a subset of E or with a set-valued tableau from X to Y, which will not cause confusion from the context. By the definition of Δ_E, a vertex $(x \not\mapsto y) \in V$ belongs to F if and only if the pair $(x \mapsto y)$ does not belong to F.

Recall that U is a set of tableaux from X to Y such that $f \subseteq E$ for each $f \in U$. So each tableau f in U is a face of Δ_E. The tableau complex $\Delta_E(X \xrightarrow{U} Y)$ is defined as the subcomplex of Δ_E such that the facets of $\Delta_E(X \xrightarrow{U} Y)$ are the tableaux in U. This means that a set-valued tableau $F \subseteq E$ is a face of $\Delta_E(X \xrightarrow{U} Y)$ if and only if F contains some tableau $f \in U$.

When X and Y are further endowed with partially ordered structures, Knutson, Miller and Yong [18] found three different expressions for the K-polynomial of a tableau complex.

Theorem 3.1 (Knutson-Miller-Yong [18]) *Let X and Y be two finite posets. For each $x \in X$, let Y_x be a totally ordered subset of Y. Let Ψ be a set of pairs (x, x') in X with $x < x'$. Let U be the set of tableaux $f\colon X \to Y$ such that*

(a) $f(x) \in Y_x$;

(b) f is weakly order preserving, that is, if $x \leq x'$, then $f(x) \leq f(x')$;

(c) if $(x, x') \in \Psi$, then $f(x) < f(x')$.

Set $E = \bigcup_{f \in U} f$. Let $\mathbf{t} = \{t_{(x \not\mapsto a)} \mid (x \not\mapsto a) \in V\}$ be the set of variables corresponding to the vertices of the tableau complex $\Delta = \Delta_E(X \xrightarrow{U} Y)$. Then, Δ is homeomorphic to a ball or a sphere. Moreover, the corresponding K-polynomial has the following expressions.

1. *Let U_1 be the set of set-valued tableaux $F \subseteq E$ such that every tableau $f \subseteq F$ lies in U. Then,*

$$K(\Bbbk[\Delta]; \mathbf{t}) = \sum_{F \in U_1} (-1)^{|F|-|X|} \prod_{x \in X} \prod_{a \in F(x)} \left(1 - t_{(x \not\mapsto a)}\right). \tag{3.1}$$

2. *Let U_2 be the set of set-valued tableaux $F \subseteq E$ each containing some tableau $f \in U$. Then,*

$$K(\Bbbk[\Delta]; \mathbf{t}) = \sum_{F\in U_2} \prod_{x\in X} \left(\prod_{a\in F(x)} \left(1 - t_{(x\mapsto\!\!\!\to a)}\right) \prod_{a\in E(x)\setminus F(x)} t_{(x\mapsto\!\!\!\to a)} \right). \tag{3.2}$$

3. *Given a tableau $f \in U$ and $x \in X$, let $Y_f(x)$ be the set of $y \in Y$ such that $f(x) < y$ and moving the label on x from $f(x)$ up to y still yields a tableau in U. Then,*

$$K(\Bbbk[\Delta]; \mathbf{t}) = \sum_{f\in U} \prod_{x\in X} \left(\left(1 - t_{(x\mapsto\!\!\!\to f(x))}\right) \prod_{a\in Y_f(x)} t_{x\mapsto\!\!\!\to a} \right). \tag{3.3}$$

We now consider the specific tableau complex such that the facets are the single-valued Rothe tableaux in $\mathrm{SRT}(w, \mathbf{f}_0)$. To be consistent with the aforementioned notation, let $X = D(w)$ and Y be the set of positive integers. Set $U = \mathrm{SRT}(w, \mathbf{f}_0)$ and

$$E = \bigcup_{T\in\mathrm{SRT}(w,\mathbf{f}_0)} T.$$

We denote the above defined tableau complex by $\Delta(w) = \Delta_E(X \xrightarrow{U} Y)$, and call $\Delta(w)$ the Rothe tableau complex for w.

Using Theorem 1.1 and Theorem 3.1, we can now give a proof of Theorem 1.4.

Proof of Theorem 1.4. We define a partial ordering on $D(w)$ as follows. For two distinct squares B and B' of $D(w)$, we use $B \to B'$ to represent that either B and B' are in the same row and B lies to the right of B', or B and B' are in the same column and B lies above B'. Define $B < B'$ if there exists a sequence $(B = B_1, B_2, \ldots, B_k = B')$ of squares of $D(w)$ such that

$$B = B_1 \to B_2 \to \cdots \to B_k = B'.$$

For each square $B = (i, j)$ of $D(w)$, let $Y_B = \{1, 2, \ldots, i\}$. Moreover, we set Ψ to be the set of pairs (B, B') with $B < B'$ such that B and B' are in the same column of $D(w)$. Now we see that the tableaux satisfying the conditions (a), (b) and (c) in Theorem 3.1 are exactly the single-valued Rothe tableaux in $\mathrm{SRT}(w, \mathbf{f}_0)$. Recall that the set U_1 defined in Theorem 3.1 consists of the set-valued tableaux $F \subseteq E$ such that every tableau in F lies in U. Clearly, $F \subseteq E$ is a set-valued tableau satisfying that every tableau contained in F lies in U if and only if F is a set-valued Rothe tableau in $\mathrm{SVRT}(w, \mathbf{f}_0)$. Thus we have $U_1 = \mathrm{SVRT}(w, \mathbf{f}_0)$. Replacing $t_{x\mapsto\!\!\!\to a}$ with $x = (i, j) \in D(w)$ by

$$\frac{x_a}{y_{m_{ij}(w)+i-a}}$$

and then replacing x_t by $1+\beta x_t$ and y_t by $\frac{1}{1+\beta y_t}$, the K-polynomial $K(\Bbbk[\Delta];\mathbf{t})$ in (3.1) becomes

$$\sum_{T\in \mathrm{SVRT}(w,\mathbf{f}_0)} (-1)^{|T|-\ell(w)}(-\beta)^{|T|} \prod_{(i,j)\in D(w)} \prod_{t\in T(i,j)} (x_t + y_{m_{ij}(w)+i-t} + \beta x_t y_{m_{ij}(w)+i-t}),$$

which, after dividing $(-\beta)^{\ell(w)}$, agrees with the formula in (1.2). Making the same substitutions in (3.2) and (3.3), we are led to (1.3) and (1.4) respectively. This completes the proof. ∎

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