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Laplacian ABC-Eigenvalues of Graphs*

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Abstract

For a graph G, the ABC-matrix of G was introduced and studied recently. A natural idea is to introduce the Laplacian ABC-matrix $\tilde{L}(G)$ of G. In this paper, some basic properties for the eigenvalues of the Laplacian ABC-matrix of a graph are explored. As one can see that they are not completely the same as those of the Laplacian matrix of a graph. More properties for the eigenvalues can be obtained by further study later.

1 Introduction

All graph considered in this paper are finite and simple. Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and size m with edge set $E(G) = \{e_1, e_2, \cdots, e_m\}$. For $v_i \in V(G)$, we use d_i to denote the degree of v_i. A dominating set in G is a subset X of $V(G)$, such that each vertex of $V(G) - X$ is adjacent to at least one vertex of X. The size

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of a smallest dominating set of G is the dominating number $\gamma(G)$. Let P_n, C_n, K_n and S_n denote the path, cycle, complete graph and star of order n, respectively. The complete bipartite graph is denoted by $K_{a,b}$. A graph is called r-regular if each of its vertices has the same degree r. A graph is (r, s)-semiregular if it is bipartite with a bipartition $\{V_1, V_2\}$ in which each vertex of V_1 has degree r and each one of V_2 has degree s. The union of two graphs G and H, denoted by $G \bigcup H$, is the graph with vertex set $V(G) \bigcup V(H)$ and edge set $E(G) \bigcup E(H)$. kG stands for the vertex-disjoint union of k copies of G.

Estrada et al. [6] proposed a topological index named atom-bond connectivity (ABC) index using a modification of Randić connectivity index. The ABC index of G is defined as

$$ABC(G) = \sum_{v_i v_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}},$$

which displays an excellent correlation with the heat of formation of alkanes. Estrada [5] also provided a probabilistic interpretation for the ABC index, which indicates that the term $\frac{d_i+d_j-2}{d_i d_j}$ represents the probability of visiting a nearest neighbor edge from one side or the other of a given edge in a graph. Then a matrix was defined from the ABC index, which is the square matrix $\tilde{A}(G) = (\tilde{a}_{ij})_{n\times n}$ of order n, whose entries $\tilde{a}_{ij}$ are given as

$$\tilde{a}_{ij} = \begin{cases} \sqrt{\frac{d_i+d_j-2}{d_i d_j}} & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $\tilde{A}(G)$ are called the ABC eigenvalues of G. The largest eigenvalue is called the ABC spectral radius of G. Actually, in 2015 Li proposed the idea to study the matrices defined from topological or chemical indices in [13].

In 2018, Chen [3] presented some results on the ABC eigenvalues and the ABC energy of a graph, which received quite a lot of attentions. Soon later, it was presented that, for any tree of order $n \geq 3$, P_n and $K_{1,n-1}$ have the smallest and the largest ABC spectral radii, respectively. Gao and Shao [8] showed that the star has the minimum ABC energy among all trees. Li and Wang [14] proved that C_n and $S_n + e$ have the smallest and the largest ABC spectral radii among unicyclic graphs, respectively. Ghorbani et al. [9] obtained some upper and lower bounds of ABC spectral radii and ABC energy.

The Laplacian matrix of graph G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of the vertex degrees of G, and $A(G)$ is the adjacency matrix of

G. The eigenvalues of $L(G)$ are denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The largest eigenvalue of the Laplacian matrix of G is called its Laplacian spectral radius. Since the Laplacian matrix of a graph can reflect more information on graph structures than its adjacency matrix, it is natural to generalize ABC-matrix of G to the Laplacian ABC-matrix.

Define the Laplacian ABC-matrix of G as $\tilde{L}(G) = \tilde{D}(G) - \tilde{A}(G)$, where $\tilde{D}(G) = (\tilde{d}_{ij})_{n\times n}$ is the ABC-diagonal matrix, whose entry $\tilde{d}_{ij}$ is

$$\tilde{d}_{ij} = \begin{cases} \sum\limits_{j=1}^{n} \tilde{a}_{ij} & \text{if i=j,} \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $\tilde{L}(G)$ are denoted by $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$, which are called Laplacian ABC-eigenvalues of G. The largest eigenvalue of Laplacian ABC-matrix of G is the Laplacian ABC-spectral radius of G. Let D be an oriented graph. The vertex-arc incidence matrix of D is an $n \times m$ matrix $R(G) = (r_{ie})$, where

$$r_{ie} = \begin{cases} -(\frac{d_i+d_j-2}{d_id_j})^{1/4} & \text{if } v_i \text{ is the initial vertex of e,} \\ 0 & \text{if } v_i \text{ and e are not incident,} \\ (\frac{d_i+d_j-2}{d_id_j})^{1/4} & \text{if } v_i \text{ is the terminal vertex of e.} \end{cases}$$

For any orientation of G, we have $\tilde{L}(G) = R(G)R^T(G)$. Then $\tilde{L}(G)$ is a positive semi-definite matrix. It is easy to see that 0 is an eigenvalue of $\tilde{L}(G)$ with eigenvector $\mathbf{1}$, which is the all 1 vector.

In this paper, we explore some basic properties of the eigenvalues of the Laplacian ABC-matrix of a graph. As one can see that they are not completely the same as those of the Laplacian matrix. More properties for the eigenvalues can be obtained by further study later.

2 Preliminary Results

Some lemmas are given as follows, which will be used in the sequel.

Lemma 2.1. *Let G be a connected graph of order $n \geq 3$. Then $rank(R(G)) = n - 1$.*

Proof. Assume that x is a vector of the left zero vector space for $R(G)$. That is,

$$x^T R(G) = \mathbf{0}.$$

where $\mathbf{0}$ is the zero vector. Suppose $v_iv_j \in E(G)$, whose corresponding direction is from v_j to v_i in the digraph D obtained from G. By above equality, we have

$$(x_i - x_j)\left(\frac{d_i + d_j - 2}{d_i d_j}\right)^{1/4} = 0.$$

As G is connected and $n \geq 3$, it yields that $x_i = x_j$ and each component of x are equal, which indicates that the dimension of the left zero vector space of $R(G)$ is at most 1. Then we have $rank(R(G)) \geq n - 1$.

On the other hand, we find that the sum of elements in each column of matrix $R(G)$ is 0, which means the rows of $R(G)$ are linearly dependent. So, $rank(R(G)) \leq n - 1$. □

Lemma 2.2. *Let G be a connected graph with $n \geq 3$ vertices. Then $\tilde{L}(G)$ has t $(2 \leq t \leq n)$ distinct eigenvalues if and only if there exist $t-1$ distinct nonzero numbers $r_1, r_2, ..., r_{t-1}$ such that*

$$\prod_{i=1}^{t-1}(\tilde{L}(G) - r_i I) = (-1)^{t-1}\frac{\prod\limits_{i=1}^{t-1} r_i}{n}J, \tag{2.1}$$

where I is the unit matrix of order n and J is the all 1 matrix of order n.

Proof. We first prove the sufficiency. Multiplying $\tilde{L}(G)$ for both sides of the equality (2.1), due to $\tilde{L}(G)J = \mathbf{0}$, we have

$$\tilde{L}(G)(\tilde{L}(G) - r_1 I)(\tilde{L}(G) - r_2 I)...(\tilde{L}(G) - r_{t-1} I) = \mathbf{0},$$

where $\mathbf{0}$ is zero matrix of size n. By the definition of minimal polynomial $\varphi(x)$ of a matrix, we get that the minimal polynomial of $\tilde{L}(G)$ is

$$\varphi(x) = x(x - r_1)(x - r_2)...(x - r_{t-1}).$$

Hence, $\tilde{L}(G)$ has t distinct eigenvalues $0, r_1, r_2, ..., r_{t-1}$.

For the necessity, except the zero eigenvalue, let $r_1, r_2, ..., r_{t-1}$ be the nonzero distinct eigenvalues of $\tilde{L}(G)$. Then we get the minimal polynomial $\varphi(x) = x(x - r_1)(x - r_2)...(x - r_{t-1})$ directly, which implies that

$$\tilde{L}(G)\prod_{i=1}^{t-1}(\tilde{L}(G) - r_i I) = \mathbf{0}.$$

Since G is connected, any eigenvector of $\tilde{L}(G)$ corresponding to the 0 eigenvalue is a scalar multiple of the vector $\mathbf{1}$. So the ith column vector of matrix $\prod\limits_{i=1}^{t-1}(\tilde{L}(G) - r_i I)$ can be written in the form $c_i\mathbf{1}$ for some $c_i(i = 1, 2, ..., n)$. Hence,

$$\prod_{i=1}^{t-1}(\tilde{L}(G) - r_i I) = \mathbf{1}(c_1, c_2, ..., c_n).$$

Multiplying $\mathbf{1}^T$ to both sides of the above equality, we get

$$(-1)^{t-1}\prod_{i=1}^{t-1} r_i \mathbf{1}^T = n(c_1, c_2, ..., c_n).$$

For $i = 1, 2, ..., n$, it is easy to see that

$$c_i = (-1)^{t-1}\frac{\prod_{i=1}^{t-1} r_i}{n}.$$

Then the result follows. □

Lemma 2.3. *[15] Let G be a graph of order $n \geq 2$. Then*

$$\mu_{n-1}(G) \leq \frac{n(n - 2\gamma(G) + 1)}{n - \gamma(G)},$$

and the equality holds if and only if $G = K_{2,2}$.

Lemma 2.4. *[2] Let G be a graph on n vertices. Then*

$$ABC(G) \leq \frac{n\sqrt{2n-4}}{2},$$

and the equality holds if and only if $G = K_n$.

Lemma 2.5. *[7] Let $f(x, y) = \sqrt{\frac{x+y-2}{xy}}$, where $n - 1 \geq x \geq 2, n - 1 \geq y \geq 1$. Then*
(i) $f(x, 1)$ is an increasing function with respect to x, and hence

$$\frac{\sqrt{2}}{2} = f(2, 1) \leq f(x, 1) \leq f(n - 1, 1) = \sqrt{\frac{n-2}{n-1}};$$

(ii) $f(x, 2) = \frac{\sqrt{2}}{2}$;
(iii) For $x \geq y \geq 3$, $f(x, y)$ is a decreasing function with respect to x and y, and hence

$$\frac{\sqrt{2n-4}}{n-1} = f(n - 1, n - 1) \leq f(x, y) \leq f(3, 3) = \frac{2}{3}.$$

Let A, B be real matrices of order n. We write $A \succeq B$ if the matrix $A - B$ is positive semi-definite.

Lemma 2.6. *[12] Let A, B be real matrices of order n. Let $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ and $\lambda_1(B) \geq \lambda_2(B) \geq \cdots \geq \lambda_n(B)$ be the ordered eigenvalues, respectively. If $A \succeq B$, then $\lambda_i(A) \geq \lambda_i(B)$ for each $i = 1, 2, \cdots, n$.*

Lemma 2.7. *[11] Let M be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Given a partition $\{1, 2, \cdots, n\} = \Delta_1 \bigcup \Delta_2 \bigcup \cdots \bigcup \Delta_m$, where $|\Delta_i| = n_i > 0$. Considering the corresponding blocking $M = (M_{ij})$, such that M_{ij} is an $n_i \times n_j$ block. Let e_{ij} be the sum of the entries in M_{ij} and put $B = (\frac{e_{ij}}{n_i})$ (i.e., $\frac{e_{ij}}{n_i}$ is an average row sum in M_{ij}). The eigenvalues of B are $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_m$. Then the inequalities*

$$\lambda_i \geq \nu_i \geq \lambda_{n-m+i},$$

hold for each $i = 1, 2, ..., m$. Moreover, if for some integer $k, 1 \leq k \leq m, \lambda_i = \nu_i (i = 1, 2, \cdots, k)$ and $\lambda_{n-m+i} = \nu_i (i = k+1, k+2, \cdots, m)$, then all the blocks M_{ij} of M have constant row and column sums.

Lemma 2.8. *Let G be a connected graph of order $n \geq 2$. Given a bipartition $\{v_1, v_2, \cdots, v_n\} = \Delta_1 \bigcup \Delta_2$, with $|\Delta_1| = n_1 > 0, |\Delta_2| = n_2 > 0, n_1 + n_2 = n$. The matrix $\tilde{L}(G)$ is composed of the block $\tilde{L}_{ij}$, which is an $n_i \times n_j$ block, for $1 \leq i, j \leq 2$. Suppose*

$$s_1 = \sum_{i=1}^{n_1} \frac{\sum\limits_{j=1, v_i \sim v_j}^{n} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}}{n_1}, \quad t_1 = \sum_{i=1}^{n_1} \frac{\sum\limits_{j=1, v_i \sim v_j}^{n_1} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}}{n_1}.$$

Then

$$\xi_1 \geq \frac{n(s_1 - t_1)}{n_2}.$$

Moreover, if the equality holds, then all the blocks $\tilde{L}_{ij}$ of $\tilde{L}(G)$ have constant row and column sums.

Proof. Assume that

$$s_2 = \sum_{i=n_1+1}^{n} \frac{\sum\limits_{j=1, v_i \sim v_j}^{n} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}}{n_2}, \quad t_2 = \sum_{i=n_1+1}^{n} \frac{\sum\limits_{j=n_1+1, v_i \sim v_j}^{n} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}}{n_2}.$$

By this partition of vertices, we rewrite $\tilde{L}(G)$ as

$$\tilde{\mathbf{L}}(\mathbf{G}) = \begin{pmatrix} \tilde{L}_{11} & \tilde{L}_{12} \\ \tilde{L}_{21} & \tilde{L}_{22} \end{pmatrix} = \begin{pmatrix} \tilde{D}_{11} - \tilde{A}_{11} & -\tilde{A}_{12} \\ -\tilde{A}_{21} & \tilde{D}_{22} - \tilde{A}_{22} \end{pmatrix}.$$

For $1 \le i, j \le 2$, let e_{ij} be the sum of the entries in $\tilde{L}_{ij}$ and put $B = (\frac{e_{ij}}{n_i})$. Then

$$\mathbf{B} = \begin{pmatrix} s_1 - t_1 & t_1 - s_1 \\ t_2 - s_2 & s_2 - t_2 \end{pmatrix}.$$

Thus, from $det(\nu I - B) = \nu(\nu - s_1 - s_2 + t_1 + t_2)$, the two eigenvalues of B are $\nu_1 = s_1 + s_2 - t_1 - t_2$ and $\nu_2 = 0$, respectively. From Lemma 2.7, we get

$$\xi_1 \ge s_1 + s_2 - t_1 - t_2.$$

Recall that $\tilde{L}(G)$ is symmetric, the sum of the entries in $\tilde{L}_{12}$ is equal to that for $\tilde{L}_{21}$. Then we have $n_1(s_1 - t_1) = n_2(s_2 - t_2)$. Hence,

$$s_1 + s_2 - t_1 - t_2 = \frac{n(s_1 - t_1)}{n_2}.$$

Then

$$\xi_1 \ge \frac{n(s_1 - t_1)}{n_2}.$$

If the equality holds, then $\xi_1 = \nu_1$. Due to $\xi_n = \nu_2 = 0$, from Lemma 2.7 again, this implies that all the blocks $\tilde{L}_{ij}$ of $\tilde{L}(G)$ have constant row and column sums. □

3 Main Results

To begin with, the Laplacian ABC-eigenvalues for several kinds of special graphs are shown below.

Theorem 3.1. *Let G be a graph of order n.*
(1) If G is r-regular, then $\xi_i = \frac{\sqrt{2r-2}}{r}\mu_i$, for $i = 1, 2, \cdots, n$. In particular, if $G = K_n$, then $\xi_1 = \xi_2 = \cdots = \xi_{n-1} = \frac{n\sqrt{2n-4}}{n-1}, \xi_n = 0$; If $G = C_n$, then $\xi_i = \sqrt{2} - \sqrt{2}\cos\frac{2\pi i}{n}$, for $i = 0, 1, \cdots, n-1$.

(2) If G is (r, s)-semiregular bipartite, then $\xi_i = \sqrt{\frac{r+s-2}{rs}}\mu_i$, for $i = 1, 2, \cdots, n$. In particular, if $G = K_{a,b}$, where $a + b = n, a \ge b$, then $\xi_1 = n\sqrt{\frac{n-2}{ab}}, \xi_2 = \xi_3 = \cdots = \xi_b = \sqrt{\frac{a^2+ab-2a}{b}}, \xi_{b+1} = \xi_{b+2} = ... = \xi_{n-1} = \sqrt{\frac{b^2+ab-2b}{a}}, \xi_n = 0$.

(3) If G has a vertex cover consisting of only the vertices of degree 2, then $\xi_i = \frac{\sqrt{2}}{2}\mu_i$, for $i = 1, 2, \cdots, n$. In particular, if $G = P_n$, then $\xi_i = \sqrt{2} - \sqrt{2}\cos\frac{\pi i}{n}$, for $i = 0, 1, \cdots, n-1$.

Proof. (1) If G is r-regular, then we can easily get $\tilde{L}(G) = \frac{\sqrt{2r-2}}{r}L(G)$, and hence $\xi_i = \frac{\sqrt{2r-2}}{r}\mu_i$, for $i = 1, 2, \cdots, n$. This together with the fact that if $G = K_n$, $\mu_1 = \mu_2 = \cdots = \mu_{n-1} = n, \mu_n = 0$ and if $G = C_n$, $\mu_i = 2 - 2\cos\frac{2\pi i}{n}$, for $i = 0, 1, \cdots, n-1$, would yield the required result.

(2) If G is (r, s)-semiregular bipartite, it is also easy to see that $\tilde{L}(G) = \sqrt{\frac{r+s-2}{rs}}L(G)$ and $\xi_i = \sqrt{\frac{r+s-2}{rs}}\mu_i$, for $i = 1, 2, \cdots, n$. Then by the fact that, if $G = K_{a,b}$, $\mu_1 = n, \mu_2 = \mu_3 = \mu_b = a, \mu_{b+1} = \mu_{b+2} = \mu_{n-1} = b, \mu_n = 0$, we obtain the desired result.

(3) If G has a vertex cover consisting of only the vertices of degree 2, it implies that every edge of G has at least one endpoint of degree 2. So, $\tilde{L}(G) = \frac{\sqrt{2}}{2}L(G)$, showing that $\xi_i = \frac{\sqrt{2}}{2}\mu_i$, for $i = 1, 2, \cdots, n$. If $G = P_n$, then $\mu_i = 2 - 2\cos\frac{\pi i}{n}$, for $i = 0, 1, \cdots, n-1$. □

Theorem 3.2. *Let G be a graph of order n. Then G has exactly one (distinct) Laplacian ABC-eigenvalue if and only if $G = rK_2 \bigcup (n-2r)K_1$, where $0 \leq r \leq \frac{n}{2}$.*

Proof. Note that $\tilde{L}(G)$ is a positive semi-definite matrix. One can see that $tr\tilde{L}(G) = 0$ if and only if $\tilde{L}(G) = \mathbf{0}$. Then for each vertex of G, its degree is 0, or 1, which means that $G = rK_2 \bigcup (n-2r)K_1$, with $0 \leq r \leq \frac{n}{2}$. □

It is well-known that the multiplicity of eigenvalue zero for the Laplacian matrix of a graph is the number of its components. Similarly, we obtain the following result.

Theorem 3.3. *Let G be a connected graph of order $n \geq 3$. Suppose G has s connected components, which have n_i vertices, respectively, where $n_i \geq 3$, $i = 1, 2, ..., s$. Then the number of components of G is equal to the multiplicity of eigenvalue 0 for $\tilde{L}(G)$.*

Proof. It is easy to get that

$$rank(\tilde{L}(G)) = rank(R(G)R(G)^T) = rank(R(G)).$$

If G is connected, by Lemma 2.1, then $rank(\tilde{L}(G)) = rank(R(G)) = n-1$. Combining with the fact that $\tilde{L}(G)$ is a real symmetric matrix, the multiplicity of eigenvalue 0 is 1. Otherwise, suppose G has s (> 1) connected components, which have n_i vertices, respectively, where $n_i \geq 3$, $i = 1, 2, \cdots, s$. Applying Lemma 2.1, we can get $rank(\tilde{L}(G)) = n-s$, which means that the multiplicity of eigenvalue 0 is s. □

Theorem 3.4. *Let G be a graph with $n \geq 3$ vertices. Then G has exactly two distinct Laplacian ABC-eigenvalues if and only if $G = K_n$.*

Proof. By Lemma 2.2, G has exactly two distinct Laplacian ABC-eigenvalues if and only if there is a non-zero number r such that

$$\tilde{L}(G) - rI = -\frac{r}{n}J.$$

That is,

$$\tilde{L}(G) = rI - \frac{r}{n}J.$$

We can see that the off-diagonal entries of $\tilde{L}(G)$ are all non-zero. Thus, we see that $G = K_n$ and $r = \frac{n\sqrt{2n-4}}{n-1}$. □

Next, we give two upper bounds on the second smallest Laplacian ABC-eigenvalue ξ_{n-1}.

Theorem 3.5. *Let G be a connected graph of order $n \geq 3$. Suppose G has a vertex cover only consisting of the vertices of degree 2. Then*

$$\xi_{n-1} \leq \frac{n(n - 2\gamma(G) + 1)}{\sqrt{2}(n - \gamma(G))},$$

with equality holding if and only if $G = K_{2,2}$.

Proof. Since G has a vertex cover only consisting of the vertices of degree 2, we get $\tilde{L}(G) = \frac{\sqrt{2}}{2}L(G)$. It is easy to check that $\xi_{n-1} = \frac{\sqrt{2}}{2}\mu_{n-1}$. By Lemma 2.3, the result holds. □

Theorem 3.6. *Let G be a connected graph of order $n \geq 3$. Then*

$$\xi_{n-1} \leq \frac{n\sqrt{2n-4}}{n-1},$$

where the equality holds if and only if $G = K_n$.

Proof. Note that

$$tr(\tilde{L}(G)) = \xi_1 + \xi_2 + ... + \xi_n = 2ABC(G). \tag{3.2}$$

As $\xi_n = 0$, we have

$$\xi_{n-1} \leq \frac{2ABC(G)}{n-1}.$$

From Lemma 2.4, we obtain

$$\xi_{n-1} \leq \frac{n\sqrt{2n-4}}{n-1} \ .$$

Then we discuss the case that the upper bound is tight. Suppose the equality holds, it means that

$$\xi_{n-1} = \frac{2ABC(G)}{n-1} = \frac{n\sqrt{2n-4}}{n-1} \ . \tag{3.3}$$

By (3.2) and $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n-1}$, we get $\xi_1 = \xi_2 = \cdots = \xi_{n-1}$. Combing with the condition that G is a connected graph of order $n \geq 3$ and Theorem 3.4, it yields $G = K_n$. From the latter equality of (3.3) and Lemma 2.4, it is easy to check that G is a complete graph. Conversely, if $G = K_n$, the equality holds by direct computation. □

For the largest Laplacian ABC-eigenvalue ξ_1, we have that $L(G) - \tilde{L}(G)$ is a positive semi-definite matrix. By Lemma 2.6, we get $\xi_1 \leq \mu_1$. It is well know that $\mu_1 \leq n$, and thus $\xi_1 \leq n$. Meantime, a lower bound on the largest Laplacian ABC-eigenvalue ξ_1 of a connected graph is obtained.

Theorem 3.7. *Let G be a connected graph of order $n \geq 2$. Then*

$$\xi_1 \geq \frac{n\sqrt{2n-4}}{(n-1)\gamma(G)} \, ,$$

with equality holding if and only if $G = K_n$.

Proof. Let X be a dominating set of G and suppose $|X| = \gamma(G)$. Assume that E_X is the set of all edges with one end vertex in X and the other one in $V - X$. According to the definition of a dominating set, we have

$$|E_X| \geq n - \gamma(G).$$

By Lemma 2.8, we know

$$\xi_1 \geq \frac{n(s_1 - t_1)}{n - \gamma(G)},$$

where

$$s_1 = \sum_{i=1}^{\gamma(G)} \frac{\sum\limits_{j=1, v_i \sim v_j}^{n} \sqrt{\frac{d_i+d_j-2}{d_i d_j}}}{\gamma(G)} \, , \ t_1 = \sum_{i=1}^{\gamma(G)} \frac{\sum\limits_{j=1, v_i \sim v_j}^{\gamma(G)} \sqrt{\frac{d_i+d_j-2}{d_i d_j}}}{\gamma(G)} \ .$$

Since G is connected, by Lemma 2.5 we get $\sqrt{\frac{d_i+d_j-2}{d_i d_j}} \geq \frac{\sqrt{2n-4}}{n-1}$. So, we can see that $s_1 - t_1 =$

$$\frac{1}{\gamma(G)} \left(\sum_{i=1}^{\gamma(G)} \left(\sum_{j=\gamma(G)+1, v_i \sim v_j}^{n} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}} \right) \right) \geq \frac{1}{\gamma(G)} \frac{\sqrt{2n-4}}{n-1} |E_X| \geq \frac{(n-\gamma(G))\sqrt{2n-4}}{\gamma(G)(n-1)}.$$

Thus,

$$\xi_1 \geq \frac{n\sqrt{2n-4}}{\gamma(G)(n-1)}.$$

Now we show that the upper bound is tight. Suppose the equality holds. Then by Lemma 2.8, we have $\xi_1 = \frac{n(s_1-t_1)}{n-\gamma(G)}$, which implies that it is according to the vertex partition $V(G) = X \bigcup (V(G) - X)$, in which every block $\tilde{L}_{ij}$ of the blocking matrix $\tilde{L}(G) = (\tilde{L}_{ij})$ has constant row and column sums, respectively. That is, $\tilde{L}_{12}$ and $\tilde{L}_{21}$ have constant row and column sums, respectively. Due to $|E_X| = n - \gamma(G)$, every column of $\tilde{L}_{12}$ has exactly one non-zero entry (i.e., every row of $\tilde{L}_{21}$ has exactly one non-zero entry.) Obviously, each column of $\tilde{L}_{12}$ has the same non-zero value, which is $\frac{\sqrt{2n-4}}{n-1}$. Thus, we obtain $d_{\gamma(G)+1} = d_{\gamma(G)+2} = ... = d_n = n-1$. Combining with the fact that X is the smallest dominating set, we get $\gamma(G) = 1, d_1 = n-1$ and $G = K_n$. Conversely, it is not hard to check that it holds for the case $G = K_n$. □

At last, we survey the interlacing property of the Laplacian ABC-eigenvalues. It is well-known that the Laplacian eigenvalues of a graph G possess the interlacing property when one of its edge is deleted.

Theorem 3.8. *[10] Let G be a graph of order n. Suppose e is an edge of G and $G' = G-e$. Then*

$$0 = \mu_n(G') = \mu_n(G) \leq \mu_{n-1}(G') \leq \mu_{n-1}(G) \leq ... \leq \mu_2(G) \leq \mu_1(G') \leq \mu_1(G).$$

However, by the example below, we find that it does not hold for our Laplacian ABC-eigenvalues of a graph.

Example. Let G_1 be the graph obtained from the star S_6 by adding an edge. By direct computation, we get $\xi_1(G_1) \approx 4.9292 < \xi_1(S_6) \approx 5.3666$.

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