SUMS OF SQUARES AND PARTITION CONGRUENCES

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ABSTRACT. For positive integers n and k, let $r_k(n)$ denote the number of representations of n as a sum of k squares, where representations with different orders and different signs are counted as distinct. For a given positive integer m, by means of some properties of binomial coefficients, we derive some infinite families of congruences for $r_k(n)$ modulo 2^m. Furthermore, in view of these arithmetic properties of $r_k(n)$, we establish many infinite families of congruences for the overpartition function and the overpartition pair function.

1. Introduction

For positive integers n and k, let $r_k(n)$ denote the number of representations of n as a sum of k squares, where representations with different orders and different signs are counted as distinct. By convention, $r_k(0) = 1$.

The purpose of this paper is to establish some arithmetic properties of $r_k(n)$ modulo powers of 2, and then find some congruences for some partition functions by using these properties.

The values of $r_k(n)$ have been widely studied in the literature. Jacobi [20] derived the values of $r_k(n)$ for $k = 2, 4, 6, 8$. Then Glaisher [15] obtained the formulas for $k = 10, 12, 14, 16, 18$. Moreover, Mordell [26] and Cooper [8] proved a general formula of $r_k(n)$ for arbitrary even values of k which was provided by Ramanujan [28]. In addition, Newman [27] presented the following recurrence formulas of $r_k(n)$. For any odd prime p,

$$r_k(np^2) = \left\{1 + p^{k-2} - (-1)^{\frac{(k-1)(p-1)}{4}} p^{\frac{k-3}{2}} \left(\frac{n}{p}\right)\right\} r_k(n) - p^{k-2} r_k\left(\frac{n}{p^2}\right), \quad k = 1, 3, 5, 7,$$

$$r_k(np^2) = \left\{1 + p^{k-2} + (-1)^{\frac{k(p-1)}{4}} p^{\frac{k-2}{2}} \left(\frac{n}{p}\right)^2\right\} r_k(n) - p^{k-2} r_k\left(\frac{n}{p^2}\right), \quad k = 2, 4, 6, 8.$$

For odd k, Hardy [16, 17] studied representations of a number as the sum of five or seven squares. Then Lomadze [23] discussed the number of representations of natural numbers by sums of nine squares. Hirschhorn and Sellers [18] obtained some properties of $r_3(n)$. Later, Cooper [9] derived the recurrence formulas of $r_k(n)$ for $k = 5, 7, 9$. Recently, Berndt et al. [2, 3] studied some transformations involving $r_k(n)$ and some Bessel functions. For more properties of $r_k(n)$, see [1, 10, 11, 29, 30].

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Using some arithmetic properties of binomial coefficients, we obtain the following theorems. Notice that all the parameters used in this paper are integers.

Theorem 1.1. *If m, k, and the prime p satisfy one of the following conditions:*

(1) $m \geq 5$, $k \equiv 0, 2 \pmod{2^{m-4}}$, *and* $p \equiv 7 \pmod 8$;
(2) $m \geq 4$, $k \equiv 1 \pmod{2^{m-3}}$, *and* $p \equiv 7 \pmod 8$;
(3) $m \geq 4$, $k \equiv 0, 2 \pmod{2^{m-3}}$, *and* $p \equiv 3 \pmod 4$;
(4) $m \geq 2$, $k \equiv 0, 1 \pmod{2^{m-2}}$, *and* $p \equiv 1 \pmod 2$,

then for $\alpha \geq 0$, $n \geq 0$, and $j = 1, 2 \ldots, p-1$,

$$r_k\left(p^{2\alpha+1}(pn+j)\right) \equiv 0 \pmod{2^m}.$$

Theorem 1.2. *If m, k, and the prime p satisfy one of the following conditions:*

(1) $m \geq 3$, $k \equiv 0, 1 \pmod{2^{m-2}}$, *and* $p \equiv \pm 1 \pmod 8$;
(2) $m \geq 2$, $k \equiv 1 \pmod{2^{m-1}}$, *and* $p \equiv 1 \pmod 2$,

then for $\alpha \geq 0$ and $n \geq 0$,

$$r_k\left(p^{2\alpha}(pn+r)\right) \equiv 0 \pmod{2^m},$$

where r is a quadratic nonresidue modulo p.

Throughout the paper, we make use of the standard q-series notation [14]

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \quad |q|<1.$$

For convenience, define

$$f_k = (q^k;q^k)_\infty.$$

Let $f(a,b)$ be Ramanujan's general theta function given by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab|<1. \tag{1.1}$$

Then Jacobi's triple product identity can be stated in Ramanujan's notation as follows

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Thus,

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}.$$

Then the generating function of $r_k(n)$ is stated as

$$\sum_{n=0}^{\infty} r_k(n) q^n = \varphi^k(q).$$

This paper is organized as follows. In Section 2, we establish some congruence properties of binomial coefficients which are used in the proofs of the above theorems.

In Section 3, in view of the lemmas given in Section 2, we prove Theorems 1.1 and 1.2. In Section 4, applying the main theorems, we derive many infinite families of congruences for the overpartition function and the overpartition pair function.

2. Preliminaries

In this section, we prove some lemmas which are used in the proofs of Theorems 1.1 and 1.2. Let k, ℓ, and m be positive integers. It is obvious that for $\ell \geq m$,

$$2^\ell \binom{k}{\ell} \equiv 0 \pmod{2^m}.$$

In the proofs of the following lemmas, set

$$\ell = \begin{cases} 2^s t, & \text{if } \ell \text{ is even;} \\ 2^s t + 1, & \text{if } \ell \text{ is odd,} \end{cases}$$

where s is a positive integer and t is an odd positive integer.

Lemma 2.1. *For $m \geq 5$, even $\ell \geq 6$, and $k \equiv 0 \pmod{2^{m-4}}$, we have*

$$2^\ell \binom{k}{\ell} \equiv 0 \pmod{2^m}.$$

Proof. For $k \equiv 0 \pmod{2^{m-4}}$, let

$$k = 2^{m-4} g,$$

where g is a positive integer. Since ℓ is even, we have $\ell = 2^s t$. Then

$$\begin{aligned} 2^\ell \binom{k}{\ell} &= 2^\ell \frac{k}{\ell} \prod_{i=1}^{\frac{\ell-2}{2}} \frac{k-2i}{2i} \prod_{j=1}^{\frac{\ell}{2}} \frac{k-2j+1}{2j-1} \\ &= 2^{\ell-s+m-4} \frac{g}{t} \prod_{i=1}^{\frac{\ell-2}{2}} \frac{2^{m-4}g-2i}{2i} \prod_{j=1}^{\frac{\ell}{2}} \frac{k-2j+1}{2j-1}. \end{aligned}$$

Notice that the numerator and denominator of

$$\prod_{j=1}^{\frac{\ell}{2}} \frac{k-2j+1}{2j-1}$$

are odd integers. For $1 \leq i \leq \frac{\ell-2}{2}$, set

$$2i = 2^{s_i} t_i,$$

where $s_i \geq 1$ and $t_i \equiv 1 \pmod 2$. We consider the following two cases (1) and (2).

(1) If $s_i \leq m-4$ for all $1 \leq i \leq \frac{\ell-2}{2}$, then

$$2^\ell \binom{k}{\ell} = 2^{\ell-s+m-4} \frac{g}{t} \prod_{i=1}^{\frac{\ell-2}{2}} \frac{2^{m-4}g-2^{s_i}t_i}{2^{s_i}t_i} \prod_{j=1}^{\frac{\ell}{2}} \frac{k-2j+1}{2j-1}$$

$$= 2^{\ell-s+m-4}\frac{g}{t}\prod_{i=1}^{\frac{\ell-2}{2}}\frac{2^{m-4-s_i}g-t_i}{t_i}\prod_{j=1}^{\frac{\ell}{2}}\frac{k-2j+1}{2j-1}, \tag{2.1}$$

where t, t_i $(1 \le i \le \frac{\ell-2}{2})$, and $2j-1$ $(1 \le j \le \frac{\ell}{2})$ in (2.1) are odd integers. Thus, to prove

$$2^\ell \binom{k}{\ell} \equiv 0 \pmod{2^m},$$

it suffices to show that

$$2^{\ell-s+m-4} \equiv 0 \pmod{2^m}$$

which means

$$\ell - s + m - 4 \ge m.$$

(a) If $s = 1$, then since $\ell \ge 5$, we have

$$\ell - s + m - 4 = \ell - 1 + m - 4 \ge m.$$

(b) If $s \ge 2$, then

$$\ell - s + m - 4 = \ell - log_2\frac{\ell}{t} + m - 4 \ge \ell - log_2\ell - 4 + m. \tag{2.2}$$

Notice that the function

$$y - log_2 y - 4$$

is monotonically increasing for $y \ge 4$. So, for

$$\ell \ge 7 = min\{y \mid y - log_2 y - 4 \ge 0 \text{ and } y \ge 4\},$$

we have

$$\ell - log_2\ell - 4 \ge 0.$$

Hence, based on (2.2) and the above inequality, we obtain

$$\ell - s + m - 4 \ge \ell - log_2\ell - 4 + m \ge m.$$

Therefore, combining the subcases (a) and (b) yields that for $s = 1$ and $\ell \ge 5$, or $s \ge 2$ and $\ell \ge 7$,

$$2^\ell \binom{k}{\ell} \equiv 0 \pmod{2^m}.$$

That is to say, in Case (1), for $m \ge 5$, even $\ell \ge 6$, and $k \equiv 0 \pmod{2^{m-4}}$,

$$2^\ell \binom{k}{\ell} \equiv 0 \pmod{2^m}.$$

(2) If there exists a positive integer d such that

$$1 \le d \le \frac{\ell-2}{2} \quad \text{and} \quad s_d > m-4,$$

then since

$$1 \le i \le \frac{\ell-2}{2} \quad \text{and} \quad 2i = 2^{s_i}t_i,$$

we have

$$\ell \ge 2i + 2 = 2^{s_i}t_i + 2.$$

So,

$$\ell \geq 2^{s_d} t_d + 2 > 2^{m-4} + 2.$$

Notice that the function

$$2^{x-4} - x + 3$$

is monotonically increasing for $x \geq 5$. Then for

$$m \geq 5 = min\{x \mid 2^{x-4} - x + 3 \geq 0 \text{ and } x \geq 5\},$$

we derive

$$2^{m-4} - m + 3 \geq 0.$$

Therefore, we obtain

$$\ell > 2^{m-4} + 2 \geq m - 3 + 2 = m - 1$$

which means $\ell \geq m$. So we have

$$2^\ell \binom{k}{\ell} \equiv 0 \pmod{2^m}.$$

In conclusion, combining Cases (1) and (2), we complete the proof. □

Lemma 2.2. *For $m \geq 6$, odd $\ell \geq 5$, and $k \equiv 0 \pmod{2^{m-4}}$, we have*

$$2^\ell \binom{k}{\ell} \equiv 0 \pmod{2^m}.$$

Proof. Set $k = 2^{m-4}g$ and $\ell = 2^s t + 1$. Then

$$\begin{aligned} 2^\ell \binom{k}{\ell} &= 2^\ell \frac{k}{k-\ell} \prod_{i=1}^{\frac{\ell-1}{2}} \frac{k-2i}{2i} \prod_{j=1}^{\frac{\ell+1}{2}} \frac{k-2j+1}{2j-1} \\ &= 2^{\ell+m-4} \frac{g}{k-\ell} \prod_{i=1}^{\frac{\ell-1}{2}} \frac{2^{m-4}g-2i}{2i} \prod_{j=1}^{\frac{\ell+1}{2}} \frac{k-2j+1}{2j-1}. \end{aligned}$$

For $1 \leq i \leq \frac{\ell-1}{2}$, let

$$2i = 2^{s_i} t_i,$$

where $s_i \geq 1$ and $t_i \equiv 1 \pmod 2$. Next, we consider the following two cases.

(1) If $s_i \leq m-4$ for all $1 \leq i \leq \frac{\ell-1}{2}$, then

$$\begin{aligned} 2^\ell \binom{k}{\ell} &= 2^{\ell+m-4} \frac{g}{k-\ell} \prod_{i=1}^{\frac{\ell-1}{2}} \frac{2^{m-4}g-2^{s_i}t_i}{2^{s_i}t_i} \prod_{j=1}^{\frac{\ell+1}{2}} \frac{k-2j+1}{2j-1} \\ &= 2^{\ell+m-4} \frac{g}{k-\ell} \prod_{i=1}^{\frac{\ell-1}{2}} \frac{2^{m-4-s_i}g-t_i}{t_i} \prod_{j=1}^{\frac{\ell+1}{2}} \frac{k-2j+1}{2j-1}. \end{aligned}$$

Since $k-\ell$, t_i ($1 \leq i \leq \frac{\ell-1}{2}$), and $2j-1$ ($1 \leq j \leq \frac{\ell+1}{2}$) are odd, when $\ell \geq 5$, we have

$$\ell + m - 4 \geq m + 1.$$

Then

$$2^{\ell}\binom{k}{\ell} \equiv 0 \pmod{2^m}.$$

(2) If there exists a positive integer d such that

$$1 \leq d \leq \frac{\ell-1}{2} \quad \text{and} \quad s_d > m-4,$$

then from

$$1 \leq i \leq \frac{\ell-1}{2} \quad \text{and} \quad 2i = 2^{s_i}t_i,$$

it follows that

$$\ell \geq 2i+1 = 2^{s_i}t_i + 1.$$

So we have

$$\ell \geq 2^{s_d}t_d + 1 > 2^{m-4} + 1.$$

The function

$$2^{x-4} - x + 2$$

is monotonically increasing for $x \geq 5$. Hence, for

$$m \geq 6 = min\{x \mid 2^{x-4} - x + 2 \geq 0 \text{ and } x \geq 5\},$$

we derive

$$2^{m-4} - m + 2 \geq 0.$$

Therefore, we obtain

$$\ell > 2^{m-4} + 1 \geq m - 1$$

which means

$$2^{\ell}\binom{k}{\ell} \equiv 0 \pmod{2^m}.$$

Based on the above two cases, we complete the proof. □

Lemma 2.3. *If m, ℓ, and k satisfy one of the following conditions:*

(1) $m \geq 5$, $\ell \geq 5$, *and* $k \equiv 0 \pmod{2^{m-4}}$;
(2) $m \geq 4$, $\ell \geq 5$, *and* $k \equiv 0 \pmod{2^{m-3}}$;
(3) $m \geq 3$, $\ell \geq 3$, *and* $k \equiv 0 \pmod{2^{m-2}}$,

then we have

$$2^{\ell}\binom{k}{\ell} \equiv 0 \pmod{2^m}.$$

Proof. First, we prove the lemma for the condition (1). When $m = 5$ and $\ell \geq 5$, it is obvious that

$$2^{\ell}\binom{k}{\ell} \equiv 0 \pmod{2^m}.$$

Therefore, combining Lemma 2.1, Lemma 2.2, and the above fact, we complete the proof of the first case.

Since the proofs of the second and third cases are similar to that of the first case, we omit the details. □

Lemma 2.4. *If m, ℓ, and k satisfy one of the following conditions:*

(1) $m \geq 4$, $\ell \geq 5$, *and* $k \equiv 1 \pmod{2^{m-3}}$*;*
(2) $m \geq 3$, $\ell \geq 3$, *and* $k \equiv 1 \pmod{2^{m-2}}$*,*

then we have

$$2^{\ell}\binom{k}{\ell} \equiv 0 \pmod{2^m}.$$

Proof. In Case (1), let

$$k = 2^{m-3}g + 1,$$

where g is a nonnegative integer. Then for even $\ell \geq 6$, we have

$$\begin{aligned}
2^{\ell}\binom{k}{\ell} &= 2^{\ell}\frac{k-1}{\ell}\prod_{i=1}^{\frac{\ell-2}{2}}\frac{(k-1)-2i}{2i}\prod_{j=1}^{\frac{\ell}{2}}\frac{k-2j+2}{2j-1}\\
&= 2^{\ell}\frac{2^{m-3}g}{2^s t}\prod_{i=1}^{\frac{\ell-2}{2}}\frac{2^{m-3}g-2i}{2i}\prod_{j=1}^{\frac{\ell}{2}}\frac{k-2j+2}{2j-1}\\
&= 2^{\ell-s+m-3}\frac{g}{t}\prod_{i=1}^{\frac{\ell-2}{2}}\frac{2^{m-3}g-2i}{2i}\prod_{j=1}^{\frac{\ell}{2}}\frac{k-2j+2}{2j-1}.
\end{aligned}$$

Since the proof is similar to that of Lemma 2.1, we derive that

$$2^{\ell}\binom{k}{\ell} \equiv 0 \pmod{2^m}, \text{ if } m \geq 4 \text{ and even } \ell \geq 6.$$

Now we turn to prove

$$2^{\ell}\binom{k}{\ell} \equiv 0 \pmod{2^m}, \text{ if } m \geq 4 \text{ and odd } \ell \geq 5.$$

Since ℓ is odd, we have $\ell - 1 = 2^s t$. Then

$$\begin{aligned}
2^{\ell}\binom{k}{\ell} &= 2^{\ell}\frac{k-1}{\ell-1}\prod_{i=1}^{\frac{\ell-3}{2}}\frac{(k-1)-2i}{2i}\prod_{j=1}^{\frac{\ell+1}{2}}\frac{k-2j+2}{2j-1}\\
&= 2^{\ell}\frac{2^{m-3}g}{2^s t}\prod_{i=1}^{\frac{\ell-3}{2}}\frac{2^{m-3}g-2i}{2i}\prod_{j=1}^{\frac{\ell+1}{2}}\frac{k-2j+2}{2j-1}\\
&= 2^{\ell-s+m-3}\frac{g}{t}\prod_{i=1}^{\frac{\ell-3}{2}}\frac{2^{m-3}g-2i}{2i}\prod_{j=1}^{\frac{\ell+1}{2}}\frac{k-2j+2}{2j-1}.
\end{aligned}$$

According to the proof of Lemma 2.1, we complete the proof for odd ℓ. So, we complete the proof of Case (1).

Similarly, we can prove Case (2). Here we omit the details. □

Lemma 2.5. *If m, ℓ, and k satisfy one of the following conditions:*

(1) $m \geq 5$, $\ell \geq 5$, *and* $k \equiv 2 \pmod{2^{m-4}}$;
(2) $m \geq 4$, $\ell \geq 5$, *and* $k \equiv 2 \pmod{2^{m-3}}$,

then we have

$$2^\ell \binom{k}{\ell} \equiv 0 \pmod{2^m}.$$

Proof. In Case (1), let

$$k = 2^{m-4}g + 2,$$

where g is a nonnegative integer. First, we prove

$$2^\ell \binom{k}{\ell} \equiv 0 \pmod{2^m}, \text{ if } m \geq 5 \text{ and even } \ell \geq 6.$$

Since ℓ is even, we have

$$\begin{aligned} 2^\ell \binom{k}{\ell} &= 2^\ell \frac{k-2}{\ell} \frac{k}{\ell-2} \prod_{i=1}^{\frac{\ell-4}{2}} \frac{(k-2)-2i}{2i} \prod_{j=1}^{\frac{\ell}{2}} \frac{k-2j+1}{2j-1} \\ &= 2^\ell \frac{2^{m-4}g}{2^s t} \frac{2^{m-4}g+2}{2^s t-2} \prod_{i=1}^{\frac{\ell-4}{2}} \frac{2^{m-4}g-2i}{2i} \prod_{j=1}^{\frac{\ell}{2}} \frac{k-2j+1}{2j-1} \\ &= 2^{\ell-s+m-4} \frac{g}{t} \frac{2^{m-4}g+2}{2^s t-2} \prod_{i=1}^{\frac{\ell-4}{2}} \frac{2^{m-4}g-2i}{2i} \prod_{j=1}^{\frac{\ell}{2}} \frac{k-2j+1}{2j-1}. \end{aligned}$$

According to the proof of Lemma 2.1, we complete the proof for even ℓ.

Next, we prove

$$2^\ell \binom{k}{\ell} \equiv 0 \pmod{2^m}, \text{ if } m \geq 5 \text{ and odd } \ell \geq 5.$$

Notice that

$$\begin{aligned} 2^\ell \binom{k}{\ell} &= 2^\ell \frac{k-2}{\ell-1} \frac{(k-2)+2}{k-\ell} \prod_{i=1}^{\frac{\ell-3}{2}} \frac{(k-2)-2i}{2i} \prod_{j=1}^{\frac{\ell+1}{2}} \frac{k-2j+1}{2j-1} \\ &= 2^\ell \frac{2^{m-4}g}{2^s t} \frac{2^{m-4}g+2}{k-\ell} \prod_{i=1}^{\frac{\ell-3}{2}} \frac{2^{m-4}g-2i}{2i} \prod_{j=1}^{\frac{\ell+1}{2}} \frac{k-2j+1}{2j-1} \\ &= 2^{\ell-s+m-4+1} \frac{g}{t} \frac{2^{m-4-1}g+1}{k-\ell} \prod_{i=1}^{\frac{\ell-3}{2}} \frac{2^{m-4}g-2i}{2i} \prod_{j=1}^{\frac{\ell+1}{2}} \frac{k-2j+1}{2j-1}. \end{aligned}$$

Then based on the proof of Lemma 2.1, we prove the lemma for odd ℓ. Therefore, we complete the proof of Case (1).

The proof of Case (2) is similar to that of Case (1). So, we omit it. ∎

3. Main results

Lemma 3.1. *For any odd prime p,*

$$\varphi(q)=\varphi(q^{p^2})+2\sum_{i=1}^{\frac{p-1}{2}}q^{i^2}f(q^{p^2+2pi},q^{p^2-2pi}).$$

Proof. We have

$$\varphi(q)=\sum_{n=-\infty}^{\infty}q^{n^2}=\sum_{i=-\frac{p-1}{2}}^{\frac{p-1}{2}}\sum_{n=-\infty}^{\infty}q^{(pn+i)^2}=\varphi(q^{p^2})+2\sum_{i=1}^{\frac{p-1}{2}}q^{i^2}\sum_{n=-\infty}^{\infty}q^{p^2n^2+2pin}.$$

Then in view of (1.1), we arrive at what we need. □

Proof of Theorem 1.1. Based on Lemma 3.1, we have

$$\begin{aligned}\varphi^k(q)&=\left(\varphi(q^{p^2})+2\sum_{i=1}^{\frac{p-1}{2}}q^{i^2}f(q^{p^2+2pi},q^{p^2-2pi})\right)^k\\&=\sum_{\ell=0}^{k}\sum_{\substack{\ell_1,\ell_2,\dots,\ell_{\frac{p-1}{2}}\geq 0\\ \ell_1+\ell_2+\cdots+\ell_{\frac{p-1}{2}}=\ell}}2^\ell\binom{k}{k-\ell,\ell_1,\ell_2,\dots,\ell_{\frac{p-1}{2}}}\varphi^{k-\ell}(q^{p^2})\\&\qquad\times\prod_{i=1}^{\frac{p-1}{2}}\left(q^{i^2}f(q^{p^2+2pi},q^{p^2-2pi})\right)^{\ell_i}.\end{aligned}\tag{3.1}$$

Noticing that

$$\binom{k}{k-\ell,\ell_1,\ell_2,\dots,\ell_t}\Big/\binom{k}{\ell}=\binom{\ell}{\ell_1,\ell_2,\dots,\ell_t},$$

where $\ell_1+\ell_2+\cdots\ell_t=\ell$, we derive

$$\binom{k}{\ell}\mid\binom{k}{k-\ell,\ell_1,\ell_2,\dots,\ell_t}.$$

It means that if

$$2^\ell\binom{k}{\ell}\equiv 0\pmod{2^m},$$

then

$$2^\ell\binom{k}{k-\ell,\ell_1,\ell_2,\dots,\ell_t}\equiv 0\pmod{2^m}.$$

Next, with the aid of Lemmas 2.3-2.5, we consider the following four cases for $\varphi^k(q)$ modulo 2^m when k, ℓ, and m satisfy some conditions.

Case (1): in view of Lemmas 2.3 and 2.5, we obtain that for $m\geq 5$, $\ell\geq 5$, and $k\equiv 0,2\pmod{2^{m-4}}$,

$$2^\ell\binom{k}{\ell}\equiv 0\pmod{2^m}.$$

Furthermore, we observe that if $m \geq 5$ and $k \equiv 0, 2 \pmod{2^{m-4}}$, then

$$\begin{aligned} 2^3 \binom{k}{k-3,1,1,1} &\equiv 2^4 \binom{k}{k-4,3,1} \equiv 2^4 \binom{k}{k-4,2,1,1} \\ &\equiv 2^4 \binom{k}{k-4,1,1,1,1} \equiv 0 \pmod{2^m}. \end{aligned}$$

Thus, from (3.1), it can be seen that for $m \geq 5$ and $k \equiv 0, 2 \pmod{2^{m-4}}$,

$$\begin{aligned} \varphi^k(q) &\equiv \varphi^k(q^{p^2}) + 2k\varphi^{k-1}(q^{p^2}) \sum_{i=1}^{\frac{p-1}{2}} q^{i^2} f(q^{p^2+2pi}, q^{p^2-2pi}) \\ &+ 2k(k-1)\varphi^{k-2}(q^{p^2}) \sum_{i=1}^{\frac{p-1}{2}} q^{2i^2} f^2(q^{p^2+2pi}, q^{p^2-2pi}) \\ &+ 4k(k-1)\varphi^{k-2}(q^{p^2}) \sum_{i,j=1,i<j}^{\frac{p-1}{2}} q^{i^2+j^2} f(q^{p^2+2pi}, q^{p^2-2pi}) f(q^{p^2+2pj}, q^{p^2-2pj}) \\ &+ \frac{4k(k-1)(k-2)}{3}\varphi^{k-3}(q^{p^2}) \sum_{i=1}^{\frac{p-1}{2}} q^{3i^2} f^3(q^{p^2+2pi}, q^{p^2-2pi}) \\ &+ 4k(k-1)(k-2)\varphi^{k-3}(q^{p^2}) \sum_{i,j=1,i\neq j}^{\frac{p-1}{2}} q^{2i^2+j^2} f^2(q^{p^2+2pi}, q^{p^2-2pi}) f(q^{p^2+2pj}, q^{p^2-2pj}) \\ &+ \frac{2k(k-1)(k-2)(k-3)}{3}\varphi^{k-4}(q^{p^2}) \sum_{i=1}^{\frac{p-1}{2}} q^{4i^2} f^4(q^{p^2+2pi}, q^{p^2-2pi}) \\ &+ 4k(k-1)(k-2)(k-3)\varphi^{k-4}(q^{p^2}) \\ &\times \sum_{i,j=1,i<j}^{\frac{p-1}{2}} q^{2i^2+2j^2} f^2(q^{p^2+2pi}, q^{p^2-2pi}) f^2(q^{p^2+2pj}, q^{p^2-2pj}) \pmod{2^m}. \end{aligned} \tag{3.2}$$

Case (2): with the aid of Lemma 2.4, when $m \geq 4$, $\ell \geq 5$, and $k \equiv 1 \pmod{2^{m-3}}$, we have

$$2^\ell \binom{k}{\ell} \equiv 0 \pmod{2^m}.$$

Meanwhile, we point out that for $m \geq 4$ and $k \equiv 1 \pmod{2^{m-3}}$,

$$\begin{aligned} 2^3 \binom{k}{k-3,1,1,1} &\equiv 2^4 \binom{k}{k-4,3,1} \equiv 2^4 \binom{k}{k-4,2,2} \\ &\equiv 2^4 \binom{k}{k-4,2,1,1} \equiv 2^4 \binom{k}{k-4,1,1,1,1} \equiv 0 \pmod{2^m}. \end{aligned}$$

So applying (3.1) yields that for $m \geq 4$ and $k \equiv 1 \pmod{2^{m-3}}$,

$$\begin{aligned}
\varphi^k(q) \equiv\ & \varphi^k(q^{p^2}) + 2k\varphi^{k-1}(q^{p^2}) \sum_{i=1}^{\frac{p-1}{2}} q^{i^2} f(q^{p^2+2pi}, q^{p^2-2pi}) \\
& + 2k(k-1)\varphi^{k-2}(q^{p^2}) \sum_{i=1}^{\frac{p-1}{2}} q^{2i^2} f^2(q^{p^2+2pi}, q^{p^2-2pi}) \\
& + 4k(k-1)\varphi^{k-2}(q^{p^2}) \sum_{i,j=1, i<j}^{\frac{p-1}{2}} q^{i^2+j^2} f(q^{p^2+2pi}, q^{p^2-2pi}) f(q^{p^2+2pj}, q^{p^2-2pj}) \\
& + \frac{4k(k-1)(k-2)}{3}\varphi^{k-3}(q^{p^2}) \sum_{i=1}^{\frac{p-1}{2}} q^{3i^2} f^3(q^{p^2+2pi}, q^{p^2-2pi}) \\
& + 4k(k-1)(k-2)\varphi^{k-3}(q^{p^2}) \sum_{i,j=1, i\neq j}^{\frac{p-1}{2}} q^{2i^2+j^2} f^2(q^{p^2+2pi}, q^{p^2-2pi}) f(q^{p^2+2pj}, q^{p^2-2pj}) \\
& + \frac{2k(k-1)(k-2)(k-3)}{3}\varphi^{k-4}(q^{p^2}) \sum_{i=1}^{\frac{p-1}{2}} q^{4i^2} f^4(q^{p^2+2pi}, q^{p^2-2pi}) \pmod{2^m}.
\end{aligned} \tag{3.3}$$

Case (3): by means of Lemmas 2.3 and 2.5, we obtain that for $m \geq 4$, $\ell \geq 5$, and $k \equiv 0, 2 \pmod{2^{m-3}}$,

$$2^\ell \binom{k}{\ell} \equiv 0 \pmod{2^m}.$$

Furthermore, we observe that if $m \geq 4$ and $k \equiv 0, 2 \pmod{2^{m-3}}$, then

$$\begin{aligned}
& 2^3\binom{k}{k-3,3} \equiv 2^3\binom{k}{k-3,2,1} \equiv 2^3\binom{k}{k-3,1,1,1} \equiv 2^4\binom{k}{k-4,3,1} \\
& \equiv 2^4\binom{k}{k-4,2,2} \equiv 2^4\binom{k}{k-4,2,1,1} \equiv 2^4\binom{k}{k-4,1,1,1,1} \equiv 0 \pmod{2^m}.
\end{aligned}$$

So, according to (3.1), we derive that for $m \geq 4$ and $k \equiv 0, 2 \pmod{2^{m-3}}$,

$$\begin{aligned}
\varphi^k(q) \equiv\ & \varphi^k(q^{p^2}) + 2k\varphi^{k-1}(q^{p^2}) \sum_{i=1}^{\frac{p-1}{2}} q^{i^2} f(q^{p^2+2pi}, q^{p^2-2pi}) \\
& + 2k(k-1)\varphi^{k-2}(q^{p^2}) \sum_{i=1}^{\frac{p-1}{2}} q^{2i^2} f^2(q^{p^2+2pi}, q^{p^2-2pi}) \\
& + 4k(k-1)\varphi^{k-2}(q^{p^2}) \sum_{i,j=1, i<j}^{\frac{p-1}{2}} q^{i^2+j^2} f(q^{p^2+2pi}, q^{p^2-2pi}) f(q^{p^2+2pj}, q^{p^2-2pj})
\end{aligned}$$

$$+\frac{2k(k-1)(k-2)(k-3)}{3}\varphi^{k-4}(q^{p^2})\sum_{i=1}^{\frac{p-1}{2}}q^{4i^2}f^4(q^{p^2+2pi},q^{p^2-2pi}) \pmod{2^m}. \tag{3.4}$$

Case (4): based on Lemmas 2.3 and 2.4, we obtain that for $m\geq 3$, $\ell\geq 3$, and $k\equiv 0,1 \pmod{2^{m-2}}$,

$$2^\ell\binom{k}{\ell}\equiv 0 \pmod{2^m}.$$

In addition, we observe that

$$2^2\binom{k}{k-2,1,1}\equiv 0 \pmod{2^m}.$$

Then from (3.1), it can be shown that for $m\geq 3$ and $k\equiv 0,1 \pmod{2^{m-2}}$,

$$\begin{aligned}\varphi^k(q)\equiv\varphi^k(q^{p^2})&+2k\varphi^{k-1}(q^{p^2})\sum_{i=1}^{\frac{p-1}{2}}q^{i^2}f(q^{p^2+2pi},q^{p^2-2pi})\\&+2k(k-1)\varphi^{k-2}(q^{p^2})\sum_{i=1}^{\frac{p-1}{2}}q^{2i^2}f^2(q^{p^2+2pi},q^{p^2-2pi}) \pmod{2^m}.\end{aligned} \tag{3.5}$$

Next, according to (3.2)-(3.5) in the above four cases, we consider the powers of q on the right-hand sides of these congruences. Notice that for $1\leq i,j\leq\frac{p-1}{2}$,

(1) if p is any odd prime, then $i^2,2i^2\not\equiv 0 \pmod{p}$;

(2) if $p\geq 5$, then $i^2,2i^2,3i^2,4i^2\not\equiv 0 \pmod{p}$;

(3) if $\left(\frac{-1}{p}\right)=-1$, namely, $p\equiv 3 \pmod{4}$, then there is no solution for $i^2+j^2\equiv 0 \pmod{p}$;

(4) if $\left(\frac{-2}{p}\right)=-1$, namely, $p\equiv 5,7 \pmod{8}$, then there is no solution for $2i^2+j^2\equiv 0 \pmod{p}$.

Therefore, according to (3.2)-(3.5), when one of the following conditions holds:

(1) $m\geq 5$, $k\equiv 0,2 \pmod{2^{m-4}}$, and $p\equiv 7 \pmod{8}$;
(2) $m\geq 4$, $k\equiv 1 \pmod{2^{m-3}}$, and $p\equiv 7 \pmod{8}$;
(3) $m\geq 4$, $k\equiv 1 \pmod{2^{m-3}}$, and $p\equiv 7 \pmod{8}$;
(4) $m\geq 4$, $k\equiv 0,2 \pmod{2^{m-3}}$, and $p\equiv 3 \pmod{4}$;
(5) $m\geq 3$, $k\equiv 0,1 \pmod{2^{m-2}}$, and $p\equiv 1 \pmod{2}$,

we establish that

$$\sum_{n=0}^{\infty}r_k(pn)q^n\equiv\varphi^k(q^p) \pmod{2^m}. \tag{3.6}$$

Then we have

$$\sum_{n=0}^{\infty} r_k(p^2n)q^n \equiv \varphi^k(q) \pmod{2^m}.$$

By induction on α, we derive that for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} r_k(p^{2\alpha}n)q^n \equiv \varphi^k(q) \pmod{2^m}. \tag{3.7}$$

Combining this with (3.6) yields that

$$\sum_{n=0}^{\infty} r_k(p^{2\alpha+1}n)q^n \equiv \varphi^k(q^p) \pmod{2^m}.$$

Hence, it can be seen that for $\alpha \geq 0$, $n \geq 0$, and $j = 1, 2, \cdots, p-1$,

$$r_k(p^{2\alpha+1}(pn+j)) \equiv 0 \pmod{2^m}. \tag{3.8}$$

Moreover, Letting k be any positive integer and $m = 2$ in (3.1), we have

$$\varphi^k(q) \equiv \varphi^k(q^{p^2}) + 2k\varphi^{k-1}(q^{p^2}) \sum_{i=1}^{\frac{p-1}{2}} q^{i^2} f(q^{p^2+2pi}, q^{p^2-2pi}) \pmod{2^m}.$$

So, for any odd prime p, we obtain (3.8) in this case. Therefore, we complete the proof. $\square$

Proof of Theorem 1.2. When $m \geq 3$ and $k \equiv 0, 1 \pmod{2^{m-2}}$, we have (3.5). Notice that i^2 and $2i^2$ are quadratic residues modulo p when $p \equiv \pm 1 \pmod 8$. Let r be a quadratic nonresidue modulo p. Then using (3.5), we derive

$$r_k(pn+r) \equiv 0 \pmod{2^m}. \tag{3.9}$$

Applying (3.7) and the above relation yields that for $\alpha \geq 0$,

$$r_k(p^{2\alpha}(pn+r)) \equiv 0 \pmod{2^m}. \tag{3.10}$$

In addition, by induction and the binomial theorem, we deduce that for $k \geq 1$,

$$f_1^{2^k} \equiv f_2^{2^{k-1}} \pmod{2^k}. \tag{3.11}$$

Then with the help of (3.11), we have that for $m \geq 1$,

$$\varphi^{2^{m-1}}(q) = \left(\frac{f_2^5}{f_1^2 f_4^2}\right)^{2^{m-1}} = \frac{f_2^{5\cdot 2^{m-1}}}{f_1^{2^m} f_4^{2^m}} \equiv \frac{f_2^{5\cdot 2^{m-1}}}{f_2^{2^{m-1}} f_2^{2^{m+1}}} = 1 \pmod{2^m}. \tag{3.12}$$

Therefore, by means of Lemma 3.1 and (3.12), we find that when $m \geq 2$, $k \equiv 1 \pmod{2^{m-1}}$, and $p \equiv 1 \pmod 2$,

$$\varphi^k(q) \equiv \varphi(q) = \varphi(q^{p^2}) + 2\sum_{i=1}^{\frac{p-1}{2}} q^{i^2} f(q^{p^2+2pi}, q^{p^2-2pi}) \pmod{2^m}$$

which implies (3.9). Then according to (3.7), we derive (3.10). This completes the proof. $\square$

4. Congruences for some partition functions

In this section, in light of the main theorems, we establish many infinite families of congruences for the overpartition function and the overpartition pair function.

4.1. **Overpartitions.** A partition of a positive integer n is any non-increasing sequence of positive integers whose sum is n. An overpartition of n is a partition of n where we may overline the first occurrence of a part. We denote the number of overpartitions of n by $\overline{p}(n)$. The generating function of $\overline{p}(n)$ is

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{1}{\varphi(-q)}.$$

Overpartitions have been used by Corteel and Lovejoy [12] in combinatorial proofs of q-series identities. Some properties of $\overline{p}(n)$ have been established. See [7, 13, 22, 24, 25, 31, 32] for examples.

Applying Theorems 1.1 and 1.2, we generalize some known results for $\overline{p}(n)$. In particular, we provide another proof of a conjecture given by Hirschhorn and Sellers [19].

Theorem 4.1. *For any odd prime p, $\alpha \geq 0$, $n \geq 0$, and $j = 1, 2, \ldots, p-1$,*

$$\overline{p}\left(p^{2\alpha+1}(pn+j)\right) \equiv \begin{cases} 0 \pmod 8, & p \equiv 1, \pm 3 \pmod 8, \\ 0 \pmod{16}, & p \equiv 7 \pmod 8. \end{cases} \tag{4.1}$$

Proof. First, according to (3.12), we have

$$\sum_{n=0}^{\infty} \overline{p}(n)(-1)^n q^n = \frac{1}{\varphi(q)} = \frac{\varphi^{2^{m-1}-1}(q)}{\varphi^{2^{m-1}}(q)} \equiv \varphi^{2^{m-1}-1}(q) = \sum_{n=0}^{\infty} r_{2^{m-1}-1}(n)q^n \pmod{2^m}. \tag{4.2}$$

When $m = 3$ in (4.2), based on the condition (4) with $m = k = 3$ and $p \equiv 1 \pmod 2$ in Theorem 1.1, we find that for any odd prime p, $\alpha \geq 0$, and $n \geq 0$,

$$\overline{p}\left(p^{2\alpha+1}(pn+j)\right) \equiv 0 \pmod 8.$$

Then when $m = 4$ in (4.2), using the condition (2) with $m = 4$, $k = 7$, and $p \equiv 7 \pmod 8$ in Theorem 1.1, we obtain that for $p \equiv 7 \pmod 8$, $\alpha \geq 0$, and $n \geq 0$,

$$\overline{p}\left(p^{2\alpha+1}(pn+j)\right) \equiv 0 \pmod{16}.$$

Hence, we derive (4.1). □

Notice that setting $\alpha = 0$ for the congruences modulo 16 in (4.1) yields the result given by Chen et al. [4].

Corollary 4.2. *For $\alpha \geq 1$ and $n \geq 0$,*

$$\overline{p}\left(5^{2\alpha+1}(5n \pm 1)\right) \equiv 0 \pmod{40}.$$

Proof. Chen et al. [6] presented that for $\alpha \geq 1$,

$$\overline{p}\left(5^{2\alpha+1}(5n \pm 1)\right) \equiv 0 \pmod{5}. \tag{4.3}$$

Then setting $p = 5$ in Theorem 4.1 yields that

$$\overline{p}\left(5^{2\alpha+1}(5n \pm 1)\right) \equiv 0 \pmod{8}. \tag{4.4}$$

Combining (4.3) with (4.4), we establish what we need. □

Theorem 4.3. *If p is an odd prime and r is a quadratic nonresidue modulo p, then for $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{p}\left(p^{2\alpha}(pn + r)\right) \equiv \begin{cases} 0 \pmod{4}, & p \equiv \pm 3 \pmod{8}, \\ 0 \pmod{8}, & p \equiv \pm 1 \pmod{8}. \end{cases} \tag{4.5}$$

Proof. When $m = 2$ in (4.2), applying the condition (2) with $m = 2$, $k = 1$, and $p \equiv 1 \pmod{2}$ in Theorem 1.2, we deduce the first congruence. Similarly, when $m = 3$ in (4.2), applying the condition (1) with $m = k = 3$ and $p \equiv \pm 1 \pmod{8}$ in Theorem 1.2, we derive the second congruence. □

Kim [21] proved the case for $\alpha = 0$ in (4.5) which was conjectured by Hirschhorn and Sellers [19]. Furthermore, setting $k = 3$ in (3.1) implies that for any odd prime p,

$$\begin{aligned} \varphi^3(q) \equiv \varphi^3(q^{p^2}) &+ 6\varphi^2(q^{p^2}) \sum_{i=1}^{\frac{p-1}{2}} q^{i^2} f(q^{p^2+2pi}, q^{p^2-2pi}) \\ &+ 12\varphi(q^{p^2}) \sum_{i=1}^{\frac{p-1}{2}} q^{2i^2} f^2(q^{p^2+2pi}, q^{p^2-2pi}) \pmod{8}. \end{aligned}$$

Combining this with (4.2) yields the following result given by Kim [21]: if n is neither a square nor twice a square, then $\overline{p}(n) \equiv 0 \pmod{8}$.

4.2. **Overpartition pairs.** An overpartition pair π of n is a pair of overpartitions (λ, μ) such that the sum of all of the parts is n. Note that we allow λ and μ to be an overpartition of zero. Let $\overline{pp}(n)$ denote the number of overpartition pairs of n. Then the generating function of $\overline{pp}(n)$ is stated as follows.

$$\sum_{n=0}^{\infty} \overline{pp}(n) q^n = \frac{1}{\varphi^2(-q)}. \tag{4.6}$$

Chen and Lin [5] established some congruences for $\overline{pp}(n)$ modulo 3 and 5. Using Theorems 1.1 and 1.2, we derive the following congruences for $\overline{pp}(n)$.

Theorem 4.4. *For $n \geq 1$,*

$$\overline{pp}(n) \equiv 0 \pmod{4}.$$

Proof. Based on (3.12) and (4.6), we obtain

$$\sum_{n=0}^{\infty} \overline{pp}(n)(-1)^n q^n = \frac{1}{\varphi^2(q)} \equiv 1 \pmod 4.$$

This completes the proof. □

Theorem 4.5. *For $\alpha \geq 0$, $n \geq 0$, and $j = 1, 2, \cdots, p-1$,*

$$\overline{pp}\left(p^{2\alpha+1}(pn+j)\right) \equiv \begin{cases} 0 \pmod{2^3}, & p \equiv 1, 5 \pmod 8, \\ 0 \pmod{2^5}, & p \equiv 3 \pmod 8, \\ 0 \pmod{2^6}, & p \equiv 7 \pmod 8. \end{cases}$$

Proof. First, with the aid of (3.12), we derive that

$$\sum_{n=0}^{\infty} \overline{pp}(n)(-1)^n q^n = \frac{1}{\varphi^2(q)} = \frac{\varphi^{2^{m-1}-2}(q)}{\varphi^{2^{m-1}}(q)} \equiv \varphi^{2^{m-1}-2}(q) = \sum_{n=0}^{\infty} r_{2^{m-1}-2}(n) q^n \pmod{2^m}. \tag{4.7}$$

When $m = 3$, $m = 5$, and $m = 6$ in (4.7), in view of the conditions (4), (3), and (1) of Theorem 1.1, respectively, we prove the theorem. □

Theorem 4.6. *Let $p \equiv \pm 1 \pmod 8$ and r be a quadratic nonresidue modulo p. For $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{pp}\left(p^{2\alpha}(pn+r)\right) \equiv 0 \pmod 8.$$

Proof. When $m = 3$ in (4.7), based on the condition (1) with $m = 3$, $k = 2$, and $p \equiv \pm 1 \pmod 8$ in Theorem 1.2, we obtain the required result. □

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