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Mixed Connectivity Properties of Random Graphs and Some Special Graphs

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Abstract For positive integers k and λ, a graph G is (k,λ)-connected if it satisfies the following two conditions: (1) $|V(G)| \geq k+1$, and (2) for any subset $S \subseteq V(G)$ and any subset $L \subseteq E(G)$ with $\lambda|S|+|L| < k\lambda$, $G-(S\cup L)$ is connected. For positive integers k and ℓ, a graph G with $|V(G)| \geq k+\ell+1$ is said to be (k,ℓ)-mixed-connected if for any subset $S \subseteq V(G)$ and any subset $L \subseteq E(G)$ with $|S| \leq k, |L| \leq \ell$ and $|S|+|L| < k+\ell$, $G-(S\cup L)$ is connected. In this paper, we investigate the (k,λ)-connectivity and (k,ℓ)-mixed-connectivity of random graphs, and generalize the results of Erdős and Rényi (1959), and Stepanov (1970). Furthermore, our argument can show that in the random graph process $\tilde{G} = (G_t)_0^N$, $N = \binom{n}{2}$, the hitting times of minimum degree at least $k\lambda$ and of G_t being (k,λ)-connected coincide with high probability, and also the hitting times of minimum degree at least $k+\ell$ and of G_t being (k,ℓ)-mixed-connected coincide with high probability. These results are analogous to the work of Bollobás and Thomassen (1986) on classic connectivity. Additionally, we obtain the (k,λ)-connectivity and (k,ℓ)-mixed-connectivity of the complete graphs and complete bipartite graphs, and characterize the minimally (k,ℓ)-mixed-connected graphs.

Keywords connectivity · edge-connectivity · mixed connectivity · random graph · threshold function · hitting time

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1 Introduction

All graphs in this paper are undirected and finite. Additionally, we make this assumption: the removal of a vertex in graph implies the removal of all its incident edges. A graph G is k-connected if $G-S$ is connected for any vertex subset S with $|S|<k$, and a graph G is ℓ-edge-connected if $G-L$ is connected for any edge subset L with $|L|<\ell$. The connectivity $\kappa(G)$ of graph G is the largest k for which the graph is k-connected. Similarly, the edge-connectivity $\lambda(G)$ of graph G is the largest ℓ for which the graph is ℓ edge-connected. There are many generalizations of connectivity and edge-connectivity, and we refer to [1,12].

In 2000, Kaneko and Ota [14] introduced the (k,λ)-*connectivity*. For a given graph G, let x and y be two distinct vertices. A pair (x,y) of vertices is said to be (k,λ)-*connected* in G if for any subset $S\subseteq V(G)-\{x,y\}$ and any subset $L\subseteq E(G)$ with $\lambda|S|+|L|<k\lambda$, the vertices x and y belong to the same component of $G-S-L$. Formally, a graph G is (k,λ)-*connected* if it satisfies the following conditions: (i) $|V(G)|\geq k+1$, and (ii) for any subset $S\subseteq V(G)$ and any subset $L\subseteq E(G)$ with $\lambda|S|+|L|<k\lambda$, $G-S-L$ is connected. The well known Menger's Theorem characterizes the relationship of graph connectivity and the minimum number of disjoint paths between any pair of vertices. For example, the pair (x,y) of vertices is k-connected if and only if there are k internally disjoint (or vertex-disjoint) paths between x and y. Let (x,y)-k-fan be a union of k internally disjoint paths, and [6, 14] showed that the pair (x,y) is (k,λ)-connected in G if and only if G contains λ edge-disjoint (x,y)-k-fans. To this end, the (k,λ)-connectivity can be considered as an extension of the classical connectivity and the edge-connectivity, considering the conclusion of Menger's theorem. More specifically, $(k,1)$-connected graphs are k-connected graphs, and $(1,\lambda)$-connected graphs are λ-edge-connected graphs. In fact, we will show that the concept of $(k,1)$-connected is equivalent to k-connected, and $(1,\lambda)$-connected is equivalent to λ-edge-connected.

As another generation of connectivity and the edge-connectivity was proposed by Beineke and Harary [3] in early 1960s. To avoid confusion with the (k,λ)-connectivity defined above, we here call this type of connectivity as (k,λ)-*mixed-connectivity.* Two distinct vertices x,y are said to be (k,ℓ)-*mixed-connected* (k, ℓ are two positive integers), if for any subset $S\subseteq V(G)-\{x,y\}$ and any subset $L\subseteq E(G)$ with $|S|+|L|<k+\ell$, the vertices x and y belong to the same component of $G-S-L$. For positive integers k and ℓ, a graph G with $|V(G)|\geq k+\ell+1$ is said to be (k,ℓ)-mixed-connected if for any subset $S\subseteq V(G)$ and any subset $L\subseteq E(G)$ with $|S|\leq k, |L|\leq\ell$ and $|S|+|L|<k+\ell$, $G-(S\cup L)$ is connected. Similarly, as generation of the conclusions in Menger's Theorem for (k,λ)-connectivity, [3] claimed to prove that a pair (x,y) is (k,ℓ)-mixed-connected if there are $(k+\ell)$ edge-disjoint paths of which k paths are vertex-disjoint. However, Mader [15] pointed out a gap in their proof. Recently, Sadeghi and Fan [17] modified the conclusion (by changing to $k+1$ vertex-disjoint paths instead of k), and then proved it.

The two generations, (k,λ)-connectivity and (k,ℓ)-mixed-connectivity, consider both connectivity and edge-connectivity, and can be applied for vulnerability analysis for network design. As pointed out in [17], the survivable networks, with robustness against both vertex and edge failures, require the concepts of mixed connectivity. In this paper, we investigate this two concepts of connectivity in the setting of random graphs and some special graphs.

The Erdős-Rényi random graph model $G(n,p)$ consists of all graphs with n vertices in which the edges are chosen independently and with probability p. We say an event $\mathscr{A}$ happens *with high probability (w.h.p.)* if the probability that it happens approaches 1 as $n \to \infty$, i.e., $\Pr[\mathscr{A}] = 1 - o_n(1)$. We will always assume that n is the variable that tends to infinity. Let G and H be two graphs on n vertices. A property P is said to be *monotone increasing* if whenever $G \subseteq H$ and G satisfies P, then H also satisfies P. For a graph property P, a function $p(n)$ is called a *threshold function* of P if:

- for every $r(n)$ with $r(n)/p(n) \to \infty$, $G(n,r(n))$ w.h.p. satisfies P; and
- for every $r'(n)$ with $r'(n)/p(n) \to 0$, $G(n,r'(n))$ w.h.p. does not satisfy P.

Furthermore, $p(n)$ is called a *sharp threshold function* of P if for any constants $0 < c < 1$ and $C > 1$, such that:

- for every $r(n) \geq C \cdot p(n)$, $G(n,r(n))$ w.h.p. satisfies P; and
- for every $r'(n) \leq c \cdot p(n)$, $G(n,r'(n))$ w.h.p. does not satisfy P.

A *random graph process* on $V = \{1,2,\cdots,n\}$, or simply a *graph process*, is a Markov chain $\tilde{G} = (G_t)_0^N$, $N = \binom{n}{2}$, which starts with the empty graph on n vertices at time $t = 0$ and where at each step one edge is added, chosen uniformly at random from those not already present in the graph, until at time N we have a complete graph. We call G_t the state of a graph process $\tilde{G} = (G_t)_0^N$ at time t. Given a graph process $\tilde{G} = (G_t)_0^N$, for a monotone graph property P, the time at which P appears is the *hitting time* of P, denote by τ_P or $\tau(P)$, i.e.,

$$\tau_P = \tau(P) = \min\{t \geq 0\colon G_t \text{ has property } P\}.$$

Denote by $\delta(G)$ the minimum degree of a graph G, and let $D_t = \{G\colon \delta(G) \geq t\}$ denote the graph property such that the graph G has minimum degree at least t.

In the extensive study of the properties of random graphs, many researchers observed that there are threshold functions for various natural graph properties. It is well known that all non-trivial monotone increasing graph properties have threshold functions (see [5] and [10]). In one of the first papers on random graphs, Erdős and Rényi [7] showed that $m = n\log n/2$ is a sharp threshold for connectivity in $G(n,m)$. Later, Stepanov [18] established a sharp threshold of connectivity for $G(n,p)$. For more results on this topic, we refer to Erdős-Rényi [8] and Ivchenko [13]. Especially, Bollobás and Thomassen [4] proved that for almost every graph process, the hitting time of the graph having the connectivity $\kappa(G)$ at least k is equal to the hitting time of the graph having the minimum degree at least k. Their result builds the bridge between the connectivity and the minimum degree. For more details on the hitting times of a random graph process, we refer to [2]. For connectivity and edge-connectivity, there are some excellent surveys on this topic, see e.g. [15,16], and for the results on connectivity and edge-connectivity of random graphs, the reader can check [2].

In this paper, we extend these results for threshold functions and hitting times to (k,λ)-connectivity and (k,ℓ)-mixed-connectivity. First, we will generalize the result of Erdős and Rényi [7] and Stepanov [18], and provide the threshold functions for (k,λ)-connectivity and (k,ℓ)-mixed-connectivity of random graphs, respectively. Second, we study the hitting time in the random graph process for both definitions of mixed connectivity. Additionally, we show mixed connectivity properties of some special graphs, such as complete graphs and complete bipartite graphs. This paper is an extended version of our work published in [11].

The remainder of this paper is organized as follows. In Section 2, we show the (k,λ)-connectivity results of random graphs and some special graphs, while the results for (k,ℓ)-mixed-connectivity are presented in Section 3.

2 (k,λ)-connectivity

2.1 (k,λ)-connectivity of random graphs

Theorem 1 *For any two positive integers k and λ, if $p=\{\log n+k\lambda\log\log n-\omega(n)\}/n$, then w.h.p. $G(n,p)$ is (k,λ)-connected, if $p=\{\log n+(k\lambda-1)\log\log n-\omega(n)\}/n$, then w.h.p. $G(n,p)$ is not (k,λ)-connected, where $\omega(n)\to\infty$ and $\omega(n)=o(\log\log n)$.*

To prove this Theorem 1, the following Lemma 1 proved by Ivchenko [13] will be needed.

Lemma 1 *[13] If $p\le\{\log n+k\log\log n\}/n$ for some fixed k, then w.h.p. we have that*

$$\kappa(G(n,p))=\lambda(G(n,p))=\delta(G(n,p)).$$

Proof of Theorem 1. Let $p_1=\{\log n+k\lambda\log\log n-\omega(n)\}/n$, $p_2=\{\log n+(k\lambda-1)\log\log n-\omega(n)\}/n$, where $\omega(n)\to\infty$ and $\omega(n)=o(\log\log n)$. When considering the (k,λ)-connectivity of a graph G, we only need to check the connectivity of $G-S-L$ such that $\lambda|S|+|L|<k\lambda$, where $S\subseteq V(G)$ and $L\subseteq E(G)$. Suppose that $|S|=i$, then it suffices to consider the case that $0\le i\le k-1$ and $|L|$ satisfying that $|L|<k\lambda-i\lambda=(k-i)\lambda$.

Note that if G is $(s+1)$-connected, i.e., $G-X$ is connected for any vertex subset X with $|X|\le s$, then we have that $G-D$ is connected for any edge subset D with $|D|\le s$. Furthermore, for any vertex subset X and edge subset D with $|S|+|D|\le s$, $G-X-D$ is still connected. It is known that the minimum degree $\delta(G(n,p_1))$ of $G(n,p_1)$ is w.h.p. equal to $k\lambda$ (see [2]). Combining with Lemma 1, we have that $G(n,p_1)$ is w.h.p. $k\lambda$-connected, so the new graph obtained by deleting any $k\lambda-1$ vertices from $G(n,p_1)$ remains connected. Hence, for any vertex subset S and edge subset L with $|S|=i$ and $|L|<(k-i)\lambda$, $G(n,p_1)-S-L$ is w.h.p. connected, where $0\le i\le k-1$. Therefore, $G(n,p_1)$ is w.h.p. (k,λ)-connected.

For the second part of Theorem 1, since the minimum degree of $G(n,p_2)$ is w.h.p. equal to $k\lambda-1$, if we let L be the set of edges incident to a vertex with minimum degree $k\lambda-1$ in $G(n,p_2)$, then $G(n,p_2)-L$ is disconnected. Notice that $|L|=k\lambda-1<k\lambda$, we have that w.h.p. $G(n,p_2)$ is not (k,λ)-connected.

∎

Considering the definition of sharp threshold functions, the following Corollary 1 is an immediate result of Theorem 1.

Corollary 1 *For any two positive integers k and λ,*

$$p = \{\log n + k\lambda \log\log n - \omega(n)\}/n$$

is a sharp threshold function for the property that $G(n,p)$ is (k,λ)-connected, where $\omega(n) \to \infty$ and $\omega(n) = o(\log\log n)$.

In fact, let $p = \{\log n + k\lambda \log\log n - \omega(n)\}/n$, where $\omega(n) \to \infty$ and $\omega(n) = o(\log\log n)$. From Theorem 1 and the monotonicity of (k,λ)-connectivity, it is easy to obtain that for every constant c_1 with $c_1 > 1$, $G(n, c_1 p)$ is w.h.p. (k,λ)-connected. And, for every constant c_2 with $0 < c_2 < 1$, we have that $c_2 p < \{\log n + (k\lambda - 1)\log\log n - \omega(n)\}/n$ for sufficiently large n. By the second part of Theorem 1, we obtain that $G(n, c_2 p)$ is w.h.p. not (k,λ)-connected. Thus, Corollary 1 follows.

If the edge probability p is between $\{\log n + (k\lambda - 1)\log\log n - \omega(n)\}/n$ and $\{\log n + k\lambda \log\log n - \omega(n)\}/n$, what is the probability of the random graph $G(n,p)$ being (k,λ)-connected? In fact, we can prove the following result. Recall that we use the notation $\Pr[\mathscr{A}]$ to denote the probability that the event $\mathscr{A}$ happens.

Theorem 2 *Let x be a fixed real number, if*

$$p = \{\log n + (k\lambda - 1)\log\log n + x\}/n,$$

then $\Pr[\ G(n,p)$ *is* (k,λ)*-connected* $] \to e^{-e^{-x/(k\lambda-1)!}}$.

To prove Theorem 2, we will make use of the following Lemma 2.

Lemma 2 (Theorem 3.5 in [2]) *Let x be a fixed real number, if*

$$p = \{\log n + k \log\log n + x\}/n,$$

then

$$\Pr[\ \delta(G(n,p)) = k] \to 1 - e^{-e^{-x/k!}}$$

and

$$\Pr[\ \delta(G(n,p)) = k+1] \to e^{-e^{-x/k!}}.$$

Proof of Theorem 2. Let $p = \{\log n + (k\lambda - 1)\log\log n + x\}/n$. From Lemma 2, we can see that $\delta(G(n,p))$ is either $k\lambda - 1$ or $k\lambda$, when n approaches to infinity.

Note that for any graph G, if G is (k,λ)-connected, then $\delta(G) \geq k\lambda$. Indeed, if $\delta(G) \leq k\lambda - 1$, let v be a vertex of minimum degree in G. Let $S = \emptyset$, L be the set of edges incident to v, we have that $\lambda|S| + |L| \leq k\lambda - 1 < k\lambda$, but $G - S - L$ is disconnected. Hence, if $G(n,p)$ is (k,λ)-connected, then $\delta(G(n,p)) \geq k\lambda$. On the other hand, since w.h.p. $p < \{\log n + k\lambda \log\log n\}/n$, from Lemma 1, we know that w.h.p. $\kappa(G(n,p)) = \lambda(G(n,p)) = \delta(G(n,p))$. If $\delta(G(n,p)) \geq k\lambda$, then w.h.p. $\kappa(G(n,p)) \geq k\lambda$. So $G(n,p) - S - L$ is still connected with $|S| = i$ and $|L| < (k-i)\lambda$, where $0 \leq i \leq k-1$. Thus, $G(n,p)$ is (k,λ)-connected. Therefore, $\Pr[\ G(n,p)$ is (k,λ)-connected $] = \Pr[\ \delta(G(n,p)) \geq k\lambda] + o(1)$, and by Lemma 2, it is equal to $\Pr[\ \delta(G(n,p)) = k\lambda] + o(1) \to e^{-e^{-x/(k\lambda-1)!}}$.

∎

Let $F_{k,\lambda} = \{G\colon G$ is (k,λ)-connected$\}$ be the graph property such that the graph G is (k,λ)-connected, we have the following result for the hitting time of the random graph process.

Theorem 3 *Given $k \in \mathbb{N}$, in the random graph process $\tilde{G} = (G_t)_0^N$, $N = \binom{n}{2}$, with high probability $\tau_{D_{k\lambda}} = \tau_{F_{k,\lambda}}$.*

Bollobás and Thomason [4] presented the following conclusion Lemma 3 on the hitting time relation between the connectivity and the minimum degree. We will use it to prove our results.

Lemma 3 *[4] For every function $k = k(n)$, $1 \leq k \leq n-1$, in the random graph process $\tilde{G} = (G_t)_0^N$, $N = \binom{n}{2}$, w.h.p.*

$$\tau(\kappa(G) \geq k) = \tau(\delta(G) \geq k).$$

Proof of Theorem 3. If G is (k,λ)-connected, then $\delta(G) \geq k\lambda$. Since otherwise, if $\delta(G) < k\lambda$, letting L be the edge subset consisting of the edges incident to a vertex with minimum degree in G, we have that $|L| < k\lambda$, and $G - L$ is disconnected, which contradicts to the assumption that G is (k,λ)-connected. So we have that w.h.p.

$$\tau_{D_{k\lambda}} \leq \tau_{F_{k,\lambda}}. \tag{1}$$

Let $t = \tau_{D_{k\lambda}}$. From Lemma 3, we have that w.h.p. $\kappa(G_t) \geq k\lambda$, so for any vertex subset S with $|S| < k\lambda$, $G_t - S$ is connected. Hence we know that for any vertex subset S and edge subset L with $|S| = i$ and $|L| < (k-i)\lambda$, w.h.p. $G_t - S - L$ is still connected, where $0 \leq i \leq k-1$. By the definition of (k,λ)-connectivity, we know that G_t has been (k,λ)-connected already, which implies that w.h.p.

$$\tau_{D_{k\lambda}} \geq \tau_{F_{k,\lambda}}. \tag{2}$$

By (1) and (2), we obtain that w.h.p.

$$\tau_{D_{k\lambda}} = \tau_{F_{k,\lambda}}.$$

∎

Remark 1 For $\lambda = 1$, we have that $F_{k,\lambda} = F_{k,1} = \{G\colon G \text{ is } k\text{-connected}\}$ from Corollary 2 in Section 2.2, thus Theorem 3 is just the same with Theorem 3 given by Bollobás and Thomason. Hence Theorem 3 is a generalization of that proved by Bollobás and Thomason.

2.2 (k,λ)-connectivity of special graphs

From the definition of (k,λ)-connected graphs, we can obtain that $(k,1)$-connected graphs are k-connected graphs, and $(1,\lambda)$-connected graphs are λ-edge-connected graphs. Let G be a $(k,1)$-connected graph, for any vertex subset S and edge subset L such that $|S| + |L| < k$, we have that $G - S - L$ is connected. In particular, letting L be an empty set, then $G - S$ is connected for every vertex subset S with $|S| < k$. That implies G is k-connected. Similarly, if G is $(1,\lambda)$-connected, then for any vertex subset S and edge subset L with $\lambda|S| + |L| < \lambda$, $G - S - L$ is connected. Note that $\lambda|S| + |L| < \lambda$ holds if and only if $|S| = 0$ and $|L| < \lambda$. Thus, we have that $G - L$ is connected for any edge subset L with $|L| < \lambda$. Therefore, G is λ-edge-connected.

Moreover, a k-connected graph G is $(k,1)$-connected. As we showed in the proof of Theorem 1, for any k-connected graph G, $G-S-L$ is connected for any vertex subset S and edge set L with $|S|+|L|<k$, which implies that G is $(k,1)$-connected. Similarly, a λ-edge-connected graph G is $(1,\lambda)$-connected. If any vertex subset S and edge set L satisfies that $\lambda|S|+|L|<\lambda$, then S must be an empty set and $|L|<\lambda$. So $G-S-L$ is connected. Thus, if G is λ-edge-connected, then for any vertex subset S and edge subset L with $\lambda|S|+|L|<\lambda$, $G-S-L$ is connected. So we can obtain the following fact.

Corollary 2 *For a graph G with more than k vertices, G is $(k,1)$-connected if and only if G is k-connected, and G is $(1,\lambda)$-connected if and only if G is λ-edge-connected.*

If a graph is k-connected, then it must be k-edge connected. Thus, from Corollary 2, we have that if a graph is $(k,1)$-connected, then it is $(1,k)$-connected. More generally, we have the following proposition:

Proposition 1 *If G is (k_1,λ_1)-connected, then G is (k_2,λ_2)-connected for $\lambda_2 \geq \lambda_1$ with $k_1\lambda_1 = k_2\lambda_2$.*

Proof. For any vertex subset S and edge subset L such that $\lambda_2|S|+|L|<k_2\lambda_2$, we have that

$$\lambda_1|S|+|L| \leq \lambda_2|S|+|L| < k_2\lambda_2 = k_1\lambda_1,$$

and $G-S-L$ is still connected since G is (k_1,λ_1)-connected, which implies that G is (k_2,λ_2)-connected. ■

It is natural to consider that if we fix k or λ, then dose G being (k,λ)-connected guarantee that G is (k',λ)-connected or (k,λ')-connected for $k'<k$ and $\lambda'<\lambda$? The answer is affirmative, and we summarize it as follows.

Proposition 2 *If G is (k,λ)-connected, then*

i) G is (k',λ)-connected for $k'<k$,

(ii) G is (k,λ')-connected for $\lambda'<\lambda$.

Proof. (i) It is apparent that G is (k',λ)-connected, since if the vertex subset S and edge subset L satisfy that $\lambda|S|+|L|<k'\lambda$, then $\lambda|S|+|L|<k\lambda$, so $G-S-L$ is connected.

(ii) For the vertex subset S and edge subset L satisfying that $\lambda'|S|+|L|<k\lambda'$, if $\lambda|S|+|L|<k\lambda$, then we have done since G is (k,λ)-connected. So let us assume that there exist vertex subset S_0 and edge subset L_0, such that

$$\lambda'|S_0|+|L_0|<k\lambda',$$

but

$$\lambda|S_0|+|L_0| \geq k\lambda.$$

So $(\lambda-\lambda')|S_0|>(\lambda-\lambda')k$. Since $\lambda-\lambda'>0$, we have $|S_0|>k$. On the other hand, since $\lambda'|S_0|+|L_0|<k\lambda'$ and $|L_0|\geq 0$, we have that $|S_0|<k$, it is a contradiction. Hence, such S_0 and L_0 do not exist, and we can obtain that G is (k,λ')-connected. ■

We know that the complete graph K_n of order n is an $(n-1)$-connected simple graph, so it is $(n-1,1)$-connected by Corollary 2. With the aid of Proposition 1, we can obtain more on the (k,λ)-connectedness of K_n.

Theorem 4 *For any two positive integers k and λ with $k\lambda = n-1$, the complete graph K_n is (k,λ)-connected. And K_n is the only (k,λ)-connected simple graph of order n.*

Proof. The fact that K_n is (k,λ)-connected follows directly from Proposition 1. To see its uniqueness, realize that if a graph G is (k,λ)-connected, then the minimum degree $\delta(G)$ of G is at least $k\lambda$. So a (k,λ)-connected simple graph of order n must have minimum degree at least $k\lambda = n-1$, hence it can only be the complete graph K_n. ∎

Remark 2 From the proof of Theorem 4, we can obtain that $K_{k\lambda+1}$ have the minimum number of vertices among the (k,λ)-connected simple graphs. Call a graph G *minimally (k,λ)-connected* if G is (k,λ)-connected and $G-e$ is not (k,λ)-connected for any edge e of G. Thus, we can obtain that $K_{k\lambda+1}$ is the only minimally (k,λ)-connected simple graph of order n as well. That is because, for any edge e of $K_{k\lambda+1}$, $\delta(K_{k\lambda+1}-e) < k\lambda$, which implies that $K_{k\lambda+1}-e$ is not (k,λ)-connected.

Let K_n^* be the graph obtained by replacing the edges of K_n with multiple edges. Let $1 \le \ell_1 \le \ell_2 \le \ldots \le \ell_N$ be the multiplicities of edges in K_n, where $N = \frac{1}{2}n(n-1)$. If K_n is (k,λ)-connected, then clearly $K_n^*$is (k,λ)-connected. Moreover, we can derive that

Theorem 5 *For the multiple graph K_n^*, if positive integers k and λ satisfying $\ell_1 + \ell_2 + \ldots + \ell_{n-1} = k\lambda$, and $\lambda \le \min\{\ell_{n-k+1}, \ell_1 + \ell_2 + \ldots + \ell_{n-k}\}$, then K_n^* is (k,λ)-connected.*

Proof. Let S be a vertex subset and L be an edge subset, such that $\lambda|S| + |L| < k\lambda$. Let $|S| = i$, then $0 \le k-1$, and $|L| < (k-i)\lambda$. Since $\lambda \le \min\{\ell_{n-k+1}, \ell_1 + \ell_2 + \ldots + \ell_{n-k}\}$ and $\ell_1 + \ell_2 + \ldots + \ell_{n-1} = k\lambda$, $K_n^* - S - L$ is connected. So K_n^* is (k,λ)-connected. ∎

If we know the minimum degree of K_n^*, we can have the following more simplified result.

Corollary 3 *If the minimum degree $\delta(K_n^*)$ of K_n^* satisfies that $\delta(K_n^*) = k\lambda$ for some $k \le n-1$ and λ, then K_n^* is (k,λ)-connected.*

Proof. Let S be a vertex subset and L be an edge subset, such that $\lambda|S| + |L| < k\lambda$. Let $|S| = i$, then $0 \le i \le k-1$, and $|L| < (k-i)\lambda \le k\lambda = \delta(K_n^*)$. Hence $K_n^* - S - L$ is connected, which means K_n^* is (k,λ)-connected. ∎

In particular, if all the edges have the same multiplicity ℓ, we can have the following result by letting $k = n-1$, $\lambda = \ell$ in Theorem 5.

Corollary 4 *Let K_n^ℓ be the graph obtained from K_n by replacing each edge with ℓ edges, then K_n^ℓ is $(n-1,\ell)$-connected.*

In [14], the authors characterized the minimally $(1,\lambda)$-connected graphs with n vertices and $\lambda(n-1)$ edges. Here, we prove that

Theorem 6 *K_n^ℓ is the only $(n-1,\ell)$-connected graph (and the only minimally $(n-1,\ell)$-connected graph) with $\frac{1}{2}\ell n(n-1)$ edges.*

Proof. Suppose G is $(n-1,\ell)$-connected, then $\delta(G) \geq (n-1)\ell$. So the number of edges of G satisfies that

$$\frac{1}{2}\ell n(n-1) = |E(G)| \geq \frac{1}{2}|V(G)|(n-1)\ell,$$

then $|V(G)| \leq n$. But $|V(G)| \geq n$ holds since G is $(n-1,\ell)$-connected, therefore $|V(G)| = n$. If there exists a vertex v with degree $d(v) > (n-1)\ell$, then $|E(G)| > \frac{1}{2}[(n-1)^2\ell + (n-1)\ell] = \frac{1}{2}\ell n(n-1)$, a contradiction. Hence, all the vertices have the same degree $(n-1)\ell$. Then we prove that the multiplicity of every edge is ℓ. If the edge uv has multiplicity less than ℓ, then let S be the set consisting of all the vertices of G other than u and v, L be the set of multiple edges between u and v, we have that $\ell|S| + |L| < \ell(n-2) + \ell = (n-1)\ell$, and $G - S - L$ is disconnected. It contradicted to the fact that G is $(n-1,\ell)$-connected. So the multiplicity of every edge is ℓ. Combining all above, we can obtain that G is K_n^ℓ.

Since $\delta(K_n^\ell - e) < (n-1)\ell$ for any edge e of K_n^ℓ, we have that $K_n^\ell - e$ is not $(n-1,\ell)$-connected, which implies that K_n^ℓ is minimally $(n-1,\ell)$-connected. ∎

If we focus on the (k,λ)-connected bipartite graphs, we can obtain the result similar to Theorem 4 as follows.

Theorem 7 *For two positive integers n_1 and n_2 with $n_1 \leq n_2$, then the complete bipartite graph K_{n_1,n_2} is (k,λ)-connected for any positive k and λ such that $k\lambda = n_1$, and K_{n_1,n_1} is the only (k,λ)-connected bipartite simple graph, which having the minimum number of vertices.*

Proof. Since K_{n_1,n_2} is n_1-connected, it is also $(n_1,1)$-connected. By Proposition 1, it is (k,λ)-connected for any $k\lambda = n_1$. If a bipartite simple graph is (k,λ)-connected, then each vertex class has at least $k\lambda = n_1$ vertices, since the minimum degree is at least $k\lambda$. So K_{n_1,n_1} is the only (k,λ)-connected bipartite simple graph, which having the minimum number of vertices. ∎

Remark 3 (i) K_{n_1,n_1} is also the only minimally (k,λ)-connected bipartite simple graph of the minimum order.

(ii) Let $K^*_{n_1,n_2}$ be the graph obtained by replacing the edges of K_{n_1,n_2} with multiple edges. Let $1 \leq \ell_1 \leq \ell_2 \leq \ldots \leq \ell_M$ be the multiplicities of edges in K_{n_1,n_2}, where $M = |E(K^*_{n_1,n_2})|$. If positive integers k and λ satisfying $\ell_1 + \ell_2 + \ldots + \ell_{n_1} = k\lambda$, and $\lambda \leq \min\{\ell_{n_1-k+2}, \ell_1 + \ell_2 + \ldots + \ell_{n-k+1}\}$, then $K^*_{n_1,n_2}$ is (k,λ)-connected.

(iii) For multipartite graphs, we can derive the similar results. For t positive integers $n_1 \leq n_2 \leq \ldots \leq n_t$, the complete t-partite graph $K_{n_1,n_2,\ldots,n_t}$ is (k,λ)-connected for any $k\lambda = n_1$, and $K_{n_1,n_1,\ldots,n_1}$ is the only (k,λ)-connected t-partite simple graph, which having the minimum number of vertices. And for the multiple graph $K^*_{n_1,n_2,\ldots,n_t}$, we can also derive a result similar to (ii), here we omit the details.

3 (k,ℓ)-mixed-connectivity

3.1 (k,ℓ)-mixed-connectivity of random graphs

For the (k,ℓ)-mixed-connectivity, Sadeghi and Fan [17] presented a necessary and sufficient condition as follows, which will be applied to prove our results.

Lemma 4 *[17] Let $n \geq k+\ell+1$; and $k,\ell \geq 1$. A graph G of order n is (k,ℓ)-mixed-connected if and only if*

(i) G is $(k+1)$-connected and
(ii) G is $(k+\ell)$-edge-connected.

Theorem 8 *For any two positive integers k and ℓ,if $p = \{\log n + (k+\ell)\log\log n - \omega(n)\}/n$, then w.h.p. $G(n,p)$ is (k,ℓ)-mixed-connected, if $p = \{\log n + (k+\ell-1)\log\log n - \omega(n)\}/n$, then w.h.p. $G(n,p)$ is not (k,ℓ)-mixed-connected, where $\omega(n) \to \infty$ and $\omega(n) = o(\log\log n)$.*

Proof. Let $p_1 = \{\log n + (k+\ell)\log\log n - \omega(n)\}/n$, $p_2 = \{\log n + (k+\ell-1)\log\log n - \omega(n)\}/n$, where $\omega(n) \to \infty$ and $\omega(n) = o(\log\log n)$. From Theorem 1 and the fact that w.h.p. $\delta(G(n,p_1)) = k+\ell$, we have that w.h.p.

$$\kappa(G(n,p_1)) = \lambda(G(n,p_1)) = \delta(G(n,p_1)) = k+\ell.$$

Namely, $G(n,p_1)$ is w.h.p. $(k+\ell)$-connected and $(k+\ell)$-edge-connected. Thus, it is clear that $G(n,p_1)$ is w.h.p. $(k+1)$-connected and $(k+\ell)$-edge-connected. By Lemma 4, it follows that $G(n,p_1)$ is w.h.p. (k,ℓ)-mixed-connected.

For the second part of Theorem 8, since $\delta(G(n,p_2))$ is w.h.p. equal to $k+\ell-1$. Combining with Lemma 1, w.h.p. we have that $\lambda(G(n,p_2)) = \delta(G(n,p_2)) = k+\ell-1 < k+\ell$. From Lemma 4, we obtain that w.h.p. $G(n,p_2)$ is not (k,ℓ)-mixed-connected. ∎

Considering the definition of sharp threshold functions, the following Corollary 5 is an immediate consequence of Theorem 8.

Corollary 5 *For any two positive integers k and ℓ,*

$$p = \{\log n + (k+\ell)\log\log n - \omega(n)\}/n$$

is a sharp threshold function for the property that $G(n,p)$ is w.h.p. (k,ℓ)-mixed-connected, where $\omega(n) \to \infty$ and $\omega(n) = o(\log\log n)$.

Let $p = \{\log n + (k+\ell)\log\log n - \omega(n)\}/n$, where $\omega(n) \to \infty$ and $\omega(n) = o(\log\log n)$. For any constant $c_1 \geq 1$, from Theorem 8 and the monotonicity of (k,ℓ)-mixed-connectivity, we know that $G(n,c_1p)$ is w.h.p. (k,ℓ)-mixed-connected. On the other hand, for any constant $0 < c_2 < 1$, since $c_2p < \{\log n + (k+\ell-1)\log\log n - \omega(n)\}/n$ for sufficiently large n, we have that w.h.p. $G(n,c_2p)$ is not (k,ℓ)-mixed-connected by Theorem 8 and the monotonicity of (k,ℓ)-mixed-connectivity. Thus we obtain that Corollary 5 holds.

Theorem 9 *Let x be a fixed real number, if*

$$p = \{\log n + (k+\ell-1)\log\log n + x\}/n,$$

then $\Pr[\ G(n,p)$ *is* (k,ℓ)*-mixed-connected* $] \to e^{-e^{-x/(k+\ell-1)!}}$.

Proof. Let $p = \{\log n + (k+\ell-1)\log\log n + x\}/n$. If $G(n,p)$ is (k,ℓ)-mixed-connected, then $G(n,p)$ is $(k+\ell)$-edge-connected. Hence, we can obtain that $\delta(G(n,p)) \geq \lambda(G(n,p)) \geq k+\ell$. On the other hand, from Lemma 1, we know that w.h.p. $\kappa(G(n,p)) = \lambda(G(n,p)) = \delta(G(n,p))$. If $\delta(G(n,p)) \geq k+\ell$, then w.h.p. $\kappa(G(n,p)) = \lambda(G(n,p)) \geq k+\ell$. So $G(n,p)$ is (k,ℓ)-mixed-connected by Lemma 4. Therefore, $\Pr[\ G(n,p)$ is (k,ℓ)-mixed-connected $] = \Pr[\ \delta(G(n,p)) \geq k+\ell] + o(1) = \Pr[\ \delta(G(n,p)) = k+\ell] + o(1) \to e^{-e^{-x/(k+\ell-1)!}}$.

∎

Let $R_{k,\ell} = \{G\colon G$ is (k,ℓ)-mixed-connected $\}$ be the graph property such that the graph G is (k,ℓ)-mixed-connected, we have the following result for the hitting time of the random graph process.

Theorem 10 *Given $k \in \mathbb{N}$, in the random graph process $\tilde{G} = (G_t)_0^N$, $N = \binom{n}{2}$, with high probability $\tau_{D_{k+\ell}} = \tau_{R_{k,\ell}}$.*

Proof. From Lemma 3, we know that

$$\tau_{D_{k+\ell}} = \tau(\kappa(G) \geq k+\ell).$$

When $\kappa(G) \geq k+\ell$, we have that $\lambda(G) \geq \kappa(G) \geq k+\ell$. Thus, we obtain that G is (k,ℓ)-mixed-connected by Lemma 4. So, w.h.p.

$$\tau_{R_{k,\ell}} \leq \tau_{D_{k+\ell}}. \tag{3}$$

Let $t = \tau_{R_{k,\ell}}$, we have that w.h.p. G_t is $(k+\ell)$-edge-connected from Lemma 4. So it must hold that $\delta(G_t) \geq k+\ell$. Since otherwise, let L be the set of edges incident to a vertex with minimum degree in G_t, then $|L| \leq k+\ell-1$ and $G_t - L$ is disconnected, a contradiction to the fact that $\lambda(G_t) \geq k+\ell$. That means the minimum degree of G_t has already been at least $k+\ell$. Hence, w.h.p.

$$\tau_{R_{k,\ell}} \geq \tau_{D_{k+\ell}}. \tag{4}$$

Therefore, by (3) and (4), we have w.h.p.

$$\tau_{R_{k,\ell}} = \tau_{D_{k+\ell}}.$$

∎

3.2 (k,ℓ)-mixed-connectivity of special graphs

The authors in [9] proved the following property concerning the mixed connection.

Lemma 5 *[9] Let $n \geq k+\ell+1$ and $k,\ell \geq 1$. If a graph remains connected after removal of any k vertices and $(\ell-1)$ edges, then this graph also remains connected after removal of any $(k-1)$ vertices and any ℓ edges.*

In [17], the authors presented the following theorem to determine whether a graph is (k,ℓ)-mixed-connected.

Lemma 6 *[17] Let $n \geq k+\ell+1$ and $k,\ell \geq 1$. A graph G is (k,ℓ)-mixed-connected if and only if the resulted graph after removal of any k vertices and $(\ell-1)$ edges is connected.*

From Lemma 5 and Lemma 6, we can have that if G is (k,ℓ)-mixed-connected, then G is also $(k-1,\ell+1)$-mixed-connected. Since G is $(k-1,\ell+1)$-mixed-connected, it is $(k-2,\ell+2)$-mixed-connected, etc. So if G is (k,ℓ)-mixed-connected, then G is (k_1,ℓ_1)-mixed-connected for any k_1,ℓ_1 with $k_1 \leq k$ and $k_1+\ell_1 = k+\ell$.

Similar to Theorem 4, the following result tells us the (k,ℓ)-mixed-connectedness of the complete graphs.

Theorem 11 *For $n \geq 3$, the complete graph K_n is $(n-2,1)$-mixed-connected. And K_n is the only $(n-2,1)$-mixed-connected simple graph of order n.*

Proof. Since K_n is $(n-1)$-connected and $(n-1)$-edge-connected, it is $(n-2,1)$-mixed-connected by Lemma 4. For any $(n-2,1)$-mixed-connected simple graph G, we have that G is $(n-1)$-connected. So $\delta(G) \geq \kappa(G) \geq n-1$. Hence, if G is an $(n-2,1)$-mixed-connected simple graph of order n, it is isomorphic to K_n. ∎

For the complete bipartite graph, we can obtain the following result. We omit the proof of Theorem 12, since it uses the same method applied in Theorem 11.

Theorem 12 *For two positive integers n_1 and n_2 with $n_1 \leq n_2$, then the complete bipartite graph K_{n_1,n_2} is $(n_1-1,1)$-mixed-connected, and K_{n_1,n_1} is the only $(n_1-1,1)$-mixed-connected bipartite simple graph, which having the minimum number of vertices.*

A graph G is said to be *minimally k-(edge)-connected*, if G is k-(edge)-connected, and $G-e$ is not k-(edge)-connected for any edge e of G. Similarly, we call G *minimally (k,ℓ)-mixed-connected*, if G is (k,ℓ)-mixed-connected and $G-e$ is not (k,ℓ)-mixed-connected for any edge e of G. For the minimally (k,ℓ)-mixed-connected graphs, we obtain the following result.

Theorem 13 *A graph G is minimally (k,ℓ)-mixed-connected if and only if G is $(k+1)$-connected and minimally $(k+\ell)$-edge-connected for $k>0$ and $\ell>1$, and G is minimally $(k,1)$-mixed-connected if and only if G is minimally $(k+1)$-connected for $k>0$.*

Proof. For $k > 0$ and $\ell > 1$, if G is $(k+1)$-connected and minimally $(k+\ell)$-edge-connected, then G is obviously minimally (k,ℓ)-mixed-connected from Lemma 4. We will show that if G is minimally (k,ℓ)-mixed-connected, then G is $(k+1)$-connected and minimally $(k+\ell)$-edge-connected. Since G is (k,ℓ)-mixed-connected, G is $(k+1)$-connected and $(k+\ell)$-edge-connected by Lemma 4. For any edge $e = xy$ of G, if $G-e$ is not $(k+1)$-connected, then there exists a vertex subset S with $|S| = k$, such that $G-e-S$ is disconnected. Suppose that x and y belong to the different components of $G-e-S$, then let the edge subset L be the single edge e, we have that $G-S-L$ is disconnected with $|S| = k$ and $|L| = 1$, it contradicts to the fact that G is (k,ℓ)-mixed-connected. Hence, x and y belong to the same component of $G-e-S$, which implies that G is disconnected after removing S. Since $|S| = k$, it is a contradiction to G being (k,ℓ)-mixed-connected. Thus, $G-e$ is $(k+1)$-connected. Since $G-e$ is not (k,ℓ)-mixed-connected, with Lemma 4, we can obtain that $G-e$ is not $(k+\ell)$-edge-connected. So G is minimally $(k+\ell)$-edge-connected as well.

For the case that $\ell = 1$, if G is minimally $(k+1)$-connected, then G is $(k+1)$-edge-connected, by Lemma 4, G is $(k,1)$-mixed-connected. For any edge e of G, $G-e$ is not $(k+1)$-connected since G is minimally $(k+1)$-connected, so $G-e$ is not $(k,1)$-mixed-connected by Lemma 4. Therefore, G is minimally $(k,1)$-mixed-connected. On the other hand, if G is minimally $(k,1)$-mixed-connected, then G is $(k+1)$-connected. And for any edge e of G, $G-e$ is not $(k,1)$-mixed-connected. From Lemma 6, we know that there is a set S of k vertices, such that $G-e-S$ is disconnected. So $G-e$ is not $(k+1)$-connected. In conclusion, G is minimally $(k+1)$-connected.

∎

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