

ON TINY ZERO-SUM SEQUENCES OVER FINITE ABELIAN GROUPS

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ABSTRACT. Let G be an additive finite abelian group and $S = g_1 \cdot \dots \cdot g_l$ be a sequence over G . Let $k(S) = \text{ord}(g_1)^{-1} + \dots + \text{ord}(g_l)^{-1}$ be its cross number. Let $t(G)$ (resp. $\eta(G)$) be the smallest integer t such that every sequence of t elements (repetition allowed) from G has a non-empty zero-sum subsequence T with $k(T) \leq 1$ (resp. $|T| \leq \exp(G)$). It is easy to see that $t(G) \geq \eta(G)$. It is known that $t(G) = \eta(G) = |G|$ when G is cyclic, and for any integer $r \geq 3$, there are infinitely many groups G of rank r such that $t(G) > \eta(G)$. It is conjectured in 2012 [G12] that $t(G) = \eta(G)$ for all finite abelian groups of rank two. This conjecture has been verified only for the groups $G \cong C_{p^\alpha} \oplus C_{p^\alpha}$, $G \cong C_2 \oplus C_{2p}$ and $G \cong C_3 \oplus C_{3p}$ with $p \geq 5$, where p is a prime. In this paper, among other results, we confirm this conjecture for more groups including the groups $G \cong C_n \oplus C_n$ with the smallest prime divisor of n not less than the number of the distinct prime divisors of n .

1. INTRODUCTION AND MAIN RESULTS

Let G be a finite abelian group, written additively. If G is cyclic of order n , it will be denoted by C_n . In the general case, we can decompose G as a direct sum of cyclic groups $C_{n_1} \oplus \dots \oplus C_{n_r}$ such that $1 < n_1 \mid \dots \mid n_r \in \mathbb{N}$ (if $n_1 = \dots = n_r = n$, it will be abbreviated as C_n^r), where r and n_r are respectively called the *rank* and *exponent* of G . Usually, the exponent of G is simply denoted by $\exp(G)$. The order of an element g of G will be written $\text{ord}(g)$.

Given a sequence $S = g_1 \cdot \dots \cdot g_l$ over G , we denote by $S_{(d)}$ the subsequence of S consisting of all terms of S of order d and S_H the subsequence of S consisting of all terms of S belonging to a subgroup H of G . And by $k(S)$ the *cross number* of S , which is defined as follows:

$$k(S) = \sum_{i=1}^l \frac{1}{\text{ord}(g_i)}.$$

The cross number is an important concept in factorization theory. For more information on the cross number we refer to ([GG09, GS94, G09, G12]).

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Let $\mathfrak{t}(G)$ denote the smallest integer $t \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq t$ has a non-empty zero-sum subsequence S' with $\mathfrak{k}(S') \leq 1$. Such a subsequence will be called a *tiny zero-sum* subsequence.

The study of $\mathfrak{t}(G)$ goes back to the late 1980s, Lemke and Kleitman [LK89] proved that $\mathfrak{t}(C_n) = n$, which confirmed a conjecture by Erdős and Lemke. More generally, Lemke and Kleitman [LK89] conjectured that $\mathfrak{t}(G) \leq |G|$ holds for every finite abelian group G . This conjecture was proved by Geroldinger [G93] in 1993. Furthermore, Elledge and Hurlbert [EH05] gave a different proof in 2005.

In 2012, Girard [G12] proved that, by using a result of Alon and Dubiner [AD95], for finite abelian groups of fixed rank, $\mathfrak{t}(G)$ grows linearly in the exponent of G , which gives the correct order of magnitude.

Let $\eta(G)$ denote the smallest integer $t \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq t$ has a non-empty zero-sum subsequence S' with $|S'| \leq \exp(G)$. Such a subsequence is called a *short zero-sum* subsequence. The constant $\eta(G)$ is one of many classical invariants in so-called zero-sum theory. For zero-sum theory and its application, the interested reader is referred to [GG06] and [GH06].

Since $\mathfrak{k}(T) \leq 1$ implies $|T| \leq \exp(G)$, we know that $\eta(G) \leq \mathfrak{t}(G)$ always holds. Girard [G12] noticed that if $\mathfrak{t}(G) = \eta(G)$ for some finite abelian group G , then $\eta(H) \leq \eta(G)$ for any subgroup H of G , and then he deduced that for any positive integer $r \geq 4$, there is a finite abelian group of rank r such that $\mathfrak{t}(G) > \eta(G)$. Concerning groups of rank three, the first author with coauthors [FGPWZ13] found that $\mathfrak{t}(G) > \eta(G)$ if $G \cong C_2 \oplus C_2 \oplus C_{2n}$, where $n > 1$ is a positive integer. Girard [G12] also proved that $\mathfrak{t}(C_{p^\alpha}^2) = \eta(C_{p^\alpha}^2) = 3p^\alpha - 2$ for any prime p and conjectured that $\mathfrak{t}(G) = \eta(G)$ for all finite abelian groups of rank two. Girard also [G12] noticed the easy fact that $\mathfrak{t}(G) = \eta(G)$ for all elementary p -groups G , since all non-zero elements of G have same order in this case, and conjectured that $\mathfrak{t}(G) = \eta(G)$ for $G \cong C_n^r$.

Conjecture 1.1. ([G12]) *For all positive integers m, n with $m \mid n$, we have*

$$\mathfrak{t}(C_m \oplus C_n) = \eta(C_m \oplus C_n) = 2m + n - 2.$$

Conjecture 1.2. ([G12]) *For all positive integers r, n , we have*

$$\mathfrak{t}(C_n^r) = \eta(C_n^r).$$

Conjectures 1.1 and 1.2 have been confirmed only for a few classes of groups.

Theorem 1.3. ([FGPWZ13, GHST07, G12, W20]) *Let G be a finite abelian group, and n, r, α, β be positive integers and p be a prime number. Then $\mathfrak{t}(G) = \eta(G)$ for the following groups.*

- (1) $G \cong C_n$,
- (2) $G \cong C_{p^\alpha} \oplus C_{p^\alpha}$,
- (3) $G \cong C_2 \oplus C_{2p}$,
- (4) $G \cong C_3 \oplus C_{3p}$ with $p \geq 5$,
- (5) $G \cong C_n^3$ with $n = 3^\alpha$ or $n = 5^\beta$,
- (6) $G \cong C_n^r$ with $n = p$ or $n = 2^\alpha$.

In this paper, we will confirm both Conjecture 1.1 and Conjecture 1.2 for more groups. Now we state our main results.

Theorem 1.4. *Let n be a positive integer, and let $G \cong C_n \oplus C_n$. If $\sum_{p|n} \frac{1}{p} < 1$, where p runs over all distinct prime divisors of n , then*

$$\mathfrak{t}(G) = \eta(G).$$

In particular, if $p(n) \geq \omega(n)$, then $\mathfrak{t}(G) = \eta(G)$, where $p(n)$ denotes the smallest prime divisor of n and $\omega(n)$ denotes the number of distinct prime divisors of n .

Theorem 1.5. *Let α, β be positive integers and p be a prime number. Then $\mathfrak{t}(G) = \eta(G)$ for the following groups.*

- (a) $G \cong C_2 \oplus C_{2^\alpha}$,
- (b) $G \cong C_2 \oplus C_{2p^\beta}$,
- (c) $G \cong C_{3^\alpha 5^\beta}^3$.

The paper is organized as follows. Section 2 provides some notation and concepts which will be used in the sequel. In Section 3 we prove the main results.

2. NOTATION AND PRELIMINARIES

Let \mathbb{N} denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any two integers $a, b \in \mathbb{N}_0$, we set $[a, b] = \{x \in \mathbb{N}_0 \mid a \leq x \leq b\}$. Throughout this paper, all abelian groups will be written additively.

Let G be an additive finite abelian group with rank r . An r -tuple (e_1, \dots, e_r) in $G \setminus \{0\}$ is called a *basis* of G if $G \cong \langle e_1 \rangle \oplus \dots \oplus \langle e_r \rangle$. We denote by $\mathcal{F}(G)$ the free (abelian, multiplicative) monoid with basis G . The elements of $\mathcal{F}(G)$ are called *sequences* over G . We write sequences $S \in \mathcal{F}(G)$ in the form

$$S = \prod_{g \in G} g^{\mathbf{v}_g(S)}, \text{ with } \mathbf{v}_g(S) \in \mathbb{N}_0 \text{ for all } g \in G.$$

We call $\mathbf{v}_g(S)$ the *multiplicity* of g in S , and we say that S contains g if $\mathbf{v}_g(S) > 0$. A sequence S' is called a *subsequence* of S if $\mathbf{v}_g(S') \leq \mathbf{v}_g(S)$ for all $g \in G$, denote by $S' \mid S$, and SS'^{-1} denotes the subsequence obtained from S by deleting S' , two subsequences S_1 and S_2 of S are called *disjoint* if $S_1 \mid SS_2^{-1}$. The unit element $1 \in \mathcal{F}(G)$ is called the *empty* sequence.

For a sequence

$$S = g_1 \cdot \dots \cdot g_l = \prod_{g \in G} g^{\mathbf{v}_g(S)} \in \mathcal{F}(G),$$

we call

- $|S| = l = \sum_{g \in G} \mathbf{v}_g(S) \in \mathbb{N}_0$ the *length* of S ,
- $\sigma(S) = \sum_{i=1}^l g_i = \sum_{g \in G} \mathbf{v}_g(S)g \in G$ the *sum* of S ,
- $\text{supp}(S) = \{g \in G \mid \mathbf{v}_g(S) > 0\} \subset G$ the *support* of S ,
- S a *zero-sum sequence* if $\sigma(S) = 0 \in G$,
- S a *zero-sum free sequence* if there is no non-empty zero-sum subsequence of S ,
- S a *minimal zero-sum sequence* if it is a non-empty zero-sum sequence and has no proper zero-sum subsequence,
- S a *short zero-sum sequence* if S is zero-sum and $1 \leq |S| \leq \exp(G)$,
- S a *tiny zero-sum sequence* if S is a non-empty zero-sum sequence and $\mathbf{k}(S) \leq 1$.

Let $D(G)$ denote the smallest integer $t \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq t$ has a non-empty zero-sum subsequence. The invariant $D(G)$ is called the *Davenport constant* of G .

Every map of abelian groups $\varphi : G \rightarrow H$ extends to a homomorphism $\varphi : \mathcal{F}(G) \rightarrow \mathcal{F}(H)$, where $\varphi(S) = \varphi(g_1) \cdot \dots \cdot \varphi(g_l)$. If φ is a homomorphism then $\varphi(S)$ is a zero-sum sequence if and only if $\sigma(S) \in \ker(\varphi)$.

Given a positive integer n , let $p(n)$ denote the smallest prime divisor of n , by convention $p(1) = 1$, let $\omega(n)$ denote the number of distinct prime divisors of n .

We list some results on $\eta(G)$ which will be used frequently in the sequel.

Lemma 2.1. ([EEGKR07, GHST07]) *Let m, n be positive integers. Then*

- (1) $\eta(C_m \oplus C_n) = 2m + n - 2$ for $m \mid n$,
- (2) $\eta(C_n^3) = 8n - 7$ for $n = 3^\alpha 5^\beta$, where $\alpha, \beta \in \mathbb{N}_0$.

Lemma 2.2. ([E04]) *If n is an odd integer, then $\eta(C_n^3) \geq 8n - 7$.*

Lemma 2.3. ([GH06, Proposition 5.7.11]) *Let G be a finite abelian group, and let H be a subgroup of G with $\exp(G) = \exp(H)\exp(G/H)$. Then*

$$\eta(G) \leq \exp(G/H)(\eta(H) - 1) + \eta(G/H).$$

Lemma 2.4. *Let m, n be odd integers. Suppose that $\eta(C_m^3) = 8m - 7$ and $\eta(C_n^3) = 8n - 7$, then $\eta(C_{mn}^3) = 8mn - 7$.*

Proof. By Lemma 2.2 we have $\eta(C_{mn}^3) \geq 8mn - 7$. Let $G \cong C_{mn}^3$ and $H \cong C_m^3$ be a subgroup of G , then $G/H \cong C_n^3$. It follows from Lemma 2.3 that

$$\eta(G) \leq \exp(G/H)(\eta(H) - 1) + \eta(G/H) = 8mn - 7.$$

Therefore, $\eta(C_{mn}^3) = 8mn - 7$. \square

Lemma 2.5. ([S12, Corollary 3.2]) *Let $H \cong C_m \oplus C_{mn}$ with integers $m \geq 2$ and $n \geq 1$. Every sequence S over H of length $|S| = \eta(H) - 1$ having not any short zero-sum subsequence has the following form*

$$S = b_1^{m-1} b_2^{sm-1} (-x b_1 + b_2)^{(n+1-s)m-1},$$

where $\{b_1, b_2\}$ is a generating set of H with $\text{ord}(b_2) = mn$, $s \in [1, n]$, $x \in [1, m]$ with $\gcd(x, m) = 1$ and either

- (1) $\{b_1, b_2\}$ is an independent generating set of H , or
- (2) $s = n$ and $x = 1$.

Lemma 2.6. ([GH06, Theorem 5.4.5]) *Let $n > 1$ be a positive integer, and let $S \in \mathcal{F}(C_n)$ be a sequence of length $n - 1$. If S is zero-sum free then $S = g^{n-1}$ for some generating element $g \in C_n$.*

Lemma 2.7. ([FGPWZ13, Lemma 2.3]) *Let $n > 1$ be a positive integer, and let $S \in \mathcal{F}(C_n)$ be a sequence of length $2n - 1$. If S has no two disjoint non-empty zero-sum subsequences then $S = g^{2n-1}$ for some generating element $g \in C_n$.*

3. PROOF OF MAIN RESULTS

In this section we shall prove Theorem 1.4 and Theorem 1.5, and we begin with some preliminary results.

Lemma 3.1. *Let G be a finite abelian group and H be a subgroup of G . Let S be a sequence over G . Suppose that SS_H^{-1} has a factorization $SS_H^{-1} = S_1 S_2 \cdots S_k S'$ such that $\sigma(S_i) \in H$ and $\mathbf{k}(S_i) \leq \mathbf{k}(\sigma(S_i))$ for every $i \in [1, k]$. If $k + |S_H| \geq \mathbf{t}(H)$, then S has a tiny zero-sum subsequence.*

Proof. By the hypothesis of this lemma, $\sigma(S_1)\sigma(S_2) \cdots \sigma(S_k)S_H$ is a sequence over H of length $k + |S_H| \geq \mathfrak{t}(H)$. Therefore, it has a tiny zero-sum subsequence $T \prod_{i \in I} \sigma(S_i)$, where $T \mid S_H$ and $I \subset [1, k]$. Let $W = T \prod_{i \in I} S_i$. Then W is a zero-sum subsequence of S with $\mathfrak{k}(W) = \mathfrak{k}(T) + \sum_{i \in I} \mathfrak{k}(S_i) \leq \mathfrak{k}(T) + \sum_{i \in I} \mathfrak{k}(\sigma(S_i)) = \mathfrak{k}(T \prod_{i \in I} \sigma(S_i)) \leq 1$. \square

Lemma 3.2. *Let G be a finite abelian group and H be a subgroup of G . Let S be a sequence over G . Suppose that SS_H^{-1} has a subsequence L such that for every $T \mid L$ with $|T| \leq \exp(G/H)$ we have $\mathfrak{k}(T) \leq \frac{1}{\exp(H)}$. If $|S_H| + \lceil \frac{|L| - (\eta(G/H) - 1)}{\exp(G/H)} \rceil \geq \mathfrak{t}(H)$, then S has a tiny zero-sum subsequence.*

Proof. Let ϕ be the projection from G onto G/H with $\ker(\phi) = H$. By applying $\eta(\phi(G)) = \eta(G/H)$ repeatedly on the sequence $\phi(L)$, we can get a factorization $L = S_1 \cdots S_k S'$ such that $\phi(S_i)$ is a short zero-sum sequence over $\phi(G) = G/H$ for every $i \in [1, k]$, and such that $\phi(S')$ has no short zero-sum subsequence over $\phi(G) = G/H$. It follows that

$$|S'| = |\phi(S')| \leq \eta(G/H) - 1.$$

Therefore,

$$k \geq \lceil \frac{|L| - (\eta(G/H) - 1)}{\exp(G/H)} \rceil.$$

By the hypothesis, $\mathfrak{k}(S_i) \leq \frac{1}{\exp(H)} \leq \frac{1}{\text{ord}(\sigma(S_i))} = \mathfrak{k}(\sigma(S_i))$ for every $i \in [1, k]$. Now the result follows from Lemma 3.1 since $k + |S_H| \geq \lceil \frac{|L| - (\eta(G/H) - 1)}{\exp(G/H)} \rceil + |S_H| \geq \mathfrak{t}(H)$. \square

Proposition 3.3. *Let c, n, r be three positive integers such that for every positive divisor $m (> 1)$ of n , we have $\eta(C_m^r) = c(m - 1) + 1$. If $\sum_{p|n} \frac{1}{p} < 1$, where p runs over all distinct prime divisors of n , then*

$$\mathfrak{t}(C_n^r) = \eta(C_n^r).$$

Proof. Let $G \cong C_n^r$. Let p_1, \dots, p_s be the all distinct prime divisors of n . By the hypothesis of this proposition,

$$\sum_{i=1}^s \frac{1}{p_i} < 1.$$

For every positive integer $m \mid n$, let $G_m = \{x \in G \mid mx = 0\}$. Clearly, G_m is a subgroup of G with $G_m \cong C_m^r$.

Let $d(n)$ denote the number of positive divisors (> 1) of n . We proceed by induction on $d(n)$. If $d(n) = 1$ then n is a prime, therefore $\mathfrak{t}(G) = \eta(G)$ follows from Theorem 1.3(6) and we are done. Suppose that the proposition is true for $d(n) < k$ ($k \geq 2$) and then we want to prove it is true also for $d(n) = k$.

As mentioned in the introduction we always have $\mathfrak{t}(G) \geq \eta(G)$. So, it suffices to prove that

$$\mathfrak{t}(G) \leq \eta(G) = c(n-1) + 1.$$

Let S be a sequence of length $|S| = c(n-1) + 1$ over G . We want to show that S has a tiny zero-sum subsequence. If $0 \mid S$, then $S' = 0$ has the required property and we are done. Next we suppose that $0 \nmid S$. Assume to the contrary that S has no tiny zero-sum subsequence. Let

$$S = TW$$

such that $\text{ord}(g) = n$ for all $g \in \text{supp}(T)$, and $\text{ord}(h) < n$ for all $h \in \text{supp}(W)$. If $S = T$, then it is easy to see that S has a tiny zero-sum subsequence, a contradiction. Next we assume that T is a proper subsequence of S . For every $i \in [1, s]$, let W_i be the subsequence of W consisting of all terms of W in $G \frac{n}{p_i}$. Then,

$$|W_1| + \dots + |W_s| \geq |W|.$$

Since for every $T' \mid T$ with $|T'| \leq \exp(G/G \frac{n}{p_i})$ we have $\mathfrak{k}(T') \leq \frac{\exp(G/G \frac{n}{p_i})}{\exp(G)} = \frac{1}{\exp(G \frac{n}{p_i})}$, by Lemma 3.2 we obtain that

$$|W_i| + \left\lceil \frac{|T| - (\eta(G/G \frac{n}{p_i}) - 1)}{\exp(G/G \frac{n}{p_i})} \right\rceil \leq \mathfrak{t}(G \frac{n}{p_i}) - 1.$$

Therefore, by induction we have

$$\frac{|T| - c(p_i - 1)}{p_i} + |W_i| \leq c \left(\frac{n}{p_i} - 1 \right)$$

for every $i \in [1, s]$, or equivalently,

$$\frac{|T|}{p_i} + |W_i| \leq \frac{c(n-1)}{p_i}.$$

So,

$$|T| \sum_{i=1}^s \frac{1}{p_i} + |W_1| + \dots + |W_s| \leq c(n-1) \sum_{i=1}^s \frac{1}{p_i},$$

it follows that $|W_1| + \dots + |W_s| \leq (cn - c - |T|) \sum_{i=1}^s \frac{1}{p_i}$. Since $|W_1| + \dots + |W_s| \geq |W|$, we deduce that

$$c(n-1) + 1 - |T| = |S| - |T| = |W| \leq |W_1| + \dots + |W_s| \leq (cn - c - |T|) \sum_{i=1}^s \frac{1}{p_i}.$$

So we have $1 \leq (cn - c - |T|) (\sum_{i=1}^s \frac{1}{p_i} - 1)$. It follows from $\sum_{i=1}^s \frac{1}{p_i} < 1$ and $|T| \leq |S| - 1 = c(n-1)$ that

$$1 \leq (cn - c - |T|) \left(\sum_{i=1}^s \frac{1}{p_i} - 1 \right) \leq 0,$$

a contradiction. \square

Proof of Theorem 1.4. Since $\eta(C_m \oplus C_m) = 3m - 2 = 3(m - 1) + 1$ for every positive integer m , the first part of this theorem follows from Proposition 3.3. If $p(n) \geq \omega(n)$, we clearly have

$$\sum_{p|n} \frac{1}{p} \leq \frac{\omega(n)}{p(n)}$$

with equality holding if and only if $\omega(n) = 1$. Therefore, we have $\sum_{p|n} \frac{1}{p} < 1$ and the result follows from the first part of this theorem. \square

Remark 3.4. Clearly, if $\omega(n) \leq 2$ then $\sum_{p|n} \frac{1}{p} < 1$. If $\omega(n) = 3$ and $n \neq 2^\alpha 3^\beta 5^\gamma$ then we also have $\sum_{p|n} \frac{1}{p} < 1$. It would be interesting to prove $\mathfrak{t}(C_n \oplus C_n) = \eta(C_n \oplus C_n)$ for $n = 2^\alpha 3^\beta 5^\gamma$.

Lemma 3.5. *Let n be a positive even integer and let $G \cong C_2 \oplus C_{2n}$. Let S be a sequence over G with $|S| = 2n + 1$. If $\text{ord}(x) = 2n$ for every $x \in \text{supp}(S)$, then S has a tiny zero-sum subsequence.*

Proof. Let (e_1, e_2) be a basis of G . If S has a short zero-sum subsequence S' , then $\mathfrak{k}(S') = \frac{|S'|}{2n} \leq 1$ and we are done. Next we assume that S has no short zero-sum subsequence. Since $|S| = 2n + 1 = \eta(G) - 1$, then by Lemma 2.5 we have

$$S = b_1 b_2^{2s-1} (-b_1 + b_2)^{2(n+1-s)-1},$$

where $\{b_1, b_2\}$ is a generating set of G with $\text{ord}(b_2) = 2n$, $s \in [1, n]$. Let $b_1 = x_1 e_1 + y_1 e_2$ and $b_2 = x_2 e_1 + y_2 e_2$, where $x_i \in [0, 1]$, $y_i \in [0, 2n - 1]$ for $i \in \{1, 2\}$. Since $\text{ord}(b_1) = \text{ord}(b_2) = 2n$ and since n is assumed to be even, y_1, y_2 are odd. It follows that $-b_1 + b_2 = (-x_1 + x_2)e_1 + (-y_1 + y_2)e_2$, since $-y_1 + y_2$ is even, we have $\text{ord}(-b_1 + b_2) \leq n$, a contradiction. \square

Lemma 3.6. *Let $G \cong C_n^3$ be a finite abelian group with $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, where p_1, \dots, p_s are distinct odd prime numbers and $\alpha_1, \dots, \alpha_s \in \mathbb{N}$. If $\sum_{i=1}^s \frac{1}{p_i} < 1$ and $\eta(C_{p_i}^3) = 8p_i^{\alpha_i} - 7$, then*

$$\mathfrak{t}(G) = \eta(G).$$

Proof. By Lemma 2.4 we have $\eta(C_n^3) = 8n - 7$. By Proposition 3.3 we have $\mathfrak{t}(G) = \eta(G)$. \square

Proof of Theorem 1.5. (a) Let $G \cong C_2 \oplus C_{2^\alpha}$ with $\alpha \in \mathbb{N}$ and (e_1, e_2) be a basis of G . The result follows from Theorem 1.3(3) for $\alpha \leq 2$. Next we may assume that $\alpha \geq 3$.

We proceed by induction on α . Suppose that $\mathfrak{t}(C_2 \oplus C_{2^l}) = \eta(C_2 \oplus C_{2^l})$ for $l \leq \alpha - 1$. Next we need to prove it holds for $l = \alpha$.

As mentioned in the introduction we always have that $\mathfrak{t}(G) \geq \eta(G)$. So, it suffices to prove that

$$\mathfrak{t}(G) \leq \eta(G) = 2^\alpha + 2.$$

Let S be a sequence of length $|S| = 2^\alpha + 2$ over G . We want to show that S has a tiny zero-sum subsequence. If $0 \mid S$, then $S' = 0$ has the required property and we are done. Next we suppose that $0 \nmid S$. Assume to the contrary that S has no tiny zero-sum subsequence.

Let us recall that we denote by $S_{(d)}$ the subsequence of S consisting of all terms of S of order d . Let H_1 be a subgroup of G isomorphic to $C_2 \oplus C_{2^{\alpha-1}}$ such that $H_2 = G/H_1$ is isomorphic to C_2 . Then $S = S_{H_1}S_{(2^\alpha)}$ and

$$(3.1) \quad |S| = |S_{H_1}| + |S_{(2^\alpha)}| = 2^\alpha + 2.$$

Since for every $T \mid S_{(2^\alpha)}$ with $|T| \leq \exp(G/H_1)$ we have $\mathfrak{k}(T) \leq \frac{\exp(G/H_1)}{\exp(G)} = \frac{1}{\exp(H_1)}$, by Lemma 3.2 we obtain that

$$|S_{H_1}| + \left\lceil \frac{|S_{(2^\alpha)}| - (\eta(G/H_1) - 1)}{\exp(G/H_1)} \right\rceil \leq \mathfrak{t}(H_1) - 1.$$

Therefore,

$$2|S_{H_1}| + |S_{(2^\alpha)}| \leq 2^\alpha + 3.$$

Combining equality (3.1), we obtain that $|S_{H_1}| \leq 1$. If $|S_{H_1}| = 0$, then $S = S_{(2^\alpha)}$. Hence S has a short zero-sum subsequence T' with $\mathfrak{k}(T') \leq 1$, a contradiction.

Next we assume that $|S_{H_1}| = 1$, by (3.1) we have $|S_{(2^\alpha)}| = 2^\alpha + 1$. By Lemma 3.5 we obtain that $S_{(2^\alpha)}$ has a tiny zero-sum subsequence, so S has a tiny zero-sum subsequence, a contradiction again.

(b) Let $G \cong C_2 \oplus C_{2p^\beta}$ with $\beta \in \mathbb{N}$ and p be a prime number and (e_1, e_2) be a basis of G . The results follow from Theorem 1.3 and (a) for $\beta = 1$ or $p = 2$. Next we may assume that $\beta \geq 2$ and $p \geq 3$.

We proceed by induction on β . Suppose that $\mathfrak{t}(C_2 \oplus C_{2p^s}) = \eta(C_2 \oplus C_{2p^s})$ for $s \leq \beta - 1$. Next we need to prove it holds for $s = \beta$.

As mentioned in the introduction we always have that $\mathfrak{t}(G) \geq \eta(G)$. So, it suffices to prove that

$$\mathfrak{t}(G) \leq \eta(G) = 2p^\beta + 2.$$

Let S be a sequence of length $|S| = 2p^\beta + 2$ over G . We want to show that S has a tiny zero-sum subsequence. If $0 \mid S$, then $S' = 0$ has the required property and we are done. Next we suppose that $0 \nmid S$. Assume to the contrary that S has no tiny zero-sum subsequence.

Let H_1 be a subgroup of G isomorphic to C_{p^β} such that $G/H_1 \cong C_2 \oplus C_2$. Let also H_2 be a subgroup of G isomorphic to $C_2 \oplus C_{2p^{\beta-1}}$ such that $G/H_2 \cong C_p$. Let φ_1 and φ_2 be projections from G to G/H_1 and G/H_2 , respectively, then $\ker(\varphi_1) = H_1 \cong C_{p^\beta}$ and $\ker(\varphi_2) = H_2 \cong C_2 \oplus C_{2p^{\beta-1}}$. Therefore,

$$S = S_{H_1} \cdot S_{(2)} \cdot S_{(2p)} \cdot \dots \cdot S_{(2p^{\beta-1})} \cdot S_{(2p^\beta)} = S_{H_2} \cdot S_{(p^\beta)} \cdot S_{(2p^\beta)}$$

and

(3.2)

$$|S| = |S_{H_1}| + |S_{(2)}| + |S_{(2p)}| + \dots + |S_{(2p^{\beta-1})}| + |S_{(2p^\beta)}| = |S_{H_2}| + |S_{(p^\beta)}| + |S_{(2p^\beta)}|.$$

Since for every $T \mid S_{(2p^\beta)}$ with $|T| \leq \exp(G/H_1)$ we have $k(T) \leq \frac{\exp(G/H_1)}{\exp(G)} = \frac{1}{\exp(H_1)}$, by Lemma 3.2 we obtain that

$$|S_{H_1}| + \lceil \frac{|S_{(2p^\beta)}| - (\eta(G/H_1) - 1)}{\exp(G/H_1)} \rceil \leq \mathfrak{t}(H_1) - 1.$$

Therefore,

$$2|S_{H_1}| + |S_{(2p^\beta)}| \leq 2p^\beta + 1.$$

Combining equality (3.2), we obtain that

(3.3)

$$|S_{H_1}| \leq 2p^\beta + 1 - (|S_{H_1}| + |S_{(2p^\beta)}|) = |S_{(2)}| + |S_{(2p)}| + \dots + |S_{(2p^{\beta-1})}| - 1 \leq |S_{H_2}| - 1.$$

Since for every $T \mid S_{(2p^\beta)}$ with $|T| \leq \exp(G/H_2)$ we have $k(T) \leq \frac{\exp(G/H_2)}{\exp(G)} = \frac{1}{\exp(H_2)}$, by Lemma 3.2 we obtain that

$$(3.4) \quad |S_{H_2}| + \lceil \frac{|S_{(2p^\beta)}| - (\eta(G/H_2) - 1)}{\exp(G/H_2)} \rceil \leq \mathfrak{t}(H_2) - 1.$$

Therefore,

$$|S_{(2p^\beta)}| + p|S_{H_2}| \leq 2p^\beta + 2p - 1.$$

Combining equality (3.2) and inequality (3.3),

$$\begin{aligned} p|S_{H_2}| &\leq 2p^\beta + 2p - 1 - |S_{(2p^\beta)}| \\ &= 2p^\beta + 2p - 1 - (|S| - |S_{H_2}| - |S_{(p^\beta)}|) \\ &= 2p - 3 + |S_{H_2}| + |S_{(p^\beta)}| \\ &\leq 2p - 3 + |S_{H_2}| + |S_{H_1}| \\ &\leq 2p - 4 + 2|S_{H_2}|. \end{aligned}$$

Therefore, $|S_{H_2}| \leq 2$ and $|S_{(p^\beta)}| \leq |S_{H_1}| \leq |S_{H_2}| - 1 \leq 1$. Hence, we have the following possibilities:

$$|S_{H_2}| = 1 \text{ and } |S_{(p^\beta)}| = 0, |S_{H_2}| = 2 \text{ and } |S_{(p^\beta)}| = 1, |S_{H_2}| = 2 \text{ and } |S_{(p^\beta)}| = 0.$$

We proceed case by case.

Case 1. $|S_{H_2}| = 1$ and $|S_{(p^\beta)}| = 0$, then $|S_{(2p^\beta)}| = 2p^\beta + 1 = D(G)$ and $S_{(2p^\beta)}$ is a minimal zero-sum subsequence.

It follows that we can decompose $S_{(2p^\beta)}$ into

$$S_{(2p^\beta)} = V_1 \cdot \dots \cdot V_n$$

such that $\sigma(\varphi_2(V_i)) = 0$ and $|V_i| \leq p$ for every $1 \leq i \leq n$, then $\sigma(V_i) \in \ker(\varphi_2) = H_2$ and $k(V_i) = \frac{|V_i|}{\exp(G)} \leq \frac{p}{\exp(G)} = \frac{1}{\exp(H_2)} \leq k(\sigma(V_i))$ for $1 \leq i \leq n$. So we have $n \geq \lceil \frac{|S_{(2p^\beta)}|}{p} \rceil = \lceil \frac{2p^\beta + 1}{p} \rceil = 2p^{\beta-1} + 1$, then

$$n + |S_{H_2}| \geq 2p^{\beta-1} + 1 + 1 = 2p^{\beta-1} + 2 = \eta(H_2) = \mathfrak{t}(H_2),$$

a contradiction with Lemma 3.1.

Case 2. $|S_{H_2}| = 2$ and $|S_{(p^\beta)}| = 1$. Recall that $|S_{H_1}| = 1$, then $|S_{(2p^\beta)}| = 2p^\beta - 1$. Let

$$S_{(2p^\beta)} = U_1 \cdot \dots \cdot U_m U' = V_1 \cdot \dots \cdot V_n V',$$

where $\sigma(\varphi_1(U_i)) = 0 \in G/H_1$ and $|U_i| = 2$ for $1 \leq i \leq m$ and $\varphi_1(U')$ has no short zero-sum subsequence over G/H_1 , $\sigma(\varphi_2(V_j)) = 0 \in G/H_2$ and $|V_j| \leq p$ for $1 \leq j \leq n$ and $\varphi_2(V')$ has no short zero-sum subsequence over G/H_2 .

By Lemmas 3.1 and 3.2 we have

$$\lceil \frac{|S_{(2p^\beta)}| - (\eta(G/H_2) - 1)}{\exp(G/H_2)} \rceil + |S_{H_2}| \leq n + |S_{H_2}| \leq \mathfrak{t}(H_2) - 1,$$

therefore $n = 2p^{\beta-1} - 1$, and every subsequence of $\varphi_2(S_{(2p^\beta)})$ of length $p - 1$ is zero-sum free. Otherwise, suppose that there exists a subsequence $S'_{(2p^\beta)} \mid S_{(2p^\beta)}$ of length $|S'_{(2p^\beta)}| \leq p - 1$ such that $\varphi_2(S'_{(2p^\beta)})$ is zero-sum, then $|\varphi_2(S_{(2p^\beta)} S'_{(2p^\beta)}^{-1})| \geq 2p^\beta - p$, we can find at least $2p^{\beta-1} - 1$ disjoint zero-sum subsequences of length at most p of $\varphi_2(S_{(2p^\beta)} S'_{(2p^\beta)}^{-1})$ by Lemma 3.2, so we can find at least $2p^{\beta-1}$ disjoint zero-sum subsequences of length at most p of $\varphi_2(S_{(2p^\beta)})$, a contradiction with $n = 2p^{\beta-1} - 1$. Therefore,

$$\varphi_2(S_{(2p^\beta)}) = h^{2p^{\beta-1}}$$

for some $h \in \varphi_2(G) = G/H_2$ by Lemma 2.7.

By Lemmas 3.1 and 3.2 we have

$$\lceil \frac{|S_{(2p^\beta)}| - (\eta(G/H_1) - 1)}{\exp(G/H_1)} \rceil + |S_{H_1}| \leq m + |S_{H_1}| \leq \mathfrak{t}(H_1) - 1,$$

therefore $m = p^\beta - 2$.

Let $S_{(2p^\beta)} = U_1 \cdot \dots \cdot U_{p^\beta-2} \cdot U_0$, where $U_0 = S_{(2p^\beta)}(U_1 \cdot \dots \cdot U_{p^\beta-2})^{-1}$. Since $\varphi_1(U_0)$ has no short zero-sum subsequence over G/H_1 and $|U_0| = 3 = D(G/H_1)$, $\sigma(\varphi_1(U_0)) = 0 \in G/H_1$ and $\text{supp}(\varphi_1(U_0)) = G/H_1 \setminus \{0\} = \{h_1, h_2, h_3\}$. Since $|S_{(p^\beta)} \cdot \sigma(U_0) \cdot \sigma(U_1) \cdot \dots \cdot \sigma(U_{p^\beta-2})| = p^\beta = \mathfrak{t}(H_1)$, $S_{(p^\beta)} \cdot \sigma(U_0) \cdot \sigma(U_1) \cdot \dots \cdot \sigma(U_{p^\beta-2})$ has a tiny zero-sum subsequence W_0 . If $|W_0| \leq$

$p^\beta - 1$, suppose that $W_0 = S'_{(p^\beta)} \prod_{i \in I} \sigma(U_i)$, where $S'_{(p^\beta)} \mid S_{(p^\beta)}$, $I \in [0, p^\beta - 2]$ and $|S'_{(p^\beta)}| + |I| \leq p^\beta - 1$, then $W'_0 = S'_{(p^\beta)} \prod_{i \in I} U_i$ is a tiny zero-sum subsequence of S , a contradiction. Therefore $S_{(p^\beta)} \cdot \sigma(U_0) \cdot \sigma(U_1) \cdot \dots \cdot \sigma(U_{p^\beta-2})$ is a minimal zero-sum sequence over C_{p^β} . So we have $S_{(p^\beta)} = \sigma(U_0) = \sigma(U_1) = \dots = \sigma(U_{p^\beta-2})$. Then

$$\varphi_1(S_{(2p^\beta)}) = h_1^{1+2l_1} h_2^{1+2l_2} h_3^{1+2l_3},$$

where $l_i \in [0, p^\beta - 2]$ and $l_1 + l_2 + l_3 = p^\beta - 2$.

Claim. Let $h_i \in \text{supp}(\varphi_1(S_{(2p^\beta)}))$ with $\mathbf{v}_{h_i}(\varphi_1(S_{(2p^\beta)})) \geq 3$ and let $g_1, g_2 \in \text{supp}(S_{(2p^\beta)})$. If $\varphi_1(g_1) = \varphi_1(g_2) = h_i$, then $g_1 = g_2$.

Proof of the Claim. Assume to the contrary that $g_1 \neq g_2$. Without loss of generality we may assume that $g_1 \mid U_1$ and $g_2 \mid U_0$. Let $U'_1 = U_1 g_1^{-1} g_2$. Thus, both $S_{(p^\beta)} \cdot \sigma(U_1) \cdot \dots \cdot \sigma(U_{p^\beta-2})$ and $S_{(p^\beta)} \cdot \sigma(U'_1) \cdot \dots \cdot \sigma(U_{p^\beta-2})$ are zero-sum free of length $p^\beta - 1$. It follows from Lemma 2.6 that $\sigma(U_1) = \sigma(U'_1)$ and hence $g_1 = g_2$, a contradiction. \square

Since $\varphi_1(S_{(2p^\beta)}) = h_1^{1+2l_1} h_2^{1+2l_2} h_3^{1+2l_3}$, where $l_i \in [0, p^\beta - 2]$ and $l_1 + l_2 + l_3 = p^\beta - 2$. For $i \in [1, 3]$, if $l_i \geq 1$, then $\mathbf{v}_{h_i}(\varphi_1(S_{(2p^\beta)})) \geq 3$, by the Claim we have that there exists a subsequence $g_i^{1+2l_i} \mid S_{(2p^\beta)}$ such that $\varphi_1(g_i^{1+2l_i}) = h_i^{1+2l_i}$, if $l_i = 0$, then there exists a subsequence $g_i \mid S_{(2p^\beta)}$ such that $\varphi_1(g_i) = h_i$.

Therefore, we have

$$S_{(2p^\beta)} = g_1^{1+2l_1} g_2^{1+2l_2} g_3^{1+2l_3},$$

where $\varphi_1(g_i) = h_i$ for $i \in [1, 3]$. If there exist $i, j \in [1, 3]$ and $i \neq j$ such that $l_i \geq 1$ and $l_j \geq 1$. Without loss of generality, we assume that $\{i, j\} = \{1, 2\}$. Since $S_{(p^\beta)} = \sigma(U_0) = \sigma(U_1) = \dots = \sigma(U_{p^\beta-2})$, we have $g_1 + g_2 + g_3 = 2g_1$ and $g_1 + g_2 + g_3 = 2g_2$, it deduces that $2g_3 = 0$, a contradiction. Therefore, there at least exist two zeros among l_1, l_2, l_3 and without loss of generality, we assume that $l_2 = l_3 = 0$. Then

$$S_{(2p^\beta)} = g_1^{2p^\beta-3} g_2(-g_2 + g_1),$$

and $S_{(p^\beta)} = 2g_1$.

Let $h \mid S_{H_2}$, then $\text{ord}(h) = 2p^l$, $l \in [0, \beta - 1]$. We write $h = a_1 e_1 + p^{\beta-l} y_1 g_1$, where $a_1 \in [0, 1]$ and $(y_1, p) = 1$. Let $U_1 = \dots = U_{p^\beta-2} = g_1^2$ and $U_{p^\beta-1} = S_{p^\beta} = 2g_1$. Without loss of generality, we assume that $\varphi_1(h) = \varphi_1(g_1)$.

Let $T_1 = h g_1$, $T_2 = h g_2(-g_2 + g_1)$, $T_3 = g_1 g_2(-g_2 + g_1)$. Then $\sigma(T_i) \in \ker(\varphi_1)$ for $i \in [1, 3]$. So, for every $i \in [1, 3]$, the sequence $\sigma(U_1) \cdot \dots \cdot \sigma(U_{p^\beta-1}) \cdot \sigma(T_i)$ has a zero-sum subsequence X_i over $\ker(\varphi_1)$, i.e., there exists a subset $J_i \subset [1, p^\beta - 1]$ such that $X_i = \sigma(T_i) \prod_{j \in J_i} \sigma(U_j)$ for each

$i \in [1, 3]$. Let $Y_i = T_i \prod_{j \in J_i} U_j$ for each $i \in [1, 3]$. Then Y_1, Y_2 and Y_3 are zero-sum subsequences of S . Let $t_i = |J_i|$ for $i \in [1, 3]$. Then $X_1 = (2g_1)^{t_1}(h + g_1)$, $X_2 = (2g_1)^{t_2}(h + g_2 + (-g_2 + g_1))$, $X_3 = (2g_1)^{t_3}(g_1 + g_2 + (-g_2 + g_1))$.

Since $k(Y_i) > 1$ for every $i \in [1, 3]$, we have

$$k(Y_1) = \frac{1}{\text{ord}(h)} + \frac{1}{\text{ord}(g_1)} + \frac{2t_1}{\text{ord}(g_1)} = \frac{p^{\beta-l} + 2t_1 + 1}{2p^\beta} > 1,$$

$$k(Y_2) = \frac{1}{\text{ord}(h)} + \frac{1}{\text{ord}(g_2)} + \frac{1}{\text{ord}(-g_2 + g_1)} + \frac{2t_2}{\text{ord}(g_1)} = \frac{p^{\beta-l} + 2t_2 + 2}{2p^\beta} > 1,$$

$$k(Y_3) = \frac{1}{\text{ord}(g_1)} + \frac{1}{\text{ord}(g_2)} + \frac{1}{\text{ord}(-g_2 + g_1)} + \frac{2t_3}{\text{ord}(g_1)} = \frac{2t_3 + 3}{2p^\beta} > 1.$$

Combining $t_i \leq p^\beta - 1$, by a straightforward computation we obtain that

$$p^\beta - \frac{p^{\beta-l} - 1}{2} \leq t_1 \leq p^\beta - 1, p^\beta - \frac{p^{\beta-l} + 1}{2} \leq t_2 \leq p^\beta - 1, t_3 = p^\beta - 1.$$

From X_i is zero-sum over $\ker(\varphi_1)$ we infer that

$$2t_1g_1 + h + g_1 = 2t_2g_1 + h + g_2 + (-g_2 + g_1) = 2(p^\beta - 1)g_1 + g_1 + g_2 + (-g_2 + g_1) = 0.$$

Therefore,

$$2t_1g_1 + h + g_1 + 2t_2g_1 + h + g_2 + (-g_2 + g_1) - 2(p^\beta - 1)g_1 - g_1 - g_2 - (-g_2 + g_1) = 0.$$

This deduces that $(2t_1 + 2t_2 + 2)g_1 + 2h = 0$. Therefore $(2t_1 + 2t_2 + 2)g_1 + 2p^{\beta-l}yg_1 = 0$, then $(t_1 + t_2 + 1 + p^{\beta-l}y) \equiv 0 \pmod{p^\beta}$, but $2p^\beta - p^{\beta-l} + 1 + p^{\beta-l}y \leq t_1 + t_2 + 1 + p^{\beta-l}y \leq 2p^\beta - 1 + p^{\beta-l}y$, a contradiction.

Case 3. $|S_{H_2}| = 2$ and $|S_{(p^\beta)}| = 0$, then $|S_{(2p^\beta)}| = 2p^\beta$. Therefore,

$$|S_{H_2}| + \left\lceil \frac{|S_{(2p^\beta)}| - (\eta(G/H_2) - 1)}{\exp(G/H_2)} \right\rceil = 2 + 2p^{\beta-1} = \eta(H_2),$$

a contradiction with inequality (3.4).

(c) The result follows from Lemma 2.1 and Lemma 3.6. \square

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