FOUR IDENTITIES RELATED TO THIRD ORDER MOCK THETA FUNCTIONS

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Abstract. Ramanujan presented four identities for third order mock theta functions in his Lost Notebook. In 2005, with the aid of complex analysis, Yesilyurt first proved these four identities. Recently, Andrews et al. proved these identities by using q-series. In this paper, in view of some identities of a universal mock theta function

$$g(x;q)=x^{-1}\left(-1+\sum_{n=0}^{\infty}\frac{q^{n^2}}{(x;q)_{n+1}(qx^{-1};q)_n}\right),$$

we provide different proofs of these four identities.

1. Introduction

Let q denote a complex number with $|q|<1$. Throughout this paper, we adopt the standard q-series notation [6]. For any positive integer n,

$$(a;q)_0:=1,\quad (a;q)_n:=\prod_{k=0}^{n-1}(1-aq^k),\quad (a;q)_\infty:=\prod_{k=0}^{\infty}(1-aq^k),$$

$$j(x;q):=(x;q)_\infty(q/x;q)_\infty(q;q)_\infty. \tag{1.1}$$

Letting a and m be integers with m positive, we define

$$J_{a,m}:=j(q^a;q^m),\qquad \overline{J}_{a,m}:=j(-q^a;q^m),\qquad J_m:=J_{m,3m}=\prod_{k=1}^{\infty}(1-q^{mk}).$$

Recall the Jacobi triple product identity [6, Eq. (1.6.1)]

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{\binom{k}{2}}x^k=(x;q)_\infty(q/x;q)_\infty(q;q)_\infty.$$

Thus, Ramanujan's theta function $\varphi(q)$ and $\psi(q)$ are defined, respectively, as follows.

$$\varphi(q):=\sum_{n=-\infty}^{\infty}q^{n^2}=(-q;q^2)_\infty^2(q^2;q^2)_\infty=\frac{J_2^5}{J_1^2J_4^2}, \tag{1.2}$$

$$\psi(q):=\sum_{n=0}^{\infty}q^{\binom{n+1}{2}}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}=\frac{J_2^2}{J_1}. \tag{1.3}$$

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For any real number a, define

$$f_a(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+aq+q^2)(1+aq^2+q^4)\cdots(1+aq^n+q^{2n})}. \tag{1.4}$$

Set

$$\tilde{\phi}(q) := f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}, \tag{1.5}$$

which is one of the third order mock theta functions included in Ramanujan's last letter to Hardy. Then Ramanujan [11] stated the following four identities related to third order mock theta functions in his Lost Notebook. For more on mock theta functions, see Andrews and Berndt's book [2].

Entry 1.1. *[11, p. 2] Suppose that a and b are real numbers such that $a^2+b^2=4$. Recall that $f_a(q)$ is defined by (1.4). Then*

$$\begin{aligned}\frac{b-a+2}{4} f_a(-q) + \frac{b+a+2}{4} f_{-a}(-q) - \frac{b}{2} f_b(q) \\ = \frac{(q^4;q^4)_\infty}{(-q;q^2)_\infty} \prod_{n=1}^{\infty} \frac{1-bq^n+q^{2n}}{1+(a^2b^2-2)q^{4n}+q^{8n}}.\end{aligned} \tag{1.6}$$

Entry 1.2. *[11, p. 2] Let a and b be real numbers with $a^2+ab+b^2=3$. Then, with $f_a(q)$ defined by (1.4),*

$$\begin{aligned}(a+1)f_{-a}(q) + (b+1)f_{-b}(q) - (a+b-1)f_{a+b}(q) \\ = 3\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty} \prod_{n=1}^{\infty} \frac{1}{1+ab(a+b)q^{3n}+q^{6n}}.\end{aligned} \tag{1.7}$$

Entry 1.3. *[11, p. 17] Let $\psi(q)$ and $f_a(q)$ be defined by* (1.3) *and* (1.4)*, respectively. Then*

$$\begin{aligned}\frac{1+\sqrt{3}}{2} f_{-1}(-q) + \frac{3+\sqrt{3}}{6} f_1(-q) - f_{\sqrt{3}}(q) \\ = \frac{2}{\sqrt{3}} \psi(-q) \frac{(q^4;q^4)_\infty}{(q^6;q^6)_\infty} \prod_{n=1}^{\infty} \frac{1}{1+\sqrt{3}q^n+q^{2n}}.\end{aligned}$$

Entry 1.4. *[11, p. 17] Let $\psi(q)$, $f_a(q)$, and $\tilde{\phi}(q)$ be defined by* (1.3), (1.4), *and* (1.5)*, respectively. Then*

$$\begin{aligned}\frac{1}{2}(1+e^{\pi i/4})\tilde{\phi}(iq) + \frac{1}{2}(1+e^{-\pi i/4})\tilde{\phi}(-iq) - f_{\sqrt{2}}(q) \\ = \frac{1}{\sqrt{2}} \psi(-q)(-q^2;q^4)_\infty \prod_{n=1}^{\infty} \frac{1}{1+\sqrt{2}q^n+q^{2n}}.\end{aligned}$$

In 1944, Dyson [5] defined the rank of a partition to be the largest part minus the number of parts. Let $N(m,n)$ denote the number of partitions of the positive integer

n with rank m. Then he showed that the generating function for $N(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}N(m,n)q^nx^m=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(qx;q)_n(qx^{-1};q)_n}=:G(x,q). \tag{1.8}$$

From (1.4) and (1.8), it can be seen that

$$f_a(q)=G\left(\frac{-a\pm\sqrt{a^2-4}}{2},q\right).$$

The proofs of Entries 1.1-1.4 which were first provided by Yesilyurt [12] rely on the following lemma due to Atkin and Swinnerton-Dyer.

Lemma 1.5. *[4] Let q, $|q|<1$, be fixed. Suppose that $\vartheta(z)$ is an analytic function of z, except for possibly a finite number of poles, in every region, $0<z_1\le|z|\le z_2$. If*

$$\vartheta(zq)=Az^k\vartheta(z)$$

for some integer k (positive, zero, or negative) and some constant A, then either $\vartheta(z)$ has k more poles than zeros in the region $|q|<|z|\le 1$, or $\vartheta(z)$ vanishes identically.

Recently, Andrews et al. [3] offered different proofs of these four identities by using q-series. The proofs of Entries 1.1-1.3 depend on a partial fraction expansion formula, and in the proof of Entry 1.4, two 2-dissentions for two special cases of the rank generating function $G(x,q)$ are the main tools.

The object of this paper is to present different proofs of these four identities by using the universal mock theta function,

$$g(x;q):=x^{-1}\left(-1+\sum_{n=0}^{\infty}\frac{q^{n^2}}{(x;q)_{n+1}(qx^{-1};q)_n}\right). \tag{1.9}$$

A transformation formula of $g(x;q)$ given by Ramanujan [11] and an identity of $g(x;q)$ due to Hickerson [7,8] play important roles in our proofs, see Propositions 2.2 and 2.3 in the next section. This paper is organized as follows. In Section 2, we show some preliminary results which are used in our proofs. In Section 3, we prove Entries 1.1-1.4.

2. Preliminaries

In this section, we present some preliminary results. For later use, we need the following identities:

$$\overline{J}_{1,2}=\frac{J_2^5}{J_1^2J_4^2},\quad J_{1,2}=\frac{J_1^2}{J_2},\quad J_{1,4}=\frac{J_1J_4}{J_2}.$$

In addition, the following general identities are frequently used in this paper.

$$j(qx;q)=-x^{-1}j(x;q), \tag{2.1}$$

$$j(x;q)=j(qx^{-1};q), \tag{2.2}$$

$$j(x^2;q^2)=j(x;q)j(-x;q)/J_{1,2}, \tag{2.3}$$

$$j(x;q)=J_1 j(x;q^2)j(qx;q^2)/J_2^2, \tag{2.4}$$

$$j(x;q)=\sum_{k=0}^{m-1}(-1)^k q^{\binom{k}{2}}x^k j\left((-1)^{m+1}q^{\binom{m}{2}+mk}x^m;q^{m^2}\right). \tag{2.5}$$

Setting $m=2$ in (2.5) yields that

$$j(x;q)=j(-qx^2;q^4)-xj(-q^3x^2;q^4). \tag{2.6}$$

Hickerson and Mortenson [8] defined Appell–Lerch sums as follows.

Definition 2.1. *Let $x,z\in\mathbb{C}^*:=\mathbb{C}\backslash\{0\}$ with neither z nor xz an integral power of q. Then*

$$m(x,q,z):=\frac{1}{j(z;q)}\sum_{r=-\infty}^{\infty}\frac{(-1)^r q^{\binom{r}{2}}z^r}{1-q^{r-1}xz}.$$

Following [8], the term "generic" means that the parameters do not cause poles in the Appell–Lerch sums or in the quotients of theta functions. The next proposition can be found in the Lost Notebook.

Proposition 2.2. *([11, p. 32], [1, Eq. (12.5.13)]). For generic $x\in\mathbb{C}^*$,*

$$g(x;q)=-x^{-1}+qx^{-3}g(-qx^{-2};q^4)-qg(-qx^2;q^4)+\frac{J_2J_{2,4}^2}{xj(x;q)j(-qx^2;q^2)}.$$

The following proposition was first proved by Hickerson [7], and then Hickerson and Mortenson [8] rewrote the identity in terms of Appell–Lerch sums.

Proposition 2.3. *([7, Theorem 2.2], [8]). For generic $x,z\in\mathbb{C}^*$,*

$$g(x;q)=-x^{-2}m(qx^{-3},q^3,x^3z)-x^{-1}m(q^2x^{-3},q^3,x^3z)+\frac{J_1^2 j(xz;q)j(z;q^3)}{j(x;q)j(z;q)j(x^3z;q^3)}. \tag{2.7}$$

Notice that a special case of (2.7) with $z=1$ reads

$$g(x;q)=-x^{-2}m(qx^{-3},q^3,x^3)-x^{-1}m(q^2x^{-3},q^3,x^3)+\frac{J_3^3}{J_1 j(x^3;q^3)}. \tag{2.8}$$

In the proof of Entry 1.4, we apply the next two propositions to simplify theta function identities.

Proposition 2.4. *[9, Proposition 3.4] Let $x\neq 0$. Then*

$$j(q^2x;q^4)j(q^5x;q^8)+\frac{q}{x}\cdot j(x;q^4)j(qx;q^8)-\frac{J_1}{J_4}\cdot j(-q^3x;q^4)j(q^3x;q^8)=0.$$

Proposition 2.5. *[10, Proposition 2.5] Let $x\neq 0$. Then*

$$j(-x;q^4)j(-q^5x;q^8)-j(-q^2x;q^4)j(-qx;q^8)-x\frac{J_1}{J_4}\cdot j(q^3x;q^4)j(-q^7x;q^8)=0.$$

Furthermore, in view of (1.8) and (1.9), we arrive at

$$G(x,q)=(1-x)\left(xg(x;q)+1\right). \tag{2.9}$$

3. Proofs of Entries 1.1-1.4

In this section, employing Propositions 2.2-2.5, we prove Entries 1.1-1.4.

Proof of Entry 1.1. Set $a=2\cos\theta$, $b=2\sin\theta$, and $t=e^{i\theta}$. Then $a=t+t^{-1}$ and $b=-i(t-t^{-1})$. So we have

$$\begin{aligned}\frac{b-a+2}{4}&=\frac{(i-1)(1-it)(1-t)}{4t},\\ \frac{b+a+2}{4}&=\frac{(i+1)(1-it)(1+t)}{4t},\\ \frac{b}{2}=\frac{i(1-t^2)}{2t},&\quad a^2b^2-2=-(t^4+t^{-4}).\end{aligned}$$

Moreover, according to (1.4) and (1.8), we obtain that

$$f_a(q)=G(-t,q),\quad f_{-a}(q)=G(t,q),\quad f_b(q)=G(it,q).$$

Thus, using the above relations, we rewrite (1.6) as

$$\begin{aligned}\frac{(i-1)(1-it)(1-t)}{4t}G(-t,-q)+\frac{(i+1)(1-it)(1+t)}{4t}G(t,-q)\\ -\frac{i(1-t^2)}{2t}G(it,q)=\frac{(q^4;q^4)_\infty(-iqt;q)_\infty(iqt^{-1};q)_\infty}{(-q;q^2)_\infty(q^4t^4;q^4)_\infty(q^4t^{-4};q^4)_\infty}.\end{aligned} \tag{3.1}$$

Multiplying both sides of (3.1) by $4t(1+it)/((i-1)(1-t^4))$, we arrive at

$$\frac{1}{1+t}G(-t,-q)-\frac{i}{1-t}G(t,-q)+\frac{i-1}{1-it}G(it,q)=-\frac{2(1+i)tJ_4^3j(-it;q)}{J_2^2j(t^4;q^4)}. \tag{3.2}$$

Hence, to prove Entry 1.1, it suffices to prove (3.2).

First, we consider the left-hand side of (3.2). Using (2.9) in the first equality and Proposition 2.2 in the third equality below, we have

$$\begin{aligned}&\frac{1}{1+t}G(-t,-q)-\frac{i}{1-t}G(t,-q)+\frac{i-1}{1-it}G(it,q)\\ &=-tg(-t;-q)+1-i(tg(t;-q)+1)+(i-1)(itg(it;q)+1)\\ &=-tg(-t;-q)-itg(t;-q)-(1+i)tg(it;q)\\ &=-1-qt^{-2}g(qt^{-2};q^4)-qtg(qt^2;q^4)+\frac{J_2J_{2,4}^2}{j(-t;-q)j(qt^2;q^2)}\\ &\quad+i+iqt^{-2}g(qt^{-2};q^4)-iqtg(qt^2;q^4)-\frac{iJ_2J_{2,4}^2}{j(t;-q)j(qt^2;q^2)}\\ &\quad-i+1+(1-i)qt^{-2}g(qt^{-2};q^4)+(1+i)qtg(qt^2;q^4)-\frac{(1-i)J_2J_{2,4}^2}{j(it;q)j(qt^2;q^2)}\\ &=(-1+i+1-i)qt^{-2}g(qt^{-2};q^4)-(1+i-1-i)qtg(qt^2;q^4)\end{aligned}$$

$$+\frac{J_2J_{2,4}^2}{j(-t;-q)j(qt^2;q^2)}-\frac{iJ_2J_{2,4}^2}{j(t;-q)j(qt^2;q^2)}-\frac{(1-i)J_2J_{2,4}^2}{j(it;q)j(qt^2;q^2)}$$
$$=\frac{J_2J_{2,4}^2}{j(qt^2;q^2)}\left(\frac{1}{j(-t;-q)}-\frac{i}{j(t;-q)}-\frac{1-i}{j(it;q)}\right). \tag{3.3}$$

Define

$$A:=\frac{1}{j(-t;-q)}-\frac{i}{j(t;-q)}-\frac{1-i}{j(it;q)}$$
$$=\frac{j(t;-q)j(it;q)-ij(-t;-q)j(it;q)-(1-i)j(-t;-q)j(t;-q)}{j(-t;-q)j(t;-q)j(it;q)}.$$

To evaluate A, by (2.6), we deduce that

$$j(-t;-q)=j(qt^2;q^4)+tj(q^3t^2;q^4),$$
$$j(t;-q)=j(qt^2;q^4)-tj(q^3t^2;q^4),$$
$$j(it;q)=j(qt^2;q^4)-itj(q^3t^2;q^4).$$

Then by means of the above three identities, we find that

$$\begin{aligned}&j(t;-q)j(it;q)-ij(-t;-q)j(it;q)-(1-i)j(-t;-q)j(t;-q)\\&=\left(j(qt^2;q^4)-tj(q^3t^2;q^4)\right)\left(j(qt^2;q^4)-itj(q^3t^2;q^4)\right)\\&\quad-i\left(j(qt^2;q^4)+tj(q^3t^2;q^4)\right)\left(j(qt^2;q^4)-itj(q^3t^2;q^4)\right)\\&\quad-(1-i)\left(j(qt^2;q^4)+tj(q^3t^2;q^4)\right)\left(j(qt^2;q^4)-tj(q^3t^2;q^4)\right)\\&=-2(1+i)tj(qt^2;q^4)j(q^3t^2;q^4).\end{aligned} \tag{3.4}$$

Hence, utilizing (2.3), (2.4), and (3.4), we obtain that

$$\begin{aligned}A&=-\frac{2(1+i)tj(qt^2;q^4)j(q^3t^2;q^4)}{j(-t;-q)j(t;-q)j(it;q)}\\&=-\frac{2(1+i)tJ_4^2j(qt^2;q^2)}{J_2j(-t;-q)j(t;-q)j(it;q)}\cdot\frac{j(-it;q)}{j(-it;q)}\\&=-\frac{2(1+i)tJ_4^2j(qt^2;q^2)j(-it;q)}{J_2J_{1,2}\overline{J}_{1,2}J_{2,4}j(t^4;q^4)}.\end{aligned}$$

Thus, the right-hand side of (3.3) becomes

$$\begin{aligned}&\frac{J_2J_{2,4}^2}{j(qt^2;q^2)}\left(-\frac{2(1+i)tJ_4^2j(qt^2;q^2)j(-it;q)}{J_2J_{1,2}\overline{J}_{1,2}J_{2,4}j(t^4;q^4)}\right)\\&=-\frac{2(1+i)tJ_4^3j(-it;q)}{J_2^2j(t^4;q^4)},\end{aligned}$$

which is the right-hand side of (3.2). Therefore, we complete the proof of Entry 1.1. □

Proof of Entry 1.2. We parameterize $a^2+ab+b^2=3$ by $a=2\cos(\theta+2\pi/3)$, $b=2\cos\theta$, and $t=e^{i\theta}$. Let $\omega=e^{2\pi i/3}$. Then we have $a=\omega t+(\omega t)^{-1}$ and $b=t+t^{-1}$. So,

$$a+b=-\omega^2t-(w^2t)^{-1},\quad a+1=\frac{1-t^3}{\omega t(1-\omega t)},\quad b+1=\frac{1-t^3}{t(1-t)},$$

$$a+b-1=-\frac{1-t^3}{\omega^2 t(1-\omega^2 t)}, \quad ab(a+b)=-t^3-t^{-3},$$

and

$$f_{-a}(q)=G(\omega t,q), \quad f_{-b}(q)=G(t,q), \quad f_{a+b}(q)=G(\omega^2 t,q).$$

Therefore, (1.7) is equivalent to the following identity:

$$\begin{aligned}&\frac{1-t^3}{\omega t(1-\omega t)}G(\omega t,q)+\frac{1-t^3}{t(1-t)}G(t,q)+\frac{1-t^3}{\omega^2 t(1-\omega^2 t)}G(\omega^2 t,q)\\&=\frac{3(q^3;q^3)_\infty^2}{(q;q)_\infty}\prod_{n=1}^{\infty}\frac{1}{1-(t^3+t^{-3})q^{3n}+q^{6n}}.\end{aligned} \tag{3.5}$$

Observing that the right-hand side of (3.5) reduces to

$$\frac{3J_3^3(1-t^3)}{J_1 j(t^3;q^3)},$$

we are required to prove that

$$\frac{G(\omega t,q)}{\omega t(1-\omega t)}+\frac{G(t,q)}{t(1-t)}+\frac{G(\omega^2 t,q)}{\omega^2 t(1-\omega^2 t)}=\frac{3J_3^3}{J_1 j(t^3;q^3)}.$$

From (2.9), it can be seen that

$$\begin{aligned}&\frac{G(\omega t,q)}{\omega t(1-\omega t)}+\frac{G(t,q)}{t(1-t)}+\frac{G(\omega^2 t,q)}{\omega^2 t(1-\omega^2 t)}\\&=g(\omega t;q)+\frac{1}{\omega t}+g(t;q)+\frac{1}{t}+g(\omega^2 t;q)+\frac{1}{\omega^2 t}\\&=g(t;q)+g(\omega t;q)+g(\omega^2 t;q).\end{aligned} \tag{3.6}$$

Then applying (2.8) to the right-hand side of (3.6) gives

$$\begin{aligned}&\frac{G(\omega t,q)}{\omega t(1-\omega t)}+\frac{G(t,q)}{t(1-t)}+\frac{G(\omega^2 t,q)}{\omega^2 t(1-\omega^2 t)}\\&=-t^{-2}m(qt^{-3},q^3,t^3)-t^{-1}m(q^2t^{-3},q^3,t^3)+\frac{J_3^3}{J_1 j(t^3;q^3)}\\&\quad-\omega t^{-2}m(qt^{-3},q^3,t^3)-\omega^2 t^{-1}m(q^2t^{-3},q^3,t^3)+\frac{J_3^3}{J_1 j(t^3;q^3)}\\&\quad-\omega^2 t^{-2}m(qt^{-3},q^3,t^3)-\omega t^{-1}m(q^2t^{-3},q^3,t^3)+\frac{J_3^3}{J_1 j(t^3;q^3)}\\&=-(1+\omega+\omega^2)t^{-2}m(qt^{-3},q^3,t^3)-(1+\omega+\omega^2)t^{-1}m(q^2t^{-3},q^3,t^3)\\&\quad+\frac{3J_3^3}{J_1 j(t^3;q^3)}\\&=\frac{3J_3^3}{J_1 j(t^3;q^3)}.\end{aligned}$$

Therefore, we complete the proof of Entry 1.2. □

It is easy to find that Entry 1.3 is a special case of Entry 1.1. One can refer to the proof in [12]. Here we omit it.

Proof of Entry 1.4. Let $\alpha = e^{\pi i/4}$. According to (1.4), (1.5), and (1.8), we have

$$f_{\sqrt{2}}(q) = G(-\alpha, q), \quad \tilde{\phi}(q) = G(i, q).$$

In view of the above two identities, we restate Entry 1.4 as

$$\frac{1+\alpha}{2}G(i, iq) + \frac{1+\alpha^{-1}}{2}G(i, -iq) - G(-\alpha, q) = \frac{1}{\sqrt{2}}\frac{\psi(-q)(-q^2; q^4)_\infty}{(-q\alpha; q)_\infty(-q\alpha^{-1}; q)_\infty}.$$

Dividing both sides of the above identity by $(1+\alpha)/2$, we deduce that

$$G(i, iq) + \alpha^{-1}G(i, -iq) - \frac{2}{1+\alpha}G(-\alpha, q) = \sqrt{2}\frac{\psi(-q)(-q^2; q^4)_\infty(q; q)_\infty}{j(-\alpha; q)}. \tag{3.7}$$

Replacing q by iq in (3.7) yields that

$$G(i, -q) + \alpha^{-1}G(i, q) - \frac{2}{1+\alpha}G(-\alpha, iq) = \sqrt{2}\frac{\psi(-iq)(q^2; q^4)_\infty(iq; iq)_\infty}{j(-\alpha; iq)}. \tag{3.8}$$

Therefore, to obtain Entry 1.4, it suffices to prove (3.8).

In the following, we first consider the left-hand side of (3.8). Using (2.9) and Proposition 2.2, we have

$$\begin{aligned} G(i, q) &= (i+1)g(i; q) + 1 - i \\ &= (i+1)\left(i + (i-1)qg(q; q^4) + \frac{J_2 J_{2,4}^2}{ij(i; q)j(q; q^2)}\right) + 1 - i \\ &= -2qg(q; q^4) + (1-i)\frac{J_2 J_{2,4}^2}{j(i; q)j(q; q^2)}. \end{aligned} \tag{3.9}$$

Similarly, by noticing that $\alpha^2 = i$ and $\alpha^3 = -\alpha^{-1}$, we arrive at

$$G(i, -q) = 2qg(-q; q^4) + (1-i)\frac{J_2 J_{2,4}^2}{j(i; -q)j(-q; q^2)}, \tag{3.10}$$

$$-\frac{2}{1+\alpha}G(-\alpha, q) = 2iqg(iq; q^4) - 2q\alpha g(-iq; q^4) - \frac{2J_2 J_{2,4}^2}{j(-\alpha; q)j(-iq; q^2)}. \tag{3.11}$$

To evaluate the right-hand side of (3.11), using (1.1) and (2.3), we find that

$$j(\alpha; q)j(-\alpha; q) = J_{1,2}j(i; q^2) = (1-i)\frac{J_1^2 J_8}{J_4}, \tag{3.12}$$

$$j(-iq; q^2) = \frac{J_4^2}{J_8}. \tag{3.13}$$

So, by (1.2), (3.12), and (3.13), we deduce that

$$\frac{J_2 J_{2,4}^2}{j(-\alpha; q)j(-iq; q^2)} = \frac{J_2 J_{2,4}^2}{j(-\alpha; q)j(-iq; q^2)} \cdot \frac{j(\alpha; q)}{j(\alpha; q)} = \frac{J_2 J_{2,4}^2 j(\alpha; q)}{(1-i)J_1^2 J_4} = \frac{\varphi(q)j(\alpha; q)}{(1-i)J_4}.$$

Therefore, we restate (3.11) as

$$-\frac{2}{1+\alpha}G(-\alpha, q) = 2iqg(iq; q^4) - 2q\alpha g(-iq; q^4) - \frac{2\varphi(q)j(\alpha; q)}{(1-i)J_4}.$$

Setting q by iq in the above identity yields that

$$-\frac{2}{1+\alpha}G(-\alpha,iq)=-2qg(-q;q^4)+2q\alpha^{-1}g(q;q^4)-\frac{2\varphi(iq)j(\alpha;iq)}{(1-i)J_4}. \tag{3.14}$$

Then substituting (3.9), (3.10), and (3.14) into the left-hand side of (3.8), we obtain

$$\begin{aligned}
&G(i,-q)+\alpha^{-1}G(i,q)-\frac{2}{1+\alpha}G(-\alpha,iq)\\
&=(1-i)\frac{J_2J_{2,4}^2}{j(i;-q)j(-q;q^2)}+(1-i)\alpha^{-1}\frac{J_2J_{2,4}^2}{j(i;q)j(q;q^2)}-\frac{2\varphi(iq)j(\alpha;iq)}{(1-i)J_4}\\
&=(1-i)\frac{J_2J_{2,4}^2\left(j(i;q)j(q,q^2)+\alpha^{-1}j(i;-q)j(-q;q^2)\right)}{j(i;-q)j(-q;q^2)j(i;q)j(q;q^2)}-\frac{2\varphi(iq)j(\alpha;iq)}{(1-i)J_4}. \qquad (3.15)
\end{aligned}$$

To continue the evaluation from (3.15), we note that from (2.2) and (2.6),

$$j(i;q)=j(q;q^4)-ij(q^3;q^4)=(1-i)j(q;q^4).$$

Substituting the above identity into (3.15), we have

$$\begin{aligned}
&G(i,-q)+\alpha^{-1}G(i,q)-\frac{2}{1+\alpha}G(-\alpha,iq)\\
&=\frac{J_2J_{2,4}^2(j(q;q^4)j(q;q^2)+\alpha^{-1}j(-q;q^4)j(-q;q^2))}{j(-q;q^4)j(q;q^4)j(-q;q^2)j(q;q^2)}-\frac{2\varphi(iq)j(\alpha;iq)}{(1-i)J_4}\\
&=\frac{j(q;q^4)j(q;q^2)+\alpha^{-1}j(-q;q^4)j(-q;q^2)}{J_4}-\frac{2\varphi(iq)j(\alpha;iq)}{(1-i)J_4}, \qquad (3.16)
\end{aligned}$$

where we use (2.3) in the last equality.

Next, we give an alternative representation for the right-hand side of (3.8).

$$\begin{aligned}
&\frac{\sqrt{2}\psi(-iq)(q^2;q^4)_\infty(iq;iq)_\infty}{j(-\alpha;iq)}\\
&=\frac{\sqrt{2}\psi(-iq)(q^2;q^4)_\infty(iq;iq)_\infty}{j(-\alpha;iq)}\cdot\frac{j(\alpha;iq)}{j(\alpha;iq)}\\
&=\frac{\sqrt{2}J_{2,4}j(\alpha;iq)}{(1-i)J_4}, \qquad (3.17)
\end{aligned}$$

where we utilize (1.3) and (3.12). Therefore, combining (3.16) and (3.17), we only need to prove that

$$(1-i)\left(j(q;q^4)j(q;q^2)+\alpha^{-1}j(-q;q^4)j(-q;q^2)\right)-2\varphi(iq)j(\alpha;iq)=\sqrt{2}j(q^2;q^4)j(\alpha;iq). \tag{3.18}$$

In view of (2.6), we have

$$j(q;q^4)=j(-q^6;q^{16})-qj(-q^{14};q^{16}), \tag{3.19}$$

$$j(q;q^2)=j(-q^4;q^8)-qj(-q^8;q^8), \tag{3.20}$$

$$\varphi(iq)=j(-iq;-q^2)=j(-q^4;q^8)+iqj(-q^8;q^8). \tag{3.21}$$

Moreover, using (2.6) twice yields

$$j(\alpha;iq)=j(q;q^4)-\alpha j(-q;q^4)$$

$$= j(-q^6;q^{16}) - qj(-q^{14};q^{16}) - \alpha\left(j(-q^6;q^{16}) + qj(-q^{14};q^{16})\right)$$
$$= (1-\alpha)j(-q^6;q^{16}) - (1+\alpha)qj(-q^{14};q^{16}). \tag{3.22}$$

Then substituting (3.19)-(3.22) into the left-hand side of (3.18), we find that

$$\begin{aligned}
&(1-i)\left(j(q;q^4)j(q;q^2) + \alpha^{-1}j(-q;q^4)j(-q;q^2)\right) - 2\varphi(iq)j(\alpha;iq)\\
&= (1-i)\left(j(-q^6;q^{16}) - qj(-q^{14};q^{16})\right)\left(j(-q^4;q^8) - qj(-q^8;q^8)\right)\\
&\quad + (1-i)\alpha^{-1}\left(j(-q^6;q^{16}) + qj(-q^{14};q^{16})\right)\left(j(-q^4;q^8) + qj(-q^8;q^8)\right)\\
&\quad - 2\left(j(-q^4;q^8) + iqj(-q^8;q^8)\right)\left((1-\alpha)j(-q^6;q^{16}) - (1+\alpha)qj(-q^{14};q^{16})\right)\\
&= (-1+\sqrt{2}-i)\left(j(-q^4;q^8)j(-q^6;q^{16}) - q^2j(-q^8;q^8)j(-q^{14};q^{16})\right)\\
&\quad + (1+\sqrt{2}+i)q\left(j(-q^4;q^8)j(-q^{14};q^{16}) - j(-q^8;q^8)j(-q^6;q^{16})\right).
\end{aligned} \tag{3.23}$$

Furthermore, setting $q \to q^2$ and $x = -1$ in Proposition 2.4 and employing (2.2), we arrive at

$$j(-q^4;q^8)j(-q^6;q^{16}) - q^2j(-q^8;q^8)j(-q^{14};q^{16}) = j(q^2;q^4)j(-q^6;q^{16}). \tag{3.24}$$

Setting $q \to q^2$ and $x = q^4$ in Proposition 2.5 and applying (2.1) and (2.2), we obtain

$$j(-q^4;q^8)j(-q^{14};q^{16}) - j(-q^8;q^8)j(-q^6;q^{16}) = -j(q^2;q^4)j(-q^{14};q^{16}). \tag{3.25}$$

Then substituting (3.24) and (3.25) into (3.23), we derive that

$$\begin{aligned}
&(1-i)\left(j(q;q^4)j(q;q^2) + \alpha^{-1}j(-q;q^4)j(-q;q^2)\right) - 2\varphi(iq)j(\alpha;iq)\\
&= (-1+\sqrt{2}-i)j(q^2;q^4)j(-q^6;q^{16}) - (1+\sqrt{2}+i)qj(q^2;q^4)j(-q^{14};q^{16})\\
&= \sqrt{2}(1-\alpha)j(q^2;q^4)j(-q^6;q^{16}) - \sqrt{2}(1+\alpha)qj(q^2;q^4)j(-q^{14};q^{16})\\
&= \sqrt{2}j(q^2;q^4)j(\alpha;iq),
\end{aligned}$$

where we use (3.22) to obtain the last equality. Now we complete the proof of Entry 1.4. □

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