Representation of zero-sum invariants by sets of zero-sum sequences over a finite abelian group

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Abstract
Let G be an additive finite abelian group. For a sequence T over G and $g \in G$, let $\mathrm{v}_g(T)$ denote the multiplicity of g in T. Let $\mathcal{B}(G)$ denote the set of all zero-sum sequences over G. For $\Omega \subset \mathcal{B}(G)$, let $\mathsf{d}_\Omega(G)$ be the smallest integer t such that every sequence S over G of length $|S| \ge t$ has a subsequence in Ω. The invariant $\mathsf{d}_\Omega(G)$ was formulated recently in [3] to take a unified look at zero-sum invariants, it led to the first results there, and some open problems were formulated as well. In this paper, we make some further study on $\mathsf{d}_\Omega(G)$. Let $\mathsf{q}'(G)$ be the smallest integer t such that every sequence S over G of length $|S| \ge t$ has two nonempty zero-sum subsequences, say T_1 and T_2, having different forms, i.e., $\mathrm{v}_g(T_1) \neq \mathrm{v}_g(T_2)$ for some $g \in G$. Let $\mathsf{q}(G)$ be the smallest integer t such that

$$\bigcap_{\mathsf{d}_\Omega(G)=t} \Omega = \emptyset.$$

The invariants $\mathsf{q}(G)$ and $\mathsf{q}'(G)$ were also introduced in [3]. We prove, among other results, that $\mathsf{q}(G) = \mathsf{q}'(G)$ in fact.

Keywords Zero-sum sequence · Zero-sum invariant · Abelian group

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1 Introduction

Zero-sum theory on abelian groups can be traced back to the 1960s and has been developed rapidly in the last three decades (see [1,6,7]). Many invariants have been formulated and we list some of these invariants, which will be used in this section. Let G be an additive finite abelian group. By the Fundamental Theorem of Finite Abelian Groups, $|G| = 1$, or $G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$, where $r = r(G)$ is the rank of G and $n_r = \exp(G)$ is the exponent of G. Set

$$\mathsf{D}^*(G) = 1 + \sum_{i=1}^{r}(n_i - 1).$$

A starting point of zero-sum theory involves the Davenport constant $\mathsf{D}(G)$, which is defined as the smallest integer t such that every sequence S over G of length $|S| \geq t$ has a nonempty zero-sum subsequence.

Let $\mathsf{Ol}(G)$ denote the smallest integer t such that every squarefree sequence S over G of length $|S| \geq t$ has a nonempty zero-sum subsequence. The invariant $\mathsf{Ol}(G)$ is called the Olson constant of G. Let $\mathsf{ol}(G)$ denote the maximal length of a squarefree zero-sum free sequence S over G. Clearly, $\mathsf{Ol}(G) = \mathsf{ol}(G) + 1$.

In 2012, Girard [8] posed the problem of determining the smallest positive integer t, denoted by $\mathrm{disc}(G)$, such that every sequence S over G of length $|S| \geq t$ has two nonempty zero-sum subsequences of distinct lengths. The invariant $\mathrm{disc}(G)$ has been studied recently by Gao et al. in [2,4,5]. Related to $\mathrm{disc}(G)$, Gao, Li, Peng and Wang [3] defined $\mathsf{q}'(G)$ to be the smallest integer t such that every sequence S over G of length $|S| \geq t$ has two nonempty zero-sum subsequences, say T_1 and T_2, with $\mathrm{v}_g(T_1) \neq \mathrm{v}_g(T_2)$ for some $g \in G$. That is to say, T_1 and T_2 have different forms. Clearly,

$$\mathsf{q}'(G) \leq \mathrm{disc}(G)$$

for every finite abelian group G.

In order to describe zero-sum invariants uniformly, Gao et al. [3] provided a unified way to formulate zero-sum invariants.

Let G_0 be a nonempty subset of G. Let $\mathcal{B}(G_0)$ denote the monoid of all zero-sum sequences over G_0, and denote by $\mathbb{1}$ the identity element of the monoid $\mathcal{B}(G_0)$, i.e., the empty sequence over G_0. For $\Omega \subset \mathcal{B}(G)$, let $\mathsf{d}_\Omega(G)$ be the smallest integer t such that every sequence S over G of length $|S| \geq t$ has a subsequence in Ω. If such a t does not exist, then let $\mathsf{d}_\Omega(G) = \infty$. Observe that $\mathsf{d}_\Omega(G) = 0$ if $\mathbb{1} \in \Omega$. So we only need to consider the case of $\Omega \subset \mathcal{B}(G)\backslash\{\mathbb{1}\}$ in what follows. Then $\mathsf{d}_\Omega(G) \geq \mathsf{D}(G)$.

Let $G^* = G \setminus \{0\}$. For each integer $t \geq \mathsf{D}(G)$, let $\Omega = (\mathcal{B}(G^*)\backslash\{\mathbb{1}\}) \cup \{0^{t-\mathsf{D}(G)+1}\}$. It is easy to see that $\mathsf{d}_\Omega(G) = t$. Therefore, for every positive integer $t \geq \mathsf{D}(G)$, there is an $\Omega \subset \mathcal{B}(G)$ such that $t = \mathsf{d}_\Omega(G)$. But this does not give us much information on the invariant t. For some classical invariants t, finding some special $\Omega \subset \mathcal{B}(G)$ with $\mathsf{d}_\Omega(G) = t$ can help us understand t better. Thus, Gao et al. [3] introduced the following concepts. A sequence S over G is a *weak-regular* sequence if $\mathrm{v}_g(S) \leq \mathrm{ord}(g)$ for every $g \in G$ and $\Omega \subset \mathcal{B}(G)$ is *weak-regular* if every sequence $S \in \Omega$ is *weak-regular*. Let $\mathcal{B}_{wr}(G)$ denote the set of all nonempty weak-regular zero-sum sequences over G. Let $\mathsf{Vol}(G)$ be the set of all positive integers $t \in [\mathsf{D}(G), 1 + \sum_{g \in G}(\mathrm{ord}(g) - 1)]$ such that $t = \mathsf{d}_\Omega(G)$ for some

$\Omega \subset \mathcal{B}_{wr}(G)$. If $\Omega \subset \mathcal{B}(G)$, a sequence S over G is Ω-*free* if S has no subsequence in Ω. Related to $\mathsf{d}_\Omega(G)$, Gao et al. [3] introduced that a zero-sum sequence S is *essential* with respect to some $t \geq \mathsf{D}(G)$ if every $\Omega \subset \mathcal{B}(G)$ with $\mathsf{d}_\Omega(G) = t$ contains S. Thus, a natural research problem is to determine the smallest integer t such that there is no essential zero-sum sequence with respect to t; denote this by $\mathsf{q}(G)$.

For every positive integer $t \geq \mathsf{D}(G)$, let

$$\mathsf{Q}_t(G) = \bigcap_{\Omega \subset \mathcal{B}(G), \mathsf{d}_\Omega(G)=t} \Omega.$$

Clearly, $S \in \mathsf{Q}_t(G)$ if and only if S is essential with respect to t, and $\mathsf{q}(G)$ is the smallest integer t with $\mathsf{Q}_t(G) = \emptyset$.

To study $\mathsf{Vol}(G)$ we introduce the following invariant. Let $\mathsf{N}(G)$ denote the smallest integer t such that every weak-regular sequence S over G of length $|S| \geq t$ has a nonempty zero-sum subsequence T of S satisfying $\mathrm{v}_g(T) = \mathrm{v}_g(S)$ for some $g \mid S$ or, equivalently, $\mathrm{supp}(ST^{-1}) \neq \mathrm{supp}(S)$.

In this paper, we make some further study on $\mathsf{d}_\Omega(G)$, $\mathsf{q}(G)$, $\mathsf{q}'(G)$ and $\mathsf{N}(G)$ for finite abelian groups. Our main results are as follows.

Theorem 1.1 *If p is a prime and G is a finite abelian group, then the following hold:*

(1) $\mathsf{N}(G) \leq 1 + \mathsf{ol}(G)(\exp(G) - 1)$.
(2) If $G = C_p$ *then* $\mathsf{N}(G) = 2p - \lfloor 2\sqrt{p} \rfloor$.

Theorem 1.2 *If G is a finite abelian group, then the following hold:*

(1) $[1 + \mathsf{ol}(G)(\exp(G) - 1), 1 + \sum_{g \in G}(\mathrm{ord}(g) - 1)] \subset \mathsf{Vol}(G)$.
(2) If $\mathsf{D}(G) = \mathsf{D}^*(G)$ *then*

$$\mathsf{Vol}(G) = [\mathsf{D}(G), 1 + \sum_{g \in G}(\mathrm{ord}(g) - 1)].$$

Theorem 1.3 *If m, n are positive integers, p is a prime, and G is a finite abelian group, then* $\mathsf{Vol}(G) = [\mathsf{D}(G), 1 + \sum_{g \in G}(\mathrm{ord}(g) - 1)]$ *if G is one of the following groups:*

(1) $r(G) \leq 2$.
(2) G is a p-group.
(3) $G = C_{mp^n} \oplus H$, *where H is a p-group with* $\mathsf{D}^*(H) \leq p^n$.

Theorem 1.4 *If G is a finite abelian group, then the following hold:*

(1) $\mathsf{D}^*(G) + \exp(G) \leq \mathsf{q}'(G) \leq \mathsf{D}(G) + \exp(G)$.
(2) $\mathsf{q}'(G) = \mathsf{q}(G)$.
(3) If $\mathsf{D}(G) = \mathsf{D}^*(G)$, *then* $\mathsf{q}'(G) = \mathsf{q}(G) = \mathsf{D}(G) + \exp(G)$.

The paper is organized as follows. Section 2 provides some notations and concepts which will be used in the sequel. In Sect. 3, we prove Theorem 1.1. In Sect. 4, we investigate $\mathsf{Vol}(G)$ for finite abelian groups and prove Theorems 1.2 and 1.3. In Sect. 5, we prove Theorem 1.4.

2 Preliminaries

Throughout this paper, our notations and terminology are consistent with [1,3,7] and we briefly present some key concepts. Let $\mathbb{Z}$ denote the set of integers, and let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a \le b$, we set $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$, $\lfloor a \rfloor = \max\{x \in \mathbb{Z} \mid x \le a\}$ and $\lceil a \rceil = \min\{x \in \mathbb{Z} \mid x \ge a\}$.

Throughout, let G be an additive finite abelian group. We denote by C_n the cyclic group of n elements and denote by C_n^r the direct sum of r copies of C_n. An r-tuple $(e_1, e_2, \dots, e_r)$ in $G \setminus \{0\}$ is called a *basis* of G if $G = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \cdots \oplus \langle e_r \rangle$.

Let G_0 be a nonempty subset of G. In Additive Combinatorics, a sequence (over G_0) means a finite unordered sequence of terms from G_0 where repetition is allowed, and (as usual) we consider sequences as elements of the free abelian monoid $\mathcal{F}(G_0)$ with basis G_0.

Let

$$S = g_1 \cdots g_l = \prod_{g \in G_0} g^{\mathrm{v}_g(S)} \in \mathcal{F}(G_0)$$

be a sequence over G_0. We call

- $\mathrm{v}_g(S)$ the *multiplicity* of g in S,
- $\mathrm{h}(S) = \max\{\mathrm{v}_g(S) \mid g \in G_0\}$ the *height* of S,
- $\mathrm{supp}(S) = \{g \in G_0 \mid \mathrm{v}_g(S) > 0\}$ the *support* of S,
- $|S| = l = \sum_{g \in G_0} \mathrm{v}_g(S) \in \mathbb{N}_0$ the *length* of S,
- $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G_0} \mathrm{v}_g(S) g \in G_0$ the *sum* of S,
- S a *zero-sum sequence* if $\sigma(S) = 0$,
- S a *squarefree sequence* if $\mathrm{v}_g(S) \le 1$ for all $g \in G_0$,
- T a *subsequence* of S if $\mathrm{v}_g(T) \le \mathrm{v}_g(S)$ for all $g \in G_0$, denote by $T|S$,
- $ST^{-1} = \prod_{g \in G_0} g^{\mathrm{v}_g(S) - \mathrm{v}_g(T)}$ the subsequence obtained from S by deleting T,
- S a *minimal zero-sum sequence* if it is a nonempty zero-sum sequence and has no proper zero-sum subsequence,
- S a *zero-sum free sequence* if S has no nonempty zero-sum subsequence,
- two subsequences T_1 and T_2 of S *disjoint* if $T_1 \mid ST_2^{-1}$,
- $\Sigma(S) = \{\sigma(T) \mid T|S, T \ne \mathbb{1}\}$ the set of subsums of S.

Let $\mathcal{A}(G_0)$ denote the set of all minimal zero-sum sequences over G_0. By the definition of minimal zero-sum sequences, the empty sequence $\mathbb{1}$ is not a minimal zero-sum sequence and therefore $\mathcal{A}(G_0) \subset \mathcal{B}(G_0) \setminus \{\mathbb{1}\}$. Let $\eta(G)$ be the smallest integer t such that every sequence S over G of length $|S| \ge t$ has a zero-sum subsequence of length in $[1, \exp(G)]$. Let $\mathsf{D}_2(G)$ denote the smallest integer t such that every sequence over G of length $|S| \ge t$ has two disjoint nonempty zero-sum subsequences. The invariant $\mathsf{D}_2(G)$ was first introduced by Halter-Koch [9] and was studied recently by Plagne and Schmid [13].

3 On $\mathsf{N}(G)$

In this section we shall prove Theorem 1.1 and we need some preliminary results beginning with the following well-known Cauchy–Davenport theorem.

Lemma 3.1 [10] *If $h \ge 2$, p is a prime number, and $A_1, \dots, A_h$ are nonempty subsets of C_p, then*

$$|A_1 + \cdots + A_h| \ge \min(p, \Sigma_{i=1}^{h} |A_i| - h + 1).$$

Lemma 3.2 *If S is a sequence over $C_p \setminus \{0\}$ with length $|S| = p - 1$, then*

$$\Sigma(S) \setminus \{0\} = C_p \setminus \{0\}.$$

Proof Let $S = g_1 \dots g_{p-1}$ and $A_i = \{0, g_i\}$ for each $i \in [1, p-1]$. By Lemma 3.1,

$$\begin{aligned}|\Sigma(S) \setminus \{0\}| &= |(A_1 + \cdots + A_{p-1}) \setminus \{0\}| \\ &\geq \min(p, \Sigma_{i=1}^{p-1} |A_i| - (p-1) + 1) - 1 \\ &= p - 1.\end{aligned}$$

Since $|\Sigma(S) \setminus \{0\}| \leq p-1$, we deduce $|\Sigma(S) \setminus \{0\}| = p-1$, therefore $\Sigma(S) \setminus \{0\} = C_p \setminus \{0\}$. □

Lemma 3.3 *Let k be a positive integer. Define $A_k := \min\{a + b \mid ab \geq k, a, b \in \mathbb{N}\}$. Then $A_k = \lceil 2\sqrt{k} \rceil$.*

Proof Let $a, b \in \mathbb{N}$, and $ab \geq k$. For $k = 1, 2, 3$, letting $a = 1$ and $b = k$ we get $A_k = 1 + k = \lceil 2\sqrt{k} \rceil$. For $k = 4$, letting $a = b = 2$ we get $A_k = \lceil 2\sqrt{k} \rceil$. From now on we assume that

$$k \geq 5.$$

If k is not a square, there is a unique positive integer c such that

$$c^2 < k < (c+1)^2.$$

We distinguish two cases:
Case 1. $c(c+1) < k$. Then

$$k \geq c(c+1) + 1 = \left(c + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

Therefore, $c + \frac{1}{2} < \sqrt{k} < c + 1$. Thus, $2c + 1 < 2\sqrt{k} < 2c + 2$. Hence,

$$\lceil 2\sqrt{k} \rceil = 2c + 2.$$

From $ab \geq k \geq c(c+1) + 1$ we deduce that $(a+b)^2 = 4ab + (a-b)^2 \geq 4c(c+1) + 4 + (a-b)^2 = (2c+1)^2 + 3 + (a-b)^2$. Therefore,

$$a + b \geq 2c + 2.$$

Letting $a = b = c + 1$ we get $A_k = 2c + 2 = \lceil 2\sqrt{k} \rceil$.
Case 2. $k \leq c(c+1)$. Then $c^2 < k \leq (c + \frac{1}{2})^2 - \frac{1}{4}$. Therefore, $c < \sqrt{k} < c + \frac{1}{2}$. Thus, $2c < 2\sqrt{k} < 2c + 1$. Hence,

$$\lceil 2\sqrt{k} \rceil = 2c + 1.$$

Since $ab \geq k > c^2$, we have $(a+b)^2 = 4ab + (a-b)^2 > 4c^2$. Therefore, $a + b \geq 2c + 1$. Letting $a = c, b = c + 1$ we get $A_k = 2c + 1 = \lceil 2\sqrt{k} \rceil$.

Now it remains to consider the case that k is a square. Let $k = m^2$ with $m \geq 3$ since $k \geq 5$. From $ab \geq k = m^2$ we deduce that $(a+b)^2 = (a-b)^2 + 4ab \geq 4m^2$ with equality holding if and only if $a = b = m$. Letting $a = b = m$ we get

$$A_k = 2m$$

as desired. □

Proof of Theorem 1.1. (1) Let S be a weak-regular sequence over G of length $|S| \geq 1 + \mathsf{ol}(G)(\exp(G) - 1)$. We need to show that there exists a zero-sum subsequence T of S such that $\mathrm{v}_g(T) = \mathrm{v}_g(S)$ for some $g \mid S$. If there exists $g \in G$ such that $\mathrm{v}_g(S) = \mathrm{ord}(g)$, then $T = g^{\mathrm{ord}(g)}$ is a zero-sum subsequence of S and $\mathrm{v}_g(T) = \mathrm{v}_g(S) = \mathrm{ord}(g) \geq 1$. Next we assume that

$$\mathrm{v}_g(S) \leq \mathrm{ord}(g) - 1 \leq \exp(G) - 1$$

for every $g \in G$.

Let

$$\mathrm{supp}(S) = \{g_1, \ldots, g_l\}.$$

Since $|S| \geq 1 + \mathsf{ol}(G)(\exp(G) - 1)$, we infer that $l \geq \frac{|S|}{\mathrm{h}(S)} \geq \frac{|S|}{\exp(G)-1} > \mathsf{ol}(G)$. Therefore, $l \geq \mathsf{ol}(G) + 1 = \mathsf{Ol}(G)$. Hence, $0 \in \Sigma(g_1 \ldots g_l)$, i.e., there is a nonempty subset $I \subset [1, l]$ such that $\sum_{i \in I} g_i = 0$. Take $j \in I$ with $\mathrm{v}_{g_j}(S) = \min\{\mathrm{v}_{g_i}(S) \mid i \in I\}$. Then

$$T = \left(\prod_{i \in I} g_i\right)^{\mathrm{v}_{g_j}(S)}$$

is a zero-sum subsequence of S with $\mathrm{v}_{g_j}(T) = \mathrm{v}_{g_j}(S)$.

(2) Let $G = C_p$. It is easy to verify that $\mathsf{N}(C_2) = 2$, $\mathsf{N}(C_3) = 3$. Now we assume that $p \geq 5$.

Let $k \geq 5$ be a positive integer. By Lemma 3.3,

$$A_k = \min\{a + b \mid ab \geq k, a, b \in \mathbb{N}\} = \lceil 2\sqrt{k} \rceil.$$

If $a \geq k - 1$ or $b \geq k - 1$, then $a, b \in \mathbb{N}$ and $ab \geq k$ imply that $a + b \geq k + 1 > 2\sqrt{k} + 1 \geq \lceil 2\sqrt{k} \rceil$. Therefore, for $k \geq 5$ we have

$$A_k = \min\{a + b \mid ab \geq k, a, b \in \mathbb{N}, 2 \leq a, b \leq k - 2\} = \lceil 2\sqrt{k} \rceil. \tag{3.1}$$

Since $p \geq 5$ is a prime, from $a, b \geq 2, a, b \in \mathbb{N}$ we infer that $ab \geq p$ if and only if $ab \geq p + 1$. Therefore, $A_p = A_{p+1} = \lceil 2\sqrt{p} \rceil$ by (3.1). So we need to show

$$\mathsf{N}(C_p) = 2p - \lfloor 2\sqrt{p} \rfloor = 2p - A_{p+1} + 1.$$

First we want to prove

$$\mathsf{N}(C_p) \leq 2p - A_{p+1} + 1.$$

Let S be a weak-regular sequence over C_p of length $|S| \geq 2p - A_{p+1} + 1 = 2p - \lfloor 2\sqrt{p} \rfloor$. We need to show that there exists a zero-sum subsequence T of S such that $\mathrm{v}_g(T) = \mathrm{v}_g(S)$ for some $g \mid S$.

Since S is weak-regular, $\mathrm{v}_g(S) \leq \mathrm{ord}(g)$ for every $g \in G$ by the definition. If $\mathrm{v}_g(S) = \mathrm{ord}(g)$ for some $g \in G$, then $T = g^{\mathrm{ord}(g)}$ is a zero-sum subsequence of S with $\mathrm{v}_g(T) = \mathrm{v}_g(S)$ and we are done. So we may assume that $\mathrm{v}_g(S) \leq \mathrm{ord}(g) - 1$ for every $g \in G$. It follows that

$$0 \nmid S,$$

and

$$\mathrm{v}_g(S) \leq p - 1$$

for every $g \mid S$.

If there exists $g_0 \mid S$ such that $\mathsf{v}_{g_0}(S) \le p - \lfloor 2\sqrt{p} \rfloor + 1$, then $|S(g_0^{\mathsf{v}_{g_0}(S)})^{-1}| \ge p-1$, by Lemma 3.2, there exists a subsequence $T \mid S(g_0^{\mathsf{v}_{g_0}(S)})^{-1}$ such that $\sigma(T) = -\mathsf{v}_{g_0}(S)g_0$, so $Tg_0^{\mathsf{v}_{g_0}(S)}$ is a zero-sum subsequence of S satisfying $\mathsf{v}_{g_0}(Tg_0^{\mathsf{v}_{g_0}(S)}) = \mathsf{v}_{g_0}(S)$. So we may assume

$$\mathsf{v}_g(S) \ge p - \lfloor 2\sqrt{p} \rfloor + 2$$

for every $g \mid S$.

If $|\mathrm{supp}(S)| \ge 3$, then we fix a $h \mid S$ for which $\mathsf{v}_h(S)$ is the smallest possible. Consider $U = S(h^{\mathsf{v}_h(S)})^{-1}$. If $|U| \ge p-1$, then by Lemma 3.2 there is a $V \mid U$ such that $\sigma(V) \equiv -\mathsf{v}_h(S)h \pmod p$, and then $T = Vh^{\mathsf{v}_h(S)}$ will be a zero-sum subsequence of S with $\mathsf{v}_h(T) = \mathsf{v}_h(S)$ as desired. If $\mathsf{v}_h(S) \ge p-2$, then $|S| \ge |\mathrm{supp}(S)|\mathsf{v}_h(S) \ge 3p-6$, therefore $|U| \ge 2p-5 > p-1$, and we are done. And if $\mathsf{v}_h(S) \le p-3$, then we refer to $|S| \ge |\mathrm{supp}(S)|\mathsf{v}_h(S) \ge 3p - 3\lfloor 2\sqrt{p} \rfloor + 6 > 3p - 6\sqrt{p} + 6$, so in this case $|U| \ge |S| - (p-3) > 2p - 6\sqrt{p} + 9 = p + (\sqrt{p} - 3)^2 > p-1$, and we are done in this case, too.

From the fact that S is weak-regular, we get

$$|\mathrm{supp}(S)| = 2.$$

Multiplying every term of S with an integer in $[1, p-1]$ we may assume

$$S = 1^{p-a}x^{p-b}$$

with $0 \le a, b \le p-1$ and $x \in [2, p-1]$.

If $\min\{a, b\} \le 1$ or $\max\{a, b\} = p-1$, then it is easy to see that S has a zero-sum subsequence T such that $\mathsf{v}_g(T) = \mathsf{v}_g(S)$ for some $g \mid S$. So we may assume

$$2 \le a, b \le p-2.$$

Assume to the contrary that S has no zero-sum subsequence T such that $\mathsf{v}_g(T) = \mathsf{v}_g(S)$ for some $g \mid S$.

Let m and c be integers with $m, c \in [1, p-1]$ such that

$$mx \equiv p-a \pmod p \text{ and } (p-b)x \equiv c \pmod p.$$

Then we deduce

$$(p-a)(p-b) \equiv mx(p-b) \equiv mc \pmod p,$$

which implies

$$p \mid (ab - mc).$$

If $m \ge b$ or $c \ge a$, then $1^{p-a}x^{p-m}$ or $1^{p-c}x^{p-b}$ is a zero-sum subsequence of S respectively, a contradiction. So

$$1 \le m \le b-1, 1 \le c \le a-1.$$

Now $p \mid (ab - mc)$ implies $p \le ab - mc \le ab - 1$. Therefore, $ab \ge p+1$. By the definition of A_{p+1} we infer

$$a + b \ge A_{p+1}.$$

On the other hand, since $|S| \geq 2p - A_{p+1} + 1$ and $|S| = 2p - a - b$, one has $a + b \leq A_{p+1} - 1$, a contradiction. This proves

$$\mathsf{N}(C_p) \leq 2p - A_{p+1} + 1.$$

So it remains to show

$$\mathsf{N}(C_p) \geq 2p - A_{p+1} + 1.$$

Let a_0 and b_0 be integers such that $2 \leq a_0, b_0 \leq p-1$, $a_0 b_0 \geq p+1$ and $a_0 + b_0 = A_{p+1}$. Let

$$S = 1^{p-a_0}(p - a_0)^{p-b_0}.$$

Then

$$|S| = 2p - A_{p+1}.$$

We claim that S has no zero-sum subsequence T such that $\mathrm{v}_g(T) = \mathrm{v}_g(S)$ for some $g \mid S$. Let T be a nonempty zero-sum subsequence of S. Assume to the contrary

$$\mathrm{v}_g(T) = \mathrm{v}_g(S)$$

for some $g \in \operatorname{supp}(S) = \{1, p - a_0\}$.

Notice that for any integer t with $0 \leq t \leq p - b_0 \leq p - 2$, one has $\sigma(1^{p-a_0}(p - a_0)^t) = (t + 1)(p - a_0) \neq 0$. Therefore, $g \neq 1$. So,

$$g = p - a_0$$

and therefore

$$T = 1^{p-d}(p - a_0)^{p-b_0}$$

for some $d \in [a_0, p - 1]$.

From $\sigma(T) = 0$ we deduce $(p - b_0)(p - a_0) \equiv d \pmod p$, i.e.,

$$a_0 b_0 \equiv d \pmod p. \tag{3.2}$$

Moreover, $a_0 \leq d < a_0 b_0$ since $a_0 b_0 \geq p + 1$. Let $d = q a_0 + r$ where q, r are integers such that $0 \leq r \leq a_0 - 1$. Then

$$1 \leq q < b_0$$

since $a_0 \leq d < a_0 b_0$. It follows from (3.2) that

$$a_0(b_0 - q) \equiv r \pmod p. \tag{3.3}$$

If $b_0 = 2$, then $q = 1$. But (3.3) yields $a_0 \equiv r \pmod p$, which is impossible since $0 \leq r \leq a_0 - 1 < p$. Hence $b_0 \geq 3$. If $r = 0$, then (3.3) implies $p \mid a_0(b_0 - q)$, which is a contradiction to $0 < a_0, b_0 - q \leq p - 1$. Hence $r \geq 1$.

Furthermore, if $q = b_0 - 1$, by (3.3), we get $a_0 \equiv r \pmod p$, a contradiction since $r < a_0 \leq p - 1$. So $1 \leq q \leq b_0 - 2$. This implies $2 \leq b_0 - q \leq p - 1$. Now, using (3.3) again, we deduce $p \mid a_0(b_0 - q) - r$. It follows that $p \leq a_0(b_0 - q) - r \leq a_0(b_0 - q) - 1$. That is, $a_0(b_0 - q) \geq p + 1$. But $a_0 + (b_0 - q) < a_0 + b_0$ since $q \geq 1$, which contradicts the minimality of $a_0 + b_0$. This proves $\mathsf{N}(C_p) \geq 2p - A_{p+1} + 1$, completing the proof. □

4 Vol(G) on finite abelian groups

In this section, we investigate $\mathsf{Vol}(G)$ for finite abelian groups and prove Theorems 1.2 and 1.3.

Lemma 4.1 [1,11,12,14] *Suppose p is a prime and m, n are positive integers. Then $\mathsf{D}(G) = \mathsf{D}^*(G)$ if G is one of the following groups:*

(1) $r(G) \le 2$.
(2) G is a finite abelian p-group.
(3) $G = C_{mp^n} \oplus H$ where H is a finite abelian p-group and $p^n \ge \mathsf{D}^*(H)$.

Lemma 4.2 [3, Proposition 3.1] *Suppose $\Omega \subset \mathcal{B}(G)\backslash\{\mathbb{1}\}$. Then $\mathsf{d}_\Omega(G) < \infty$ if and only if, for every $g \in G$, $g^{k\,\mathrm{ord}(g)} \in \Omega$ for some positive integer $k = k(g)$.*

Lemma 4.3 *If G is a finite abelian group, then $1 + \sum_{g\in G}(\mathrm{ord}(g) - 1) \in \mathsf{Vol}(G)$.*

Proof Let

$$\Omega = \{g^{\mathrm{ord}(g)} \mid g \in G\}.$$

We want to show

$$\mathsf{d}_\Omega(G) = 1 + \sum_{g\in G}(\mathrm{ord}(g) - 1).$$

Let

$$T = \prod_{g\in G} g^{\mathrm{ord}(g)-1}.$$

It is obvious that T is Ω-free. Therefore,

$$\mathsf{d}_\Omega(G) \ge |T| + 1 = 1 + \sum_{g\in G}(\mathrm{ord}(g) - 1).$$

It remains to show

$$\mathsf{d}_\Omega(G) \le 1 + \sum_{g\in G}(\mathrm{ord}(g) - 1).$$

Let S be any sequence over G of length $1 + \sum_{g\in G}(\mathrm{ord}(g) - 1)$. We need to show that S has a zero-sum subsequence in Ω. Assume to the contrary that S is Ω-free. Then $g^{\mathrm{ord}(g)} \nmid S$ for every $g \in G$. Hence, $\mathrm{v}_g(S) \le \mathrm{ord}(g) - 1$ for every $g \in G$. It follows that

$$|S| = \sum_{g\in G}\mathrm{v}_g(S) \le \sum_{g\in G}(\mathrm{ord}(g) - 1) < |S|,$$

which is a contradiction. This proves $\mathsf{d}_\Omega(G) = 1 + \sum_{g\in G}(\mathrm{ord}(g) - 1)$. Therefore, $1 + \sum_{g\in G}(\mathrm{ord}(g) - 1) \in \mathsf{Vol}(G)$ follows from $\Omega \subset \mathcal{B}_{wr}(G)$. □

Proof of Theorem 1.2. For $|G| = 1$, it is trivial. So we may assume

$$|G| \ge 2.$$

(1) We need to show that for every $l \in [1+\mathsf{ol}(G)(\exp(G)-1), 1+\sum_{g\in G}(\mathrm{ord}(g)-1)]$, there exists a weak-regular Ω such that

$$\mathsf{d}_\Omega(G) = l.$$

We proceed by induction on l. By Lemma 4.3, $1+\sum_{g\in G}(\mathrm{ord}(g)-1) \in \mathsf{Vol}(G)$. Now suppose $l \in \mathsf{Vol}(G)$, where $l \in [2+\mathsf{ol}(G)(\exp(G)-1), 1+\sum_{g\in G}(\mathrm{ord}(g)-1)]$. We want to prove

$$l-1 \in \mathsf{Vol}(G).$$

By the induction hypothesis, there exists an $\Omega \subset \mathcal{B}_{wr}(G)$ such that $\mathsf{d}_\Omega(G) = l$. By Lemma 4.2, $\{g^{\mathrm{ord}(g)} \mid g \in G\} \subset \Omega$. Choose a sequence S over G of length $|S| = l-1$ such that S is Ω-free. Then

$$\mathrm{v}_g(S) \le \mathrm{ord}(g) - 1$$

for every $g \in G$. Therefore, S is weak-regular. Since $|S| = l-1 \ge 1+\mathsf{ol}(G)(\exp(G)-1)$, by Theorem 1.1 (1), there exists a zero-sum subsequence W of S such that $\mathrm{v}_g(W) = \mathrm{v}_g(S) \ge 1$ for some $g \in G$. Let

$$\Omega_1 = \Omega \cup \{W\} \subset \mathcal{B}_{wr}(G).$$

It is clear that $g^{-1}S$ is Ω_1-free. Hence,

$$l-1 = |g^{-1}S| + 1 \le \mathsf{d}_{\Omega_1}(G) \le \mathsf{d}_\Omega(G) = l.$$

So $\mathsf{d}_{\Omega_1}(G) = l-1$ or l, and $\Omega \subsetneq \Omega_1 \subset \mathcal{B}_{wr}(G)$. If $\mathsf{d}_{\Omega_1}(G) = l-1$, then $l-1 \in \mathsf{Vol}(G)$ and we are done. If $\mathsf{d}_{\Omega_1}(G) = l$, repeat the above steps, then we can find $\Omega_2 \subset \mathcal{B}_{wr}(G)$ such that $\mathsf{d}_{\Omega_2}(G) = l-1$ or l, and $\Omega \subsetneq \Omega_1 \subsetneq \Omega_2 \subset \mathcal{B}_{wr}(G)$. Note that $\mathcal{B}_{wr}(G)$ is finite, we finally get an integer $m < |\mathcal{B}_{wr}(G)|$, and m subsets $\Omega_1, \Omega_2, \ldots, \Omega_m$ of $\mathcal{B}_{wr}(G)$ such that $\Omega \subsetneq \Omega_1 \subsetneq \Omega_2 \subsetneq \cdots \subsetneq \Omega_m \subset \mathcal{B}_{wr}(G)$, $\mathsf{d}_{\Omega_i}(G) = l$ for every $i \in [1, m-1]$ and $\mathsf{d}_{\Omega_m}(G) = l-1$. This proves $l-1 \in \mathsf{Vol}(G)$. Therefore, $[1+\mathsf{ol}(G)(\exp(G)-1), 1+\sum_{g\in G}(\mathrm{ord}(g)-1)] \subset \mathsf{Vol}(G)$.

(2) By the definition of $\mathsf{Vol}(G)$ we know

$$\mathsf{Vol}(G) \subset [\mathsf{D}(G), 1+\sum_{g\in G}(\mathrm{ord}(g)-1)].$$

So we need to show

$$[\mathsf{D}(G), 1+\sum_{g\in G}(\mathrm{ord}(g)-1)] \subset \mathsf{Vol}(G).$$

By Lemma 4.3, $1+\sum_{g\in G}(\mathrm{ord}(g)-1) \in \mathsf{Vol}(G)$. So it suffices to prove

$$[\mathsf{D}(G), \sum_{g\in G}(\mathrm{ord}(g)-1)] \subset \mathsf{Vol}(G). \tag{4.1}$$

Let

$$G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$$

with $1 < n_1 | n_2 | \cdots | n_r$.

Let G_2 be the maximal elementary 2-subgroup of G. Then $G_2 = \{0\}$ if $|G|$ is odd. When $|G|$ is even, let $r' = |\{i \in [1, r] \mid 2|n_i\}|$. Then, $G_2 = C_2^{r'}$. So we always have $2 \mid (|G| - |G_2|)$. Let

$$m = \frac{|G| - |G_2|}{2}.$$

If $G = C_2^r$ then $\mathsf{ol}(G) = \mathsf{D}(G) - 1 = r$ and $\exp(G) = 2$. It follows from (1) that $[\mathsf{D}(G), 1 + \sum_{g \in G}(\mathrm{ord}(g) - 1)] = \mathsf{Vol}(G)$. From now on we assume

$$G \neq C_2^r.$$

Next we want to show that there are two intervals I_1 and I_2 such that

$$I_1 \cup I_2 = [\mathsf{D}(G), \sum_{g \in G}(\mathrm{ord}(g) - 1)] \text{ and } I_j \subset \mathsf{Vol}(G) \text{ for } j = 1, 2, \tag{4.2}$$

and then (4.1) follows.

Now we want to construct I_1. Let $j \in [1, m]$, and let $\{g_1, \ldots, g_j\} \subset G \setminus G_2$ with

$$\{g_1, \ldots, g_j\} \cap \{-g_1, \ldots, -g_j\} = \emptyset.$$

Let $k_i \in [1, \mathrm{ord}(g_i) - 1]$ for each $i \in [1, j]$, and let

$$\Omega_{j,k_1,\ldots,k_j} = \{g^{\mathrm{ord}(g)} \mid g \in G\} \cup \{g_1^{k_1}(-g_1)^{k_1}, \ldots, g_j^{k_j}(-g_j)^{k_j}\}.$$

Put

$$\Omega = \Omega_{j,k_1,\ldots,k_j}.$$

We now show

$$\mathsf{d}_\Omega(G) = \sum_{g \in G}(\mathrm{ord}(g) - 1) - \sum_{i=1}^{j}(\mathrm{ord}(g_i) - k_i) + 1.$$

Let

$$T_j = g_1^{k_1 - 1} \ldots g_j^{k_j - 1} \prod_{g \in G \setminus \{0, g_1, \ldots, g_j\}} g^{\mathrm{ord}(g) - 1}.$$

It is easy to see that T_j is an Ω-free sequence of length $|T_j| = \sum_{g \in G}(\mathrm{ord}(g) - 1) - \sum_{i=1}^{j}(\mathrm{ord}(g_i) - k_i)$. Therefore,

$$\mathsf{d}_\Omega(G) \geq |T_j| + 1 = \sum_{g \in G}(\mathrm{ord}(g) - 1) - \sum_{i=1}^{j}(\mathrm{ord}(g_i) - k_i) + 1.$$

So it remains to show

$$\mathsf{d}_\Omega(G) \leq \sum_{g \in G}(\mathrm{ord}(g) - 1) - \sum_{i=1}^{j}(\mathrm{ord}(g_i) - k_i) + 1.$$

Let S_j be any sequence over G with

$$|S_j| = \sum_{g \in G}(\mathrm{ord}(g) - 1) - \sum_{i=1}^{j}(\mathrm{ord}(g_i) - k_i) + 1.$$

We only need to show that there is a zero-sum subsequence of S_j in Ω. If there exists $g \in G$ such that $\mathrm{v}_g(S_j) \geq \mathrm{ord}(g)$, then $g^{\mathrm{ord}(g)} \in \Omega$, and we are done. Hence, we next assume

$$\mathrm{v}_g(S_j) \leq \mathrm{ord}(g) - 1$$

for every $g \in G$.

If there exists $i \in [1, j]$ such that $\mathrm{v}_{g_i}(S_j) \geq k_i$ and $\mathrm{v}_{-g_i}(S_j) \geq k_i$, then $g_i^{k_i}(-g_i)^{k_i} \in \Omega$. So we assume that, for every $i \in [1, j]$, there exists $g_i' \in \{g_i, -g_i\}$ such that $\mathrm{v}_{g_i'}(S_j) \leq k_i - 1$. Since

$$|S_j| = \sum_{g \in G \setminus \{0\}} \mathrm{v}_g(S_j) \leq \sum_{g \in G} (\mathrm{ord}(g) - 1) - \sum_{i=1}^{j} (\mathrm{ord}(g_i) - k_i) < |S_j|,$$

we get a contradiction. Therefore

$$\mathsf{d}_\Omega(G) = \sum_{g \in G} (\mathrm{ord}(g) - 1) - \sum_{i=1}^{j} (\mathrm{ord}(g_i) - k_i) + 1 \in \mathsf{Vol}(G)$$

follows from the fact that Ω is weak-regular.

Let

$$f(j, k_1, \ldots, k_j) = \sum_{g \in G} (\mathrm{ord}(g) - 1) - \sum_{i=1}^{j} (\mathrm{ord}(g_i) - k_i) + 1.$$

When j runs over $[1, m]$ and k_i runs over $[1, \mathrm{ord}(g_i) - 1]$ for every $i \in [1, j]$, $f(j, k_1, \ldots, k_j)$ takes its maximal value $\sum_{g \in G} (\mathrm{ord}(g) - 1)$ when $j = 1$ and $k_1 = \mathrm{ord}(g_1) - 1$, and $f(j, k_1, \ldots, k_j)$ takes its minimal value

$$\frac{\sum_{g \in G} (\mathrm{ord}(g) - 1) - 2^{r'} + 1}{2} + 2^{r'}$$

when $j = m$ and $k_i = 1$ for every $i \in [1, m]$. It is easy to see that $f(j, k_1, \ldots, k_j)$ can take any integer in between the minimal value and the maximal value. So

$$I_1 = [\frac{\sum_{g \in G} (\mathrm{ord}(g) - 1) - 2^{r'} + 1}{2} + 2^{r'}, \sum_{g \in G} (\mathrm{ord}(g) - 1)] \subset \mathsf{Vol}(G). \tag{4.3}$$

Next we construct I_2. Let $r_0 \in [0, r - 1]$ be the smallest integer such that

$$n_{r_0 + 1} > 2.$$

Let $(e_1, \ldots, e_r)$ be a basis of G with $\mathrm{ord}(e_i) = n_i$ and $g_i = e_i$ for every $i \in [1, r]$. Let $j \in [r, m + r_0]$ and $\{g_{r+1}, \ldots, g_j\} \subset G \setminus G_2$ with $\{g_{r_0+1}, \ldots, g_j\} \cap \{-g_{r_0+1}, \ldots, -g_j\} = \emptyset$. Let $k_i \in [1, \mathrm{ord}(g_i) - 1]$ for every $i \in [r_0 + 1, j]$,

$$\begin{aligned} A_{j, k_1, \ldots, k_j} = &\{S \in \mathcal{A}(G) \mid \mathrm{supp}(S) \not\subset \{g_1, \ldots, g_j, (-g_{r_0+1}), \ldots, (-g_j)\}\} \\ &\cup \{g_{r_0+1}^{k_{r_0+1}} (-g_{r_0+1})^{k_{r_0+1}}, \ldots, g_j^{k_j} (-g_j)^{k_j}\}, \end{aligned}$$

and

$$\Omega' = \{g^{\mathrm{ord}(g)} \mid g \in G\} \cup A_{j, k_1, \ldots, k_j}.$$

We now show

$$\mathsf{d}_{\Omega'}(G) = \sum_{i=1}^{j} (\mathrm{ord}(g_i) - 1) + \sum_{i=r_0+1}^{j} (k_i - 1) + 1.$$

Let

$$T_j' = g_1^{\mathrm{ord}(g_1) - 1} \cdots g_j^{\mathrm{ord}(g_j) - 1} (-g_{r_0+1})^{k_{r_0+1} - 1} \cdots (-g_j)^{k_j - 1}.$$

It is easy to see that T'_j is an Ω'-free sequence of length $|T'_j| = \sum_{i=1}^{j}(\text{ord}(g_i) - 1) + \sum_{i=r_0+1}^{j}(k_i - 1)$. Therefore,

$$\mathsf{d}_{\Omega'}(G) \geq |T'_j| + 1 = \sum_{i=1}^{j}(\text{ord}(g_i) - 1) + \sum_{i=r_0+1}^{j}(k_i - 1) + 1.$$

So it remains to show

$$\mathsf{d}_{\Omega'}(G) \leq \sum_{i=1}^{j}(\text{ord}(g_i) - 1) + \sum_{i=r_0+1}^{j}(k_i - 1) + 1.$$

Let S'_j be any sequence over G with $|S'_j| = \sum_{i=1}^{j}(\text{ord}(g_i) - 1) + \sum_{i=r_0+1}^{j}(k_i - 1) + 1$. We only need to show that there is a zero-sum subsequence of S'_j in Ω'. If there exists $g \in G$ such that $\mathsf{v}_g(S'_j) \geq \text{ord}(g)$, then $g^{\text{ord}(g)} \in \Omega'$, and we are done. Hence, we next assume

$$\mathsf{v}_g(S'_j) \leq \text{ord}(g) - 1$$

for every $g \in G$.

If there exists $i \in [r_0 + 1, j]$ such that $\mathsf{v}_{g_i}(S'_j) \geq k_i$ and $\mathsf{v}_{-g_i}(S'_j) \geq k_i$, then $g_i^{k_i}(-g_i)^{k_i} \in \Omega'$. So we assume that, for every $i \in [r_0 + 1, j]$, there exists $g''_i \in \{g_i, -g_i\}$ such that $\mathsf{v}_{g''_i}(S'_j) \leq k_i - 1$. By renumbering, we may assume

$$\mathsf{v}_{-g_i}(S'_j) \leq k_i - 1$$

for every $i \in [r_0 + 1, j]$. Let

$$T = g_{r+1}^{\mathsf{v}_{g_{r+1}}(S'_j)} \cdots g_j^{\mathsf{v}_{g_j}(S'_j)} (-g_{r_0+1})^{\mathsf{v}_{-g_{r_0+1}}(S'_j)} \cdots (-g_j)^{\mathsf{v}_{-g_j}(S'_j)}.$$

Then

$$S'_j T^{-1} = g_1^{\mathsf{v}_{g_1}(S'_j)} \cdots g_r^{\mathsf{v}_{g_r}(S'_j)} T_1$$

with $\text{supp}(T_1) \cap \{g_1, \ldots, g_j, -g_{r_0+1}, \ldots, -g_j\} = \emptyset$.

Since

$$|S'_j T^{-1}| \geq \mathsf{D}^*(G) = \mathsf{D}(G),$$

$S'_j T^{-1}$ contains a minimal zero-sum subsequence W (say). Because $g_1 = e_1, \ldots, g_r = e_r$ is a basis of G, we infer that $g_1^{\mathsf{v}_{g_1}(S'_j)} \cdots g_r^{\mathsf{v}_{g_r}(S'_j)}$ is zero-sum free. This implies $\text{supp}(W) \cap \text{supp}(T_1) \neq \emptyset$. Now $W \in \Omega'$ follows from $\text{supp}(T_1) \cap \{g_1, \ldots, g_j, -g_{r_0+1}, \ldots, -g_j\} = \emptyset$ and the definition of Ω'. Therefore

$$\mathsf{d}_{\Omega'}(G) = \sum_{i=1}^{j}(\text{ord}(g_i) - 1) + \sum_{i=r_0+1}^{j}(k_i - 1) + 1 \in \mathsf{Vol}(G)$$

follows from the fact that Ω' is weak-regular.

Let

$$g(j, k_1, \ldots, k_j) = \sum_{i=1}^{j}(\text{ord}(g_i) - 1) + \sum_{i=r_0+1}^{j}(k_i - 1) + 1.$$

Note that $g_1 = e_1, \ldots, g_r = e_r$. When j runs over $[r, m+r_0]$ and k_i runs over $[1, \mathrm{ord}(g_i)-1]$ for every $i \in [r_0+1, j]$, $g(j, k_1, \ldots, k_j)$ takes its maximal value $\sum_{g\in G}(\mathrm{ord}(g)-1) - 2^{r'} + 2 - m + r_0$ when $j = m + r_0$ and $k_i = \mathrm{ord}(g_i) - 1$ for every $i \in [r_0+1, m+r_0]$, and $g(j, k_1, \ldots, k_j)$ takes its minimal value $1 + \sum_{i=1}^{r}(n_i - 1)$ when $j = r$ and $k_i = 1$ for every $i \in [r_0+1, r]$. It is easy to see that $g(j, k_1, \ldots, k_j)$ can take any integer in between the minimal value and the maximal value. So

$$I_2 = [1 + \sum_{i=1}^{r}(n_i - 1), \sum_{g\in G}(\mathrm{ord}(g) - 1) - 2^{r'} + 2 - m + r_0] \subset \mathsf{Vol}(G). \tag{4.4}$$

Let

$$A = \sum_{g\in G}(\mathrm{ord}(g) - 1).$$

Now it remains to show

$$I_1 \cup I_2 = [\mathsf{D}(G), \sum_{g\in G}(\mathrm{ord}(g) - 1)].$$

This is equivalent to the inequality

$$A - 2^{r'} + 2 - m + r_0 \geq \frac{A - 2^{r'} + 1}{2} + 2^{r'}.$$

Next we show the following stronger inequality:

$$A - 2^{r'} + 2 - m \geq \frac{A - 2^{r'} + 1}{2} + 2^{r'}. \tag{4.5}$$

Note that $2m = |G| - |G_2|$ and $|G_2| = 2^{r'}$. We obtain that the inequality of (4.5) is equivalent to $A - |G| \geq 2^{r'+1} - 3$. Since $|G| = \sum_{g\in G} 1$, $A - |G| \geq 2^{r'+1} - 3$ is equivalent to

$$\sum_{g\in G}(\mathrm{ord}(g) - 2) \geq 2^{r'+1} - 3,$$

and this is equivalent to

$$\sum_{g\in G\setminus G_2}(\mathrm{ord}(g) - 2) \geq 2^{r'+1} - 2.$$

So we only need to prove the above inequality.

If $r' = 0$, then it is obvious. Next we suppose that $r' \geq 1$. Take $h \in C_{n_r}$ with $\mathrm{ord}(h) = n_r$. Note that $n_r \geq 4$ since $G \neq C_2^r$ and $r' \geq 1$. It follows that

$$\begin{aligned}
\sum_{g\in G\setminus G_2}(\mathrm{ord}(g) - 2) &\geq \sum_{g\in C_{n_1}\oplus\cdots\oplus C_{n_{r-1}}\oplus\{h,-h\}}(\mathrm{ord}(g) - 2) \\
&= \sum_{g\in C_{n_1}\oplus\cdots\oplus C_{n_{r-1}}\oplus\{h,-h\}}(n_r - 2) \\
&= 2n_1 \ldots n_{r-1}(n_r - 2) \geq 2^{r+1} \\
&\geq 2^{r'+1} > 2^{r'+1} - 2.
\end{aligned}$$

This proves the inequality of (4.5), completing the proof. □

Proof of Theorem 1.3. Now the result follows from Lemma 4.1 and Theorem 1.2 (2). □

5 Proof of Theorem 1.4

In this section we will derive some properties on $\mathsf{Q}_t(G)$ and prove Theorem 1.4. We need the following lemmas.

Lemma 5.1 *If G is a finite abelian group with $r(G) \le 2$, then $\mathsf{D}_2(G) = \mathsf{D}(G) + \exp(G)$.*

Proof The result follows from [5, Lemma 3.2] and [7, Theorem 5.8.3]. □

Lemma 5.2 *Let G be a finite abelian group. For any positive integer $t \ge \mathsf{D}_2(G)$, we have $\mathsf{Q}_t(G) = \emptyset$.*

Proof Let $G^* = G \setminus \{0\}$, and $t \ge \mathsf{D}_2(G)$ be an integer. Let

$$\Omega = \{0^{t-\mathsf{D}(G)+1}\} \cup \mathcal{A}(G^*)$$

and

$$\Omega' = \{0^{t-\mathsf{D}_2(G)+1}\} \cup (\mathcal{B}(G^*) \setminus \mathcal{A}(G^*)).$$

It is easy to see that $\mathsf{d}_\Omega(G) = t = \mathsf{d}_{\Omega'}(G)$. On the other hand, note that a minimal zero-sum sequence over G of length $\mathsf{D}(G)$ has no two disjoint nonempty zero-sum subsequences, so we deduce that $\mathsf{D}_2(G) > \mathsf{D}(G)$. Therefore,

$$\Omega \cap \Omega' = \emptyset.$$

Hence, $\mathsf{Q}_t(G) = \cap_{\Omega \subset \mathcal{B}(G), \mathsf{d}_\Omega(G)=t} \Omega = \emptyset$. □

Lemma 5.3 *Let $\Omega \subset \mathcal{B}(G) \setminus \{\mathbb{1}\}$ and $S_1, S_2 \in \Omega$ with $S_1 \neq S_2$. If $S_1 \mid S_2$, then $\mathsf{d}_\Omega(G) = \mathsf{d}_{\Omega \setminus \{S_2\}}(G)$.*

Proof It is clear that $\mathsf{d}_\Omega(G) \le \mathsf{d}_{\Omega\setminus\{S_2\}}(G)$. We next show $\mathsf{d}_{\Omega\setminus\{S_2\}}(G) \le \mathsf{d}_\Omega(G)$. Let U be a sequence over G with $|U| = \mathsf{d}_\Omega(G)$. We only need to show that there is a nonempty zero-sum subsequence in $\Omega\setminus\{S_2\}$. Since $|U| = \mathsf{d}_\Omega(G)$, there exists a nonempty zero-sum subsequence S in Ω. If $S \neq S_2$, then $S \in \Omega\setminus\{S_2\}$, and we are done. Otherwise $S = S_2$. Then $S_2|U$. It follows that $S_1|S_2|U$. Therefore, $S_1 \in \Omega$. Thus, $S_1 \in \Omega\setminus\{S_2\}$ since $S_1 \neq S_2$, completing the proof. □

Lemma 5.4 *Let G be a finite abelian group with $|G| \ge 4$. If S is an essential zero-sum sequence over G with respect to some integer $t \ge \mathsf{D}(G) + 1$, then $S \neq 0$ is a minimal zero-sum sequence.*

Proof Let $G^* = G \setminus \{0\}$ and

$$\Omega = \mathcal{A}(G^*) \cup \{0^{t-\mathsf{D}(G)+1}\}.$$

It is easy to see that

$$\mathsf{d}_\Omega(G) = t.$$

We next distinguish two cases.

Case 1. $G = C_n$, where $n \ge 4$. Take an element $g \in G$ with $\mathrm{ord}(g) = n$. Let

$$\Omega' = (\mathcal{A}(G^*)\setminus\{g^{n-2}(2g)\}) \cup \{0^{t-\mathsf{D}(G)}\}.$$

We want to show

$$\mathsf{d}_{\Omega'}(G) = t.$$

Let

$$U = 0^{t-\mathsf{D}(G)-1}g^{n-1}(2g).$$

It is clear that U is Ω'-free. Therefore, $\mathsf{d}_{\Omega'}(G) \ge |U| + 1 = t$. So it suffices to show $\mathsf{d}_{\Omega'}(G) \le t$. Let

$$U_1 = 0^{t-|T_1|}T_1$$

be a sequence over G of length t, where $0 \notin \mathrm{supp}(T_1)$ and $t - |T_1| \ge 0$. We only need to show that there exists a zero-sum subsequence of U_1 in Ω'. If $t - |T_1| \ge t - \mathsf{D}(G)$, then $0^{t-\mathsf{D}(G)}$ is a zero-sum subsequence of U_1 in Ω', and we are done. Hence, we assume that $t - |T_1| \le t - \mathsf{D}(G) - 1$. Then $|T_1| \ge \mathsf{D}(G) + 1 = n + 1$, and T_1 has a minimal zero-sum subsequence. If $g^{n-2}(2g) \nmid T_1$, we are done. So we may assume that

$$T_1 = g^{n-2}(2g)T_2,$$

where $|T_2| \ge 2$.

If $\mathrm{v}_g(T_2) \ge 2$, then g^n is a minimal zero-sum subsequence of T_1 in Ω'. If $\mathrm{v}_{2g}(T_2) \ge 1$, then $g^{n-4}(2g)^2$ is a minimal zero-sum subsequence of T_1 in Ω'. So we may assume that $\mathrm{v}_g(T_2) \le 1$ and $\mathrm{v}_{2g}(T_2) = 0$. Since $|T_2| \ge 2$, we infer that $\mathrm{v}_{mg}(T_2) \ge 1$ for some $m \in [3, n-1]$, then $(mg)g^{n-m}$ is a minimal zero-sum subsequence of T_1 in Ω'. This proves that $\mathsf{d}_{\Omega'}(G) = t$. Since S is essential with respect to t, we have $S \in \Omega \cap \Omega' \subset \mathcal{A}(G^*)$, completing the proof in this case.

Case 2. G is not cyclic. Then $\mathsf{D}(G) \ge \mathsf{D}^*(G) > \exp(G) \ge \mathrm{ord}(g)$ for every $g \in G$. Let T be a minimal zero-sum sequence over G of length $|T| = \mathsf{D}(G)$, and let

$$\Omega'' = (\mathcal{A}(G^*)\setminus\{T\}) \cup \{0^{t-\mathsf{D}(G)}\},$$

We now show $\mathsf{d}_{\Omega''}(G) = t$. Let

$$U = 0^{t-\mathsf{D}(G)-1}T.$$

Then U is Ω''-free. Therefore, $\mathsf{d}_{\Omega''}(G) \ge |U| + 1 = t$. So it remains to show $\mathsf{d}_{\Omega''}(G) \le t$. Let

$$U_1 = 0^{t-|T_1|}T_1$$

be a sequence over G of length t, where $0 \notin \mathrm{supp}(T_1)$ and $t - |T_1| \ge 0$. We need to show that there exists a zero-sum subsequence of U_1 in Ω''. If $t - |T_1| \ge t - \mathsf{D}(G)$, then we are done. Hence, we assume that $t - |T_1| \le t - \mathsf{D}(G) - 1$. Then $|T_1| \ge \mathsf{D}(G) + 1$. Assume to the contrary that T_1 is an Ω''-free sequence. Let

$$T_2 = g_1 g_2 \cdots g_{\mathsf{D}(G)+1}$$

be a subsequence of T_1 of length $\mathsf{D}(G)+1$. Take an arbitrary subsequence T_3 of T_2 with length $|T_3| = |T_2| - 1 = \mathsf{D}(G)$. Then, T_3 has a minimal zero-sum subsequence T_0. If $|T_0| < \mathsf{D}(G)$, then $T_0 \in \Omega''$, a contradiction. Therefore, $|T_0| = \mathsf{D}(G)$ and $T_3 = T_0$ follows. This proves that $\sigma(T_3) = 0$ for every subsequence T_3 of T_2 with length $|T_3| = |T_2| - 1$. It follows that

$$g_1 = g_2 = \cdots = g_{\mathsf{D}(G)+1} = g_0.$$

Now $g_0^{\mathrm{ord}(g_0)}$ is a minimal zero-sum subsequence of U_1 in Ω'', a contradiction. This proves that $\mathsf{d}_{\Omega''}(G) = t$. Since S is essential with respect to t, we have $S \in \Omega \cap \Omega'' \subset \mathcal{A}(G^*)$, completing the proof. □

Remark 5.5 It is easy to check that $\mathsf{Q}_2(C_2) = \{1^2, 0\}$, $\mathsf{Q}_3(C_2) = \{1^2, 0^2\}$. Moreover, by Lemma 4.1, Lemmas 5.1 and 5.2, we obtain $\mathsf{Q}_t(C_2) = \emptyset$ when $t \ge 4$. Thus,

$$\mathsf{Q}_{t+1}(C_2) \subset \mathsf{Q}_t(C_2)$$

for any positive integer $t \ge \mathsf{D}(C_2) + 1 = 3$. Note that $\mathsf{Q}_3(C_2) \not\subset \mathsf{Q}_2(C_2)$. We will show that this is the only exception that does not satisfy $\mathsf{Q}_{t+1}(G) \subset \mathsf{Q}_t(G)$.

Lemma 5.6 *Let G be a finite abelian group with $|G| \ge 3$. For any positive integer $t \ge \mathsf{D}(G)$, we have $\mathsf{Q}_{t+1}(G) \subset \mathsf{Q}_t(G)$.*

Proof If $|G| = 3$, then $G = C_3$. It is easy to see that

$$\begin{aligned} \mathsf{Q}_3(G) &= \{0, 1^3, 2^3, 12\}, \\ \mathsf{Q}_4(G) &= \{1^3, 2^3\}, \text{ and} \\ \mathsf{Q}_5(G) &= \{1^3, 2^3\}, \end{aligned}$$

and, by Lemmas 4.1, 5.1 and 5.2, we obtain

$$\mathsf{Q}_t(G) = \emptyset$$

when $t \ge 6$. Therefore, $\mathsf{Q}_{t+1}(G) \subset \mathsf{Q}_t(G)$ follows from $|G| = 3$. From now on we assume that

$$|G| \ge 4.$$

Let $S \in \mathsf{Q}_{t+1}(G)$. For every $\Omega \subset \mathcal{B}(G)$ with $\mathsf{d}_\Omega(G) = t \ge \mathsf{D}(G)$, define $k = k(\Omega)$ as the smallest positive integer such that $0^k \in \Omega$. For any $0 \le i \le t - k + 1$, define

$$\Omega^{(i)} = (\Omega \setminus \{0^k, 0^{k+1}, \ldots, 0^{i+k-1}\}) \cup \{0^{i+k}\},$$

where, $\Omega^{(0)} = \Omega$. We next show

$$\mathsf{d}_{\Omega^{(1)}}(G) = t \text{ or } t + 1.$$

By Lemma 5.3, we have $\mathsf{d}_{\Omega \cup \{0^{k+1}\}}(G) = \mathsf{d}_\Omega(G) = t$. Therefore, $\mathsf{d}_{\Omega^{(1)}}(G) \ge \mathsf{d}_{\Omega \cup \{0^{k+1}\}}(G) = t$. So it remains to show $\mathsf{d}_{\Omega^{(1)}}(G) \le t + 1$.

Let U be a sequence over G of length $t+1$. We only need to show that there is a nonempty zero-sum subsequence of U in $\Omega^{(1)}$. Since $|U| = t + 1 > \mathsf{d}_\Omega(G)$, there exists a nonempty zero-sum subsequence T in Ω. If $k > t + 1$, then $|T| \le t + 1 < k$. Hence, $T \ne 0^k$ and $T \in \Omega^{(1)}$, we are done. We may assume that $k \le t + 1$. If $T \ne 0^k$, then $T \in \Omega^{(1)}$, and we are done. Now we assume that $T = 0^k$. Let $U = 0^k U_1$. If $0 \in \mathrm{supp}(U_1)$, then $0^{k+1} \in \Omega^{(1)}$, and we are done. Hence we may assume that $0 \notin \mathrm{supp}(U_1)$. Since $|0^{k-1} U_1| = t$, there is a nonempty zero-sum subsequence T_1 in Ω and $T_1 \ne 0^k$. Therefore, $T_1 \in \Omega^{(1)}$. This proves that $\mathsf{d}_{\Omega^{(1)}}(G) = t$ or $t + 1$.

We argue by induction on i that for each such i, $\mathsf{d}_{\Omega^{(i)}}(G)$ is either t or $t+1$. Based on the fact that 0^t has no nonempty zero-sum subsequence in $\Omega^{(t-k+1)} = (\Omega \setminus \{0^k, 0^{k+1}, \ldots, 0^t\}) \cup \{0^{t+1}\}$, we have $\mathsf{d}_{\Omega^{(t-k+1)}}(G) \geq t+1$. We conclude that there is an $i \leq t-k$ such that $\mathsf{d}_{\Omega^{(i)}}(G) = t$ and $\mathsf{d}_{\Omega^{(i+1)}} = t+1$. Next we argue that, by Lemma 5.4, $S \neq 0^{k+i+1}$, and therefore $S \in \Omega$. From the arbitrariness of Ω we conclude that $S \in \mathsf{Q}_t(G)$. □

Lemma 5.7 *Let G be a finite abelian group with $|G| \geq 3$. A zero-sum sequence S over G is essential with respect to $t \geq \mathsf{D}(G)$ if and only if there exists a sequence W with length $|W| = t$ such that every nonempty zero-sum subsequence of W has the same form with S.*

Proof Sufficiency. Let W be a sequence with $|W| = t$ such that every nonempty zero-sum subsequence of W has the same form with S. Let Ω be any subset of $\mathcal{B}(G)$ such that $\mathsf{d}_\Omega(G) = t$. Then we infer that $S \in \Omega$. Therefore, S is essential with respect to t.

Necessity. Assume to the contrary that every sequence W with length $|W| = t$ has a nonempty zero-sum subsequence S_W with $\mathsf{v}_g(S) \neq \mathsf{v}_g(S_W)$ for some $g \in G$. Let

$$\Omega = \{S_W \mid |W| = t\}.$$

Then it is clear that $\mathsf{d}_\Omega(G) \leq t$ and $S \notin \Omega$. Let $\mathsf{d}_\Omega(G) = t_0$. Then $S \notin \mathsf{Q}_{t_0}(G)$. By Lemma 5.6, we have $S \notin \mathsf{Q}_t(G)$. Therefore, S is not essential with respect to t, a contradiction. □

Lemma 5.8 [3, Theorem 4.4] *If G is a finite abelian group, then $\mathsf{q}(G) \leq \mathsf{D}_2(G)$.*

Proposition 5.9 *If G is a finite abelian group and H is a proper subgroup of G, then*

$$\mathsf{q}'(G) \geq \mathsf{q}'(H) + \mathsf{D}(G/H) - 1.$$

In particular, $\mathsf{q}'(G) > \mathsf{q}'(H)$.

Proof Let S be a sequence over H of length $\mathsf{q}'(H) - 1$ such that every nonempty zero-sum subsequence has the same form. Moreover, let T be a sequence over $G \setminus H$ avoiding a nonempty zero-sum subsequence modulo H with length $|T| = \mathsf{D}(G/H) - 1$. Clearly, each nonempty zero-sum subsequence of ST is in fact a subsequence of S, and therefore has the same form. Hence, $\mathsf{q}'(G) \geq |ST| + 1 = |S| + |T| + 1 = \mathsf{q}'(H) + \mathsf{D}(G/H) - 1$.

Obviously, $\mathsf{D}(G/H) \geq 2$ since H is a proper subgroup of G. Therefore, $\mathsf{q}'(G) > \mathsf{q}'(H)$. □

Proof of Theorem 1.4. (1) Let

$$G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r} = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \cdots \oplus \langle e_r \rangle$$

with $1 < n_1 \mid n_2 \mid \cdots \mid n_r$, and $\mathrm{ord}(e_i) = n_i$ for each $i \in [1, r]$ and

$$S = e_1^{n_1-1} e_2^{n_2-1} \cdots e_{r-1}^{n_{r-1}-1} e_r^{2n_r-1}.$$

It is clear that every nonempty zero-sum subsequence of S has the same form $e_r^{n_r}$. Therefore, $\mathsf{q}'(G) \geq |S| + 1 = \mathsf{D}^*(G) + \exp(G)$. So it remains to show

$$\mathsf{q}'(G) \leq \mathsf{D}(G) + \exp(G).$$

Let S be a sequence over G of length $\mathsf{D}(G) + \exp(G)$. We need to show that S has two nonempty zero-sum subsequences of different forms. Since $|S| > \mathsf{D}(G)$, there exists a nonempty zero-sum subsequence T of S. We now distinguish two cases.

Case 1. $|T| \le \exp(G)$. Then $|ST^{-1}| \ge \mathsf{D}(G)$. Therefore, there is a nonempty zero-sum subsequence T_1 of ST^{-1}. Hence T and TT_1 are two nonempty zero-sum subsequences of S with different forms.

Case 2. $|T| > \exp(G)$. If there is an element $g \in G$ such that $\mathsf{v}_g(T) \ge \exp(G)$, then $g^{\exp(G)}$ and T are two nonempty zero-sum subsequences of S with different forms. If $\mathsf{v}_g(T) < \exp(G)$ for every $g \in G$, then let

$$T = g_1^{k_1} g_2^{k_2} \cdots g_l^{k_l},$$

where $\exp(G) > k_1 \ge \cdots \ge k_l \ge 1$. Since $|S(g_i^{k_i})^{-1}| = |S| - k_i > \mathsf{D}(G)$ for any $i \in [1, l]$, there exists a nonempty zero-sum subsequence T_2 of $S(g_i^{k_i})^{-1}$. If T_2 and T have different forms, we are done. Otherwise, all nonempty zero-sum subsequences of $S(g_i^{k_i})^{-1}$ have the same form with T, then $\mathsf{v}_{g_i}(S) \ge 2\mathsf{v}_{g_i}(T)$ for every $i \in [1, l]$. Therefore, there are two nonempty zero-sum subsequences T, T^2 of S with different forms.

(2) Consider first $G = C_2$, then $\mathsf{q}(G) = 4$ by Remark 5.5. One readily checks that $\mathsf{q}'(C_2) = 4$ also holds, so in the sequel $|G| \ge 3$ may be assumed. Then $\mathsf{q}'(G) \ge \mathsf{q}(G)$ by Lemma 5.7, so one only has to show the reverse inequality

$$\mathsf{q}(G) \ge \mathsf{q}'(G).$$

Note that no minimal zero-sum sequence over G of length $\mathsf{D}(G)$ has two nonempty zero-sum subsequences with different forms. Thus, the inequality $\mathsf{q}'(G) \ge \mathsf{D}(G) + 1$ holds. Let S be a sequence with $|S| = \mathsf{q}'(G) - 1$ such that every nonempty zero-sum subsequence of S has the same form with T. Let $t = |S|$. Then $t \ge \mathsf{D}(G)$. By Lemma 5.7, we obtain that T is essential with respect to t. Therefore, $\mathsf{Q}_t(G) \ne \emptyset$. We assert that $\mathsf{Q}_k(G) \ne \emptyset$ holds for every $k \in [\mathsf{D}(G), t]$. In fact, if there exists $k \in [\mathsf{D}(G), t-1]$ such that $\mathsf{Q}_k(G) = \emptyset$, then by Lemma 5.6, we have $\mathsf{Q}_t(G) \subset \mathsf{Q}_k(G) = \emptyset$, a contradiction. Hence, $\mathsf{q}(G) \ge t + 1 = |S| + 1 = \mathsf{q}'(G)$.

(3). The result follows from (1) and (2). □

By Theorem 1.4 and [5, Lemma 3.2], we obtain the following result.

Corollary 5.10 *If $\mathsf{D}(G) = \mathsf{D}^*(G)$ and $\eta(G) \le \mathsf{D}(G) + \exp(G)$, then*

$$\mathsf{q}(G) = \mathsf{q}'(G) = \mathrm{disc}(G) = \mathsf{D}_2(G) = \mathsf{D}(G) + \exp(G).$$

We end this section with the following

Conjecture 5.11 *For any finite abelian group G,*

$$\mathsf{Vol}(G) = [\mathsf{D}(G), 1 + \sum_{g \in G} (\mathrm{ord}(g) - 1)].$$

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