On list 3-dynamic coloring of near-triangulations

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Abstract

An r-dynamic k-coloring of a graph G is a proper k-coloring such that for any vertex v, there are at least $\min\{r, \deg_G(v)\}$ distinct colors in $N_G(v)$. The *r-dynamic chromatic number* $\chi_r^d(G)$ of a graph G is the least k such that there exists an r-dynamic k-coloring of G. The *list r-dynamic chromatic number* of a graph G is denoted by $ch_r^d(G)$. Loeb et al. [10] showed that $ch_3^d(G) \leq 10$ for every planar graph G, and there is a planar graph G with $\chi_3^d(G) = 7$.

In this paper, we study a special class of planar graphs which have better upper bounds of $ch_3^d(G)$. We prove that $ch_3^d(G) \leq 6$ if G is a planar graph which is a near-triangulation, where a near-triangulation is a planar graph whose bounded faces are all 3-cycles.

Keywords: list r-dynamic coloring, planar graphs, triangulation, near-triangulation

1. Introduction

Let k be a positive integer. A proper k-coloring $\phi : V(G) \to \{1, 2, \ldots, k\}$ of a graph G is an assignment of colors to the vertices of G so that any two adjacent vertices receive distinct colors. The *chromatic number* $\chi(G)$ of a graph G is the least k such that there exists a proper k-coloring of G. An *r-dynamic k-coloring* of a graph G is a proper k-coloring ϕ such that for each vertex $v \in V(G)$, either the number of distinct colors in its neighborhood is at least r or the colors in its neighborhood are all distinct, that is, $|\phi(N_G(v))| \geq \min\{r, \deg_G(v)\}$. The *$r$-dynamic chromatic number* $\chi_r^d(G)$ of a graph G is the least k such that there exists an r-dynamic k-coloring of G.

A *list assignment* on a graph G is a function L that assigns each vertex v a set $L(v)$ which is a list of available colors at v. For a list assignment L of a graph G, we say G is

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L-colorable if there exists a proper coloring ϕ such that $\phi(v) \in L(v)$ for every $v \in V(G)$. A graph G is said to be *k-choosable* if for any list assignment L such that $|L(v)| \geq k$ for every vertex v, G is L-colorable.

For a list assignment L of G, we say that G is *r-dynamically L-colorable* if there exists an r-dynamic coloring ϕ such that $\phi(v) \in L(v)$ for every $v \in V(G)$. A graph G is *r-dynamically k-choosable* if for any list assignment L with $|L(v)| \geq k$ for every vertex v, G is r-dynamically L-colorable. The *list r-dynamic chromatic number* or the *r-dynamic choice number* $ch_r^d(G)$ of a graph G is the least k such that G is r-dynamically k-choosable.

An interesting property of dynamic coloring is as follows.

$$\chi(G) \leq \chi_2^d(G) \leq \cdots \leq \chi_\Delta^d(G) = \chi(G^2),$$

where G^2 is the square of the graph G.

The dynamic coloring was first introduced in [8, 11]. On the other hand, Wegner [14] conjectured that if G is a planar graph, then

$$\chi_\Delta^d(G) \leq \begin{cases} 7, & if\ \Delta(G) = 3; \\ \Delta(G) + 5, & if\ 4 \leq \Delta(G) \leq 7; \\ \lfloor \frac{3\Delta(G)}{2} \rfloor + 1, & if\ \Delta(G) \geq 8. \end{cases}$$

Lai et al. [12] posed a similar conjecture about dynamic coloring of planar graphs as follows.

Conjecture 1.1 *Let G be planar graph. Then*

$$\chi_r^d(G) \leq \begin{cases} r + 3, & if\ 1 \leq r \leq 2; \\ r + 5, & if\ 3 \leq r \leq 7; \\ \lfloor \frac{3r}{2} \rfloor + 1, & if\ r \geq 8. \end{cases}$$

Lai et al. [13] showed that conjecture 1.1 is true for planar graphs with girth at least 6. For the special case $r = 2$, Kim et al. [6] proved that $\chi_2^d(G) \leq 4$ for every planar graph except C_5 and $ch_2^d(G) \leq 5$ for every planar graph. And it was shown in [10] that $ch_3^d(G) \leq 10$ if G is a planar graph. Besides, some special classes of graphs are also investigated, such as sparse graphs [2], bipartite graphs [3], grids [4, 5], $K_{1,3}$-free graphs [9] and K_4-minor free graphs [12]. In terms of the maximum average degree, there is also a result published in [7].

Loeb et al. [10] showed $ch_3^d(G) \leq 10$ if G is a planar graph. On the other hand, there is a planar graph F such that $\chi_3^d(F) = 7$. So Loeb et al. [10] proposed the following problem.

Problem 1 ([10]) What is $\chi_3^d(G)$ if G is a planar graph? And what is $ch_3^d(G)$ if G is a planar graph?

Currently, we have the following bounds.

$$7 \leq \max \{\chi_3^d(G) : G \text{ is a planar graph } \} \leq 10. \quad (1)$$

It is natural to consider a special class of planar graphs for Problem 1. Recently, Asayama et al. [1] showed that $\chi_3^d(G) \leq 5$ if G is a triangulated planar graph, and the upper bound is sharp. But, we do not know yet whether $ch_3^d(G) \leq 5$ or not, if G is a triangulated planar graph. So the following question is still open and it would be interested to answer.

Question 1 Is it true that $ch_3^d(G) \leq 5$ if G is a triangulated planar graph?

Since there is a gap (1) for the general case of planar graphs, it would be interesting to study list 3-dynamic chromatic number $ch_3^d(G)$ for a special class of planar graphs. In this paper, we consider a near-triangulation where a *near-triangulation* is a planar graph whose bounded faces are all 3-cycles and outer face is bounded by a cycle. Note that a triangulated planar graph is a special case of a near-triangulation. First, we show the following theorem.

Theorem 1.2 *If G is a near-triangulation, then $ch_3^d(G) \leq 6$.*

And we obtain the following corollary.

Corollary 1.3 *If G is a triangulated planar graph, then $ch_3^d(G) \leq 6$.*

Let W_n be the wheel with $n+1$ vertices such that W_n is obtained from an n-cycle by adding a new vertex u and joining u and every vertex on the n-cycle. The following can be easily checked.

Proposition 1.4 *$ch_3^d(W_n) \leq 6$ for every positive integer $n \geq 3$ and $ch_3^d(W_5) = 6$.*

Note that Proposition 1.4 and Theorem 1.2 imply that the upper bound of list 3-dynamic chromatic number of near triangulations is tight. And Corollary 1.3 and [1] imply that

$$5 \leq \max \{ch_3^d(G) : G \text{ is a triangulated planar graph } \} \leq 6.$$

2. Proof of Theorem 1.2

Suppose that Theorem 1.2 does not hold, and let G be a minimal counterexample in terms of the number $\sigma(G) = |V(G)| + |E(G)|$ to Theorem 1.2. Let $C : v_0v_1 \cdots v_{t-1}v_0$ in counter-clockwise order be the boundary of the outer face of a plane graph G. If $|V(G)| \leq 6$, then it is easy to obtain $ch_3^d(G) \leq 6$, a contradiction. Hence we have $|V(G)| \geq 7$.

First, we prove the following Claim.

Claim 1 For any $v \in V(C)$, we have that $d_G(v) \geq 4$.

Proof. Suppose that there is a vertex $v_k \in V(C)$ with $d_G(v_k) \leq 3$. Let $u_0, u_1, \ldots, u_{s-1}$ denote the neighbors of v_k in counter-clockwise order such that $v_ku_iu_{i+1}$ is a 3-face for each $i \in \{0, 1, \ldots, s-2\}$. And let $u_0 = v_{k+1}$ ($k+1$ are computed by modulo t).

If $d_G(v_k) = 2$ or $d_G(v_k) = 3$ with $u_0u_2 \in E(G)$, then we remove v_k from G and call the resulting graph by G'. If $d_G(v_k) = 3$ and $u_0u_2 \notin E(G)$, then we remove v_k from G and add the edge u_0u_2 in the outer face, and call the resulting graph by G'. Clearly, for all cases above, G' is a near-triangulation.

Let $L'(v) = L(v)$ for every $v \in V(G')$. Since G is a minimal counterexample, G' has a 3-dynamic L'-coloring ϕ.

If $d_G(v_k) = 2$, then there exists a vertex u_0' such that $u_0' \in (N_G(u_0) \cap N_G(u_1)) \setminus \{v_k\}$ since $|V(G)| \geq 7$. Then we color v_k by a color $c \in L(v_k) \setminus \{\phi(u_0'), \phi(u_0), \phi(u_1)\}$, and we obtain that G has a 3-dynamic coloring from the list assignment L, a contradiction.

If $d_G(v_k) = 3$, then the vertices u_0, u_1 and u_2 receive distinct colors under the coloring ϕ. Suppose $u_0u_2 \in E(G)$. Then we color v_k by a color $c \in L(v_k) \setminus \{\phi(u_0), \phi(u_1), \phi(u_2)\}$, and we obtain that G has a 3-dynamic coloring from the list assignment L. This is a contradiction. Hence suppose that $u_0u_2 \notin E(G)$. If there is a vertex u_i for $i \in \{0, 2\}$ such that $\phi(N_G(u_i))$ has at most two different colors, then we must color v_k by a color $c \in L(v_k) \setminus (\phi(N_G(v_k)) \cup \phi(N_G(u_i)))$ so that vertex u_i satisfies the conditions of 3-dynamic coloring. Then one can easily check that the number of forbidden colors at v_k is at most 5 as follows.

Let S be the set consisting of the forbidden colors at v_k. If $|\phi(N_G(u_0))| \geq 3$ and $|\phi(N_G(u_2))| \geq 3$, then $S = \{\phi(u_0), \phi(u_1), \phi(u_2)\}$. If $|\phi(N_G(u_i))| \leq 2$ and $|\phi(N_G(u_j))| \geq 3$ for $\{i, j\} = \{0, 2\}$, then $S = \{\phi(u_0), \phi(u_1), \phi(u_2)\} \cup \phi(N_G(u_i))$. If $|\phi(N_G(u_0))| \leq 2$ and $|\phi(N_G(u_2))| \leq 2$, then $S = \{\phi(u_0), \phi(u_1), \phi(u_2)\} \cup \phi(N_G(u_0)) \cup \phi(N_G(u_2))$. Since $u_1 \in N_G(u_0) \cap N_G(u_2)$, we can easily obtain $|S| \leq 5$ for all cases above.

Thus we can color v_k by a color $c \in L(v_k) \setminus S$ so that G has a 3-dynamic coloring from the list assignment L, and it implies that G is 3-dynamically L-colorable. This is a contradiction, which completes the proof of Claim 1. $\square$

Next, we prove the following Claim.

Claim 2 For any $w \in V(G) \setminus V(C)$, we have that $d_G(w) \geq 6$.

Proof. Suppose that there is a vertex w with $d_G(w) \leq 5$. Let $w_0, w_1, \ldots, w_{s-1}$ denote the neighbors of w in counter-clockwise order.

Suppose $d_G(w) = 3$. We remove w from G and call the resulting graph by G'. Let $L'(v) = L(v)$ for every $v \in V(G')$. Since G is a minimal counterexample, G' has a 3-dynamic L'-coloring ϕ. So, we can color w by a color $c \in L(w) \setminus \phi(N_G(w))$ so that G has a 3-dynamic coloring from the list assignment L since $|L(v)| \geq 6$ for each $v \in V(G)$, a contradiction.

Now we suppose $4 \leq d_G(w) \leq 5$. With Claim 1 and the preceding paragraph, we suppose $d_G(v) \geq 4$ for each $v \in V(G)$. Then we remove w from G and add edges in the face formed by $\{w_0, w_1, \ldots, w_{s-1}\}$ so that the resulting graph, denoted by G', is a near-triangulation. Let $L'(v) = L(v)$ for every $v \in V(G')$. Since G is a minimal counterexample, G' has a 3-dynamic L'-coloring ϕ. Clearly, we have that $|\phi(N_G(w))| \geq 3$. If $N_G(w) = \{w_0, w_1, \ldots, w_{s-1}\}$ has all different colors in the coloring ϕ, then we color w by a color $c \in L(w) \setminus \{\phi(w_i) : 0 \leq i \leq s-1\}$. Then this gives a 3-dynamic coloring from its list assignment L, a contradiction.

Next, we consider the case when $N_G(w) = \{w_0, w_1, \ldots, w_{s-1}\}$ has less than s colors. Let $S = \{w_i \in N_G(w) | \phi(w_{i-1}) = \phi(w_{i+1})\}$. Since $4 \leq d_G(w) \leq 5$ and $|\phi(N_G(w))| < s$, we have that $S \neq \emptyset$ in this case. Note that G is a near-triangulation and $d_G(v) \geq 4$ for each $v \in V(G)$. So for each $w_i \in S$, we can select a vertex w_i' such that $w_i' \in \big(N_G(w_i) \cap (N_G(w_{i-1}) \cup N_G(w_{i+1}))\big) \setminus \{w, w_{i-1}, w_{i+1}\}$ ($i-1$ and $i+1$ are computed by modulo s). Clearly, $\phi(w_i') \neq \phi(w_{i-1})$ or $\phi(w_i') \neq \phi(w_{i+1})$ since ϕ is a proper coloring. Now let $S' = \{w_i' | w_i \in S\}$.

Since $S \neq \emptyset$, we have that $|S'| \geq 1$. And it is easy to check that $|S| \leq 2$ and $|S'| \leq 2$ since $4 \leq d_G(w) \leq 5$ and $|\phi(N_G(w))| \geq 3$. Moreover, if $|S| = 1$, then $|S'| = 1$ and $|\phi(N_G(w))| \leq 4$. If $|S| = 2$, then $|S'| \leq 2$ and $|\phi(N_G(w))| \leq 3$. So for both cases above, we obtain that $L(w) \setminus (\{\phi(w_i') : w_i' \in S'\} \cup \phi(N_G(w)) \neq \emptyset$ since $|L(w)| \geq 6$. Then we color w by a color $c \in L(w) \setminus (\{\phi(w_i') : w_i' \in S'\} \cup \phi(N_G(w))$. Clearly, there are at least 3 distinct colors in $N_G(w)$ and at least 3 distinct colors in $N_G(w_i)$ for each $w_i \in N_G(w) \setminus S$. For each vertex $w_i \in S$, we have that $\phi(w_{i-1}) = \phi(w_{i+1})$ and then $\phi(w_i') \neq \phi(w_{i-1}) \neq c$. So each $w_i \in S$ also satisfies the conditions of 3-dynamic coloring. Thus we obtain that G has a 3-dynamic coloring from the list assignment L, which is a contradiction since G is a counterexample. This completes the proof of Claim 2. □

Let k be the number of vertices in $V(G) \setminus V(C)$. Then $n(G) = t + k$ since $|V(C)| = t$. Now from Claim 1 and Claim 2, we have

$$2e(G) = \sum_{v \in V(G)} d_G(v) = \sum_{v \in V(C)} d_G(v) + \sum_{v \in V(G) \setminus V(C)} d_G(v) \geq 4t + 6k. \tag{2}$$

And since G is a near-triangulation, we have

$$e(G) = 3n(G) - 6 - (|V(C)| - 3) = 3n(G) - t - 3 = 2t + 3k - 3. \tag{3}$$

So, by (2) and (3)

$$4t + 6k - 6 = 2e(G) \geq 4t + 6k \implies -6 \geq 0,$$

which is a contradiction. This completes the proof of Theorem 1.2. □

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