GENERALIZATIONS OF MOCK THETA FUNCTIONS

SU-PING CUI, NANCY S.S. GU, CHEN-YANG SU, AND MATTHEW H.Y. XIE

Abstract. Mock theta functions were first introduced by Ramanujan in his last letter to Hardy. Moreover, some other mock theta functions were presented in his lost notebook. It is well known that all the classical mock theta functions can be expressed by the universal mock theta functions $g_2(x,q)$ and $g_3(x,q)$. In this paper, we establish some generalized mock theta functions and express them in terms of Appell–Lerch sums. Meanwhile, we show that some of Ramanujan's two-parameter mock theta functions are the special cases of these functions.

1. Introduction

In this paper, the following standard q-series notation [20] are needed. For positive integers n and m,

$$\begin{aligned}&(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),\\&(a_1,a_2,\ldots,a_m;q)_n := (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,\\&(a_1,a_2,\ldots,a_m;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.\end{aligned}$$

Define

$$j(x;q) := (x;q)_\infty(q/x;q)_\infty(q;q)_\infty,$$

$$J_{a,m} := j(q^a;q^m), \quad \overline{J}_{a,m} := j(-q^a;q^m), \quad J_m := J_{m,3m} = \prod_{i=1}^{\infty}(1-q^{mi}).$$

In his last letter to Hardy, Ramanujan [33] provided a list of seventeen functions which were called mock theta functions. This list includes the third, fifth, and seventh order mock theta functions. Later, more mock theta functions were found in Ramanujan's lost notebook, such as the sixth and tenth order mock theta functions. Furthermore, some other mock theta functions were discovered. In [3], Andrews established some third order mock theta functions by using q-orthogonal polynomials. Later, Bringmann, Hikami, and Lovejoy [8] found two new third order mock theta functions. In [6], Berndt and Chan found two new sixth order mock theta functions and built some linear relations between these two functions and the other sixth order mock theta functions given by Ramanujan. In 2000, Motivated by asymptotics of some q-series, Gordon and McIntosh [21] discovered eight eighth order mock theta functions. Later, some identities relate to the second and eighth order mock theta functions were established by McIntosh [28]. For the development of the classical mock theta functions, one can refer to the survey [22] and the book [4]. Generally, mock theta functions can be expressed in terms of Eulerian forms, Hecke-type double sums, Appell-Lerch sums, and

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Fourier coefficients of meromorphic Jacobi forms. Until now, properties and representations of mock theta functions have attracted many mathematicians.

In this paper, we mainly focus on multiple parameter mock theta functions. First, we introduce $g_2(x,q)$ and $g_3(x,q)$, where

$$g_2(x,q)=\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{(n^2+n)/2}}{(x,x^{-1}q;q)_{n+1}}, \tag{1.1}$$

$$g_3(x,q)=x^{-1}\left(-1+\sum_{n=0}^{\infty}\frac{q^{n^2}}{(x;q)_{n+1}(x^{-1}q;q)_n}\right)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(x,x^{-1}q;q)_{n+1}}.$$

It is well known that the odd order mock theta functions can be expressed by $g_3(x,q)$ and the even order mock theta functions are related to $g_2(x,q)$. In 2012, the fact that $g_3(x,q)$ can be represented in terms of $g_2(x,q)$ was provided by Gordon and McIntosh [22, Eq. (6.1)]. Customarily, $g_2(x,q)$ and $g_3(x,q)$ are called universal mock theta functions. In addition, some other two-parameter mock theta functions were considered in the literature [19, 26, 29]. For example,

$$N(x,q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(xq,x^{-1}q;q)_n}, \qquad K(x,q)=\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n q^{n^2}}{(xq^2,x^{-1}q^2;q^2)_n},$$

$$K_1(x,q)=\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n q^{(n+1)^2}}{(xq,x^{-1}q;q^2)_{n+1}}, \qquad K_2(x,q)=\sum_{n=0}^{\infty}\frac{(-1;q)_n q^{(n^2+n)/2}}{(xq,x^{-1}q;q)_n},$$

$$S_2(x;q)=(1+x^{-1})\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{n+1}}{(xq,x^{-1}q;q^2)_{n+1}}, \quad S_4(x;q)=(1+x^{-1})\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n}{(-x;q)_{n+1}(-x^{-1}q;q)_n}.$$

For more properties of these mock theta functions, one can refer to [7, 9–11, 18, 27, 30].

At the beginning of the research on mock theta functions, Hecke-type double sums play a very important role in the proofs of mock theta function identities. For example, Hickerson [23, 24] proved the mock theta conjectures by using Hecke-type double sums for the fifth and seventh order mock theta functions given by Andrews [2]. For Hecke-type double sums for the other mock theta functions, one can see [3, 5, 6, 12–16, 19, 31]. In 2014, Hickerson and Mortenson [25] provided the following definition of Appell–Lerch sums.

Definition 1.1. *Let $x,\ z\in\mathbb{C}^*:=\mathbb{C}\backslash\{0\}$ with neither z nor xz an integral power of q. Then*

$$m(x,q,z)=\frac{1}{j(z;q)}\sum_{r=-\infty}^{\infty}\frac{(-1)^r q^{\binom{r}{2}}z^r}{1-q^{r-1}xz}.$$

Changing r to $r+1$ in the above series yields another form of $m(x,q,z)$:

$$m(x,q,z)=\frac{-z}{j(z;q)}\sum_{r=-\infty}^{\infty}\frac{(-1)^r q^{\binom{r+1}{2}}z^r}{1-q^r xz}. \tag{1.2}$$

They [25] expressed all the classical mock theta functions and the universal mock theta functions in terms of Appell–Lerch sums. For example,

$$g_2(x,q)=\frac{1}{J_{1,2}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1-xq^n}=-x^{-1}m(x^{-2}q,q^2,x). \tag{1.3}$$

From the viewpoint of Appell-Lerch sums, identities for mock theta functions are more easily obtained than before. Moreover, Hickerson and Mortenson [25] showed some properties of Appell–Lerch sums.

Proposition 1.2. *[25] For generic $x, z \in \mathbb{C}^*$,*

$$m(x,q,z) = m(x,q,zq), \tag{1.4}$$

$$m(x,q,z) = x^{-1}m(x^{-1},q,z^{-1}). \tag{1.5}$$

Following [25], the term "generic" means that the parameters do not cause poles in the Appell–Lerch sums.

In [32], Mortenson established some two-parameter mock theta functions in terms of Appell–Lerch sums. Some of the identities were first proved in [1]. For example, Mortenson showed that

$$(1+x^{-1})\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^nq^{(n+1)^2}}{(-xq,-x^{-1}q;q^2)_{n+1}} = m(x,q,-1) - \frac{J_{1,2}^2}{2j(-x;q)}, \tag{1.6}$$

$$(1+x^{-1})\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{n+1}}{(xq,x^{-1}q;q^2)_{n+1}} = -m(x,q^2,q), \tag{1.7}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^nq^{n^2}}{(-x;q^2)_{n+1}(-x^{-1}q^2;q^2)_n} &= m(x,q,-1) + \frac{J_{1,2}^2}{2j(-x;q)}\\ &= 2m(x,q,-1) - m(x,q,\sqrt{-x^{-1}q})\\ &= m(-x^2q,q^4,-q^{-1}) - xq^{-1}m(-x^2q^{-1},q^4,-q),\end{aligned}$$

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n}{(-x;q)_{n+1}(-x^{-1}q;q)_n} = m(x,q,-1),$$

$$\sum_{n=0}^{\infty}\frac{(q^2;q^4)_n(-1)^nq^{2n^2}}{(-x;q^4)_{n+1}(-x^{-1}q^4;q^4)_n} = m(x,q^2,q) + \frac{\overline{J}_{1,4}^2j(-xq^2;q^4)}{j(-x;q^4)j(xq;q^2)}.$$

In this paper, let m, s, and t be integers. In view of some q-series identities and properties of Appell–Lerch sums, we establish some generalized mock theta functions and express them in terms of Appell–Lerch sums. The main results are stated as follows.

Theorem 1.3. *For $m \geq 1$ and $s \leq 1$, we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_{n+s-1}q^{n^2+(2m-2s+1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}\\ &= \sum_{k=0}^{m-2s}(-1)^{\frac{m-s-\delta+1}{2}}q^{\frac{\delta^2-(m-s+1)^2}{2}+(-m+3s+2k-2)\delta}A_{k,m,2s}^{(1)}m(q^{-2m+6s+4k-2\delta},q^4,-q^{m-3s-2k-\delta+1})\\ &\quad+(-1)^{\frac{m-s+\delta+1}{2}}x^{\delta}q^{\frac{\delta^2-(m-s+1)^2}{2}+(s-2)\delta}B_{m,2s}^{(1)}m(x^2q^{2s-2\delta},q^4,x^{-1}q^{1-s-\delta}),\end{aligned} \tag{1.8}$$

and for $m \geq 1$ and $s \leq 0$, we have

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+s}q^{n^2+(2m-2s)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}$$

$$
\begin{aligned}
&= \sum_{k=0}^{m-2s-1} (-1)^{\frac{m-s-\delta+1}{2}} q^{\frac{-(m-s)^2-(m-s)}{2}} A^{(1)}_{k,m,2s+1} m(q^{-2m+6s+4k+3}, q^4, q^{1+2\delta}) \\
&\quad + \sum_{k=0}^{m-2s-1} (-1)^{\frac{m-s+\delta-1}{2}} q^{\frac{-(m-s)^2-3m+7s+4k}{2}} A^{(1)}_{k,m,2s+1} m(q^{-2m+6s+4k+1}, q^4, q^{3-2\delta}) \\
&\quad + (-1)^{\frac{m-s-\delta+1}{2}} q^{\frac{-(m-s)^2-(m-s)}{2}} B^{(1)}_{m,2s+1} m(x^2 q^{2s+1}, q^4, q^{1+2\delta}) \\
&\quad + (-1)^{\frac{m-s+\delta+1}{2}} x q^{\frac{-(m-s)^2-m+3s-2}{2}} B^{(1)}_{m,2s+1} m(x^2 q^{2s-1}, q^4, q^{3-2\delta}), \qquad (1.9)
\end{aligned}
$$

where

$$
\delta := \begin{cases} 1, & m \equiv s \pmod 2, \\ 0, & m \equiv s+1 \pmod 2, \end{cases}
$$

$$
A^{(1)}_{k,m,s} := \frac{(-q^{-s-2k+2}; q^2)_{m-1}}{(q^{-2k}; q^2)_k (q^2; q^2)_{m-s-k} (1 + xq^{m-s-2k})}, \qquad (1.10)
$$

$$
B^{(1)}_{m,s} := \frac{(x^{-1} q^{-m+2}; q^2)_{m-1}}{(-x^{-1} q^{-m+s}; q^2)_{m-s+1}}. \qquad (1.11)
$$

Remark: Notice that here and in what follows, we define $\sum_{i=k}^{j} = 0$ if $k > j$.

Theorem 1.4. *For $m \geq \max\{1, 2s, 2t, 2s+2t-1, s+t+1\}$, we have*

$$
\begin{aligned}
&\frac{q^{(m-s-t)^2+(m-s-t)}}{2(-q^2; q^2)_{m-s-t-1}} \sum_{n=0}^{\infty} \frac{(-q; q^2)_{n+s} (q; q^2)_{n+t} (-1)^n q^{(2m-2s-2t)n}}{(xq^m, x^{-1} q^m; q^2)_{n+1}} \\
&= \sum_{k=1}^{m-2s} A^{(2)}_{k,m,s,t} m(q^{-2m+4s+2t+2k-1}, q^2, -1) + \sum_{k=1}^{m-2t} B^{(2)}_{k,m,s,t} m(-q^{-2m+2s+4t+2k-1}, q^2, -1) \\
&\quad + C^{(2)}_{m,s,t} m(-xq^{2s+2t-m}, q^2, -1),
\end{aligned}
$$

where

$$
A^{(2)}_{k,m,s,t} := \frac{(-q^{-2s-2k+3}; q^2)_{m-1}}{(q^{-2k+2}; q^2)_{k-1} (q^2; q^2)_{m-2s-k} (-q^{-2s+2t-2k+2}; q^2)_{m-2t} (1 + xq^{m-2s-2k+1})}, \qquad (1.12)
$$

$$
B^{(2)}_{k,m,s,t} := \frac{(q^{-2t-2k+3}; q^2)_{m-1}}{(q^{-2k+2}; q^2)_{k-1} (q^2; q^2)_{m-2t-k} (-q^{2s-2t-2k+2}; q^2)_{m-2s} (1 - xq^{m-2t-2k+1})}, \qquad (1.13)
$$

$$
C^{(2)}_{m,s,t} := \frac{(x^{-1} q^{-m+2}; q^2)_{m-1}}{(-x^{-1} q^{-m+2s+1}; q^2)_{m-2s} (x^{-1} q^{-m+2t+1}; q^2)_{m-2t}}. \qquad (1.14)
$$

Theorem 1.5. *For $m \geq \max\{1, 2s-1, 2t, 2s+2t-2, s+t\}$, we have*

$$
\begin{aligned}
&\frac{(-1)^{m-s-t+1} q^{(m-s-t)^2+2(m-s-t)+1}}{(q; q^2)_{m-s-t}} \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_{n+s-1} (-q; q^2)_{n+t} q^{(2m-2s-2t+1)n}}{(xq^m, x^{-1} q^m; q^2)_{n+1}} \\
&= \sum_{k=0}^{m-2s} A^{(3)}_{k,m,s,t} m(-q^{-2m+4s+2t+2k-1}, q^2, q) + \sum_{k=1}^{m-2t} B^{(3)}_{k,m,s,t} m(-q^{-2m+2s+4t+2k-2}, q^2, q) \\
&\quad + C^{(3)}_{m,s,t} m(xq^{-m+2s+2t-1}, q^2, q),
\end{aligned}
$$

where

$$
A^{(3)}_{k,m,s,t} := \frac{(-q^{-2s-2k+2}; q^2)_{m-1}}{(q^{-2k}; q^2)_k (q^2; q^2)_{m-2s-k} (q^{-2s+2t-2k+1}; q^2)_{m-2t} (1 + xq^{m-2s-2k})}, \qquad (1.15)
$$

$$B^{(3)}_{k,m,s,t} := \frac{(-q^{-2t-2k+3};q^2)_{m-1}}{(q^{-2k+2};q^2)_{k-1}(q^2;q^2)_{m-2t-k}(q^{2s-2t-2k+1};q^2)_{m-2s+1}(1+xq^{m-2t-2k+1})}, \quad (1.16)$$

$$C^{(3)}_{m,s,t} := \frac{(x^{-1}q^{-m+2};q^2)_{m-1}}{(-x^{-1}q^{-m+2s};q^2)_{m-2s+1}(-x^{-1}q^{-m+2t+1};q^2)_{m-2t}}. \quad (1.17)$$

Notice that the left-hand side of (1.8) generalizes $g_2(x;q)$ which is defined in (1.1). Letting $(m,s,q,x) \to (1,1,q^{1/2},xq^{-1/2})$ in (1.8), and then using (1.5), we obtain (1.3). Meanwhile, another representation of (1.6) can be derived by setting $(m,s) \to (1,0)$ in (1.9). In Theorem 1.5, we set $(m,s,t) \to (1,1,0)$ to obtain (1.7).

In addition, in view of the q-Zeilberger algorithm, Cui, Gu, Hou, and Su [17] derived some generalizations of mock theta functions. It should be pointed out that the main results in this paper are the generalizations of Theorems 1.3-1.7 in [17]. For example, by (1.8) with $s=1$, we see that for $m \geq 1$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2+(2m-1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}} \\
&= \sum_{k=0}^{m-2} (-1)^{\frac{m-\delta}{2}} q^{\frac{\delta^2-m^2}{2}+(-m+2k+1)\delta} A^{(1)}_{k,m,2} m(q^{-2m+4k-2\delta+6},q^4,-q^{m-2k-\delta-2}) \\
&\quad + (-1)^{\frac{m+\delta}{2}} x^{\delta} q^{\frac{\delta^2-m^2}{2}-\delta} B^{(1)}_{m,2} m(x^2q^{2-2\delta},q^4,x^{-1}q^{-\delta}) \\
&= \sum_{k=1}^{m-1} (-1)^{\frac{m-\delta}{2}} q^{\frac{\delta^2-m^2}{2}+(-m+2k-1)\delta} A^{(1)}_{k-1,m,2} m(q^{-2m+4k-2\delta+2},q^4,-q^{m-2k-\delta}) \\
&\quad + (-1)^{\frac{m+\delta}{2}} x^{\delta} q^{\frac{\delta^2-m^2}{2}-\delta} B^{(1)}_{m,2} m(x^2q^{2-2\delta},q^4,x^{-1}q^{-\delta}), \quad (1.18)
\end{aligned}$$

where we derive the last step by replacing k by $k-1$. Then from the definition of δ, we deduce that

$$\delta = \frac{1-(-1)^m}{2}. \quad (1.19)$$

So,

$$\delta^2 = \frac{1-(-1)^m}{2}. \quad (1.20)$$

Thus, substituting (1.19) and (1.20) into (1.18) yields Theorem 1.4 in [17]. Similarly, setting $s=0$ in (1.9), we derive Theorem 1.3 in [17] for $m \geq 1$. Moreover, Theorem 1.4 with $s=t=0$ implies Theorem 1.5 in [17] for $m \geq 1$. Furthermore, by setting $(s,t)=(1,0)$ and $(s,t)=(0,0)$ in Theorem 1.5, respectively, we obtain Theorems 1.6 and 1.7 in [17].

This paper is organized as follows. In Section 2, some lemmas are stated. In Section 3, we prove Theorems 1.3-1.5.

2. Preliminaries

In this section, we provide some preliminary results. In order to prove the main results, the following identities are needed. Let n and k be integers.

$$j(x;q) = j(q/x;q), \quad (2.1)$$

$$j(q^n x;q) = (-1)^n q^{-\binom{n}{2}} x^{-n} j(x;q), \quad (2.2)$$

$$(a;q)_{-n}=\frac{1}{(aq^{-n};q)_n},\quad n>0, \tag{2.3}$$
$$(a;q)_{n+k}=(a;q)_n(aq^n;q)_k,\quad n,k\geq 0, \tag{2.4}$$
$$(aq^{-n};q)_n=(-a/q)^n q^{-\binom{n}{2}}(q/a;q)_n,\quad n\geq 0. \tag{2.5}$$

The (unilateral) basic hypergeometric series ${}_r\phi_s$ is stated as

$${}_r\phi_s\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r}x^n.$$

Lemma 2.1. *[20, p. 185] If $abcq^2=efg$, then*

$$\begin{aligned}&{}_5\phi_4\left(\begin{matrix}q^{-2N}, & \rho_1, & \rho_2, & b, & c;\\ & \rho_1\rho_2q^{-2N}/a, & e, & f, & g\end{matrix};q^2,q^2\right)\\
&=\frac{(aq^2/\rho_1,aq^2/\rho_2;q^2)_N}{(aq^2,aq^2/\rho_1\rho_2;q^2)_N}\times\sum_{n=0}^{N}\frac{(1-aq^{4n})(a,\rho_1,\rho_2,q^{-2N};q^2)_n}{(1-a)(q^2,aq^2/\rho_1,aq^2/\rho_2,aq^{2N+2};q^2)_n}\left(\frac{aq^{2N+2}}{\rho_1\rho_2}\right)^n\\
&\quad\times{}_4\phi_3\left(\begin{matrix}q^{-2n}, & aq^{2n}, & b, & c\\ & e, & f, & g\end{matrix};q^2,q^2\right).\end{aligned}$$

Lemma 2.2. *[20, p. 186] The big q-Jacobi polynomials*

$$P_n(x)=P_n(x;a,b,c;q^2)={}_3\phi_2\left(\begin{matrix}q^{-2n}, & abq^{2n+2}, & x\\ & aq^2, & cq^2\end{matrix};q^2,q^2\right)$$

satisfy the following relation

$$A_nP_{n+1}(x)+(1-x-A_n-C_n)P_n(x)+C_nP_{n-1}(x)=0$$

with

$$\begin{aligned}A_n&=\frac{(1-aq^{2n+2})(1-cq^{2n+2})(1-abq^{2n+2})}{(1-abq^{4n+2})(1-abq^{4n+4})},\\
C_n&=-\frac{(1-q^{2n})(1-bq^{2n})(1-abc^{-1}q^{2n})acq^{2n+2}}{(1-abq^{4n})(1-abq^{4n+2})}.\end{aligned}$$

The following lemma was given in [17]. Here for completeness, we use Lemmas 2.1 and 2.2 to state a little bit different proof.

Lemma 2.3. *[17] For $m\geq 1$, we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q^2)_n}{(xq^m,x^{-1}q^m;q^2)_{n+1}}\left(\frac{q^{2m+2}}{\rho_1\rho_2}\right)^n\\
&=\frac{(q^{2m+2}/\rho_1,q^{2m+2}/\rho_2;q^2)_\infty}{(q^2,q^{2m+2}/\rho_1\rho_2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(q^{2n+2};q^2)_{m-1}(\rho_1,\rho_2;q^2)_n(-1)^nq^{n^2+2mn+3n}}{(q^{2m+2}/\rho_1,q^{2m+2}/\rho_2;q^2)_n(1-xq^{2n+m})}\left(\frac{1}{\rho_1\rho_2}\right)^n.\end{aligned}$$

Proof. Setting $(a,b,c,e,f,g)\to(q^{2m},q^2,0,xq^{m+2},x^{-1}q^{m+2},0)$ and $N\to\infty$ in Lemma 2.1, and then using Lemma 2.2, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q^2)_n}{(xq^{m+2},x^{-1}q^{m+2};q^2)_n}\left(\frac{q^{2m+2}}{\rho_1\rho_2}\right)^n&=\frac{(q^{2m+2}/\rho_1,q^{2m+2}/\rho_2;q^2)_\infty}{(q^{2m+2},q^{2m+2}/\rho_1\rho_2;q^2)_\infty}\\
\times\sum_{n=0}^{\infty}\frac{(1-q^{4n+2m})(q^{2m},\rho_1,\rho_2;q^2)_n(-1)^nq^{n^2+(2m+1)n}}{(1-q^{2m})(q^2,q^{2m+2}/\rho_1,q^{2m+2}/\rho_2;q^2)_n}&\left(\frac{1}{\rho_1\rho_2}\right)^nP_n'(q^2),\end{aligned}\tag{2.6}$$

where

$$P'_n(q^2) = P_n(q^2; xq^m, x^{-1}q^{m-2}, x^{-1}q^m; q^2)$$

and

$$A'_n P'_{n+1}(q^2) + B'_n P'_n(q^2) + C'_n P'_{n-1}(q^2) = 0 \tag{2.7}$$

with

$$\begin{aligned}
A'_n &= \frac{(1-q^{2n+2m})(1-xq^{2n+m+2})(1-x^{-1}q^{2n+m+2})}{(1-q^{4n+2m})(1-q^{4n+2m+2})},\\
B'_n &= \frac{(-q^2+q^{2n+2m}+q^{2n+2m+2}-q^{4n+2m+2})(1-xq^{2n+m})(1-x^{-1}q^{2n+m})}{(1-q^{4n+2m-2})(1-q^{4n+2m+2})},\\
C'_n &= \frac{-(1-q^{2n})(1-xq^{2n+m-2})(1-x^{-1}q^{2n+m-2})q^{2n+2m+2}}{(1-q^{4n+2m})(1-q^{4n+2m-2})}.
\end{aligned}$$

Define

$$\overline{P}_n = \frac{(1-xq^m)(1-x^{-1}q^m)q^{2n}}{(1-xq^{2n+m})(1-x^{-1}q^{2n+m})}.$$

We find that $P'_n(q^2)$ and $\overline{P}_n$ satisfy the same recurrence relation (2.7) and have the same initial values. Thus

$$P'_n(q^2) = \frac{(1-xq^m)(1-x^{-1}q^m)q^{2n}}{(1-xq^{2n+m})(1-x^{-1}q^{2n+m})}. \tag{2.8}$$

So, substituting (2.8) into (2.6) yields that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\rho_1,\rho_2;q^2)_n}{(xq^m,x^{-1}q^m;q^2)_{n+1}} \left(\frac{q^{2m+2}}{\rho_1\rho_2}\right)^n\\
&= \frac{(q^{2m+2}/\rho_1, q^{2m+2}/\rho_2;q^2)_\infty}{(q^2,q^{2m+2}/\rho_1\rho_2;q^2)_\infty}\\
&\quad\times \sum_{n=0}^{\infty} \frac{(1-q^{4n+2m})(q^{2n+2};q^2)_{m-1}(\rho_1,\rho_2;q^2)_n(-1)^n q^{n^2+2mn+3n}}{(q^{2m+2}/\rho_1,q^{2m+2}/\rho_2;q^2)_n(1-xq^{2n+m})(1-x^{-1}q^{2n+m})} \left(\frac{1}{\rho_1\rho_2}\right)^n\\
&= \frac{(q^{2m+2}/\rho_1, q^{2m+2}/\rho_2;q^2)_\infty}{(q^2,q^{2m+2}/\rho_1\rho_2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q^{2n+2};q^2)_{m-1}(\rho_1,\rho_2;q^2)_n(-1)^n q^{n^2+2mn+3n}}{(q^{2m+2}/\rho_1,q^{2m+2}/\rho_2;q^2)_n} \left(\frac{1}{\rho_1\rho_2}\right)^n\\
&\quad\times \left(\frac{x^{-1}q^{2n+m}}{1-x^{-1}q^{2n+m}} + \frac{1}{1-xq^{2n+m}}\right)\\
&= \frac{(q^{2m+2}/\rho_1, q^{2m+2}/\rho_2;q^2)_\infty}{(q^2,q^{2m+2}/\rho_1\rho_2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q^{2n+2};q^2)_{m-1}(\rho_1,\rho_2;q^2)_n(-1)^{n+1} q^{n^2+2mn+3n}}{(q^{2m+2}/\rho_1,q^{2m+2}/\rho_2;q^2)_n(1-xq^{-2n-m})} \left(\frac{1}{\rho_1\rho_2}\right)^n\\
&\quad+ \frac{(q^{2m+2}/\rho_1, q^{2m+2}/\rho_2;q^2)_\infty}{(q^2,q^{2m+2}/\rho_1\rho_2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q^{2n+2};q^2)_{m-1}(\rho_1,\rho_2;q^2)_n(-1)^n q^{n^2+2mn+3n}}{(q^{2m+2}/\rho_1,q^{2m+2}/\rho_2;q^2)_n(1-xq^{2n+m})} \left(\frac{1}{\rho_1\rho_2}\right)^n\\
&= \frac{(q^{2m+2}/\rho_1, q^{2m+2}/\rho_2;q^2)_\infty}{(q^2,q^{2m+2}/\rho_1\rho_2;q^2)_\infty}\\
&\quad\times \left(\sum_{n=-\infty}^{-m} + \sum_{n=0}^{\infty}\right) \frac{(q^{2n+2};q^2)_{m-1}(\rho_1,\rho_2;q^2)_n(-1)^n q^{n^2+2mn+3n}}{(q^{2m+2}/\rho_1,q^{2m+2}/\rho_2;q^2)_n(1-xq^{2n+m})} \left(\frac{1}{\rho_1\rho_2}\right)^n\\
&= \frac{(q^{2m+2}/\rho_1, q^{2m+2}/\rho_2;q^2)_\infty}{(q^2,q^{2m+2}/\rho_1\rho_2;q^2)_\infty}
\end{aligned}$$

$$\times\left(\sum_{n=-\infty}^{\infty}-\sum_{n=-m+1}^{-1}\right)\frac{(q^{2n+2};q^2)_{m-1}(\rho_1,\rho_2;q^2)_n(-1)^n q^{n^2+2mn+3n}}{(q^{2m+2}/\rho_1,q^{2m+2}/\rho_2;q^2)_n(1-xq^{2n+m})}\left(\frac{1}{\rho_1\rho_2}\right)^n,$$

where the penultimate equality is obtained by setting $n\to -n-m$ in the first sum, and then using (2.3), (2.4), and (2.5). Since $(q^{2n+2};q^2)_{m-1}=0$ for $m\geq 2$ and $-m+1\leq n\leq -1$, we complete the proof. □

In the following lemmas, let n be any nonnegative integer.

Lemma 2.4. *For $m\geq 1$ and $s\leq 2$, we have*

$$\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n+s};q^2)_{m-s+1}(1-xq^{2n+m})}=\sum_{k=0}^{m-s}\frac{A^{(1)}_{k,m,s}}{1+q^{2n+s+2k}}+\frac{B^{(1)}_{m,s}}{1-xq^{2n+m}},$$

where $A^{(1)}_{k,m,s}$ and $B^{(1)}_{m,s}$ are defined in (1.10) *and* (1.11), *respectively.*

Proof. Set

$$f(y):=\frac{(yq^2;q^2)_{m-1}}{(-yq^s;q^2)_{m-s+1}(1-xyq^m)}.$$

Then for $m\geq 1$ and $s\leq 2$, using partial fractional decomposition, we have

$$f(y)=\sum_{k=0}^{m-s}\frac{A^{(1)}_{k,m,s}}{1+yq^{s+2k}}+\frac{B^{(1)}_{m,s}}{1-xyq^m}.$$

Now we compute $A^{(1)}_{k,m,s}$ for $k=0,1,\ldots,m-s$ and $B^{(1)}_{m,s}$.

$$\begin{aligned}A^{(1)}_{k,m,s}&=\lim_{y\to -q^{-s-2k}}(1+yq^{s+2k})f(y)\\&=\lim_{y\to -q^{-s-2k}}\frac{(yq^2;q^2)_{m-1}}{(-yq^s;q^2)_k(-yq^{s+2k+2};q^2)_{m-s-k}(1-xyq^m)}\\&=\frac{(-q^{-s-2k+2};q^2)_{m-1}}{(q^{-2k};q^2)_k(q^2;q^2)_{m-s-k}(1+xq^{m-s-2k})},\end{aligned}$$

$$B^{(1)}_{m,s}=\lim_{y\to x^{-1}q^{-m}}(1-xyq^m)f(y)=\lim_{y\to x^{-1}q^{-m}}\frac{(yq^2;q^2)_{m-1}}{(-yq^s;q^2)_{m-s+1}}=\frac{(x^{-1}q^{-m+2};q^2)_{m-1}}{(-x^{-1}q^{-m+s};q^2)_{m-s+1}}.$$

Therefore, we complete the proof. □

Lemma 2.5. *For $m\geq\max\{1,2s,2t,2s+2t-1\}$, we have*

$$\begin{aligned}&\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n+2s+1};q^2)_{m-2s}(q^{2n+2t+1};q^2)_{m-2t}(1-xq^{2n+m})}\\&=\sum_{k=1}^{m-2s}\frac{A^{(2)}_{k,m,s,t}}{1+q^{2n+2s+2k-1}}+\sum_{k=1}^{m-2t}\frac{B^{(2)}_{k,m,,s,t}}{1-q^{2n+2t+2k-1}}+\frac{C^{(2)}_{m,s,t}}{1-xq^{2n+m}},\end{aligned}$$

where $A^{(2)}_{k,m,s,t}$, $B^{(2)}_{k,m,s,t}$, and $C^{(2)}_{m,s,t}$ are defined in (1.12), (1.13), *and* (1.14), *respectively.*

Proof. First, set

$$f(y)=\frac{(yq^2;q^2)_{m-1}}{(-yq^{2s+1};q^2)_{m-2s}(yq^{2t+1};q^2)_{m-2t}(1-xyq^m)}.$$

Then performing partial fraction decomposition on $f(y)$ yields

$$f(y)=\sum_{k=1}^{m-2s}\frac{A^{(2)}_{k,m,s,t}}{1+yq^{2s+2k-1}}+\sum_{k=1}^{m-2t}\frac{B^{(2)}_{k,m,s,t}}{1-yq^{2t+2k-1}}+\frac{C^{(2)}_{m,s,t}}{1-xyq^{m}}.$$

Now we compute $A^{(2)}_{k,m,s,t}$ for $k=1,2,\dots,m-2s$, $B^{(2)}_{k,m,s,t}$ for $k=1,2,\dots,m-2t$, and $C^{(2)}_{m,s,t}$.

$$\begin{aligned}A^{(2)}_{k,m,s,t}&=\lim_{y\to -q^{-2s-2k+1}}(1+yq^{2s+2k-1})f(y)\\&=\lim_{y\to -q^{-2s-2k+1}}\frac{(yq^2;q^2)_{m-1}}{(-yq^{2s+1};q^2)_{k-1}(-yq^{2s+2k+1};q^2)_{m-2s-k}(yq^{2t+1};q^2)_{m-2t}(1-xyq^m)}\end{aligned}$$

which implies (1.12).

Similarly, we derive (1.13) and (1.14). So, we complete the proof. □

Lemma 2.6. *For $m\ge \max\{1,2s-1,2t,2s+2t-2\}$, we have*

$$\begin{aligned}&\frac{(q^{2n+2};q^2)_{m-1}}{(-q^{2n+2s};q^2)_{m-2s+1}(-q^{2n+2t+1};q^2)_{m-2t}(1-xq^{2n+m})}\\&=\sum_{k=0}^{m-2s}\frac{A^{(3)}_{k,m,s,t}}{1+q^{2n+2s+2k}}+\sum_{k=1}^{m-2t}\frac{B^{(3)}_{k,m,s,t}}{1+q^{2n+2t+2k-1}}+\frac{C^{(3)}_{m,s,t}}{1-xq^{2n+m}},\end{aligned}$$

where $A^{(3)}_{k,m,s,t}$, $B^{(3)}_{k,m,s,t}$, and $C^{(3)}_{m,s,t}$ are defined in (1.15), (1.16), *and* (1.17), *respectively.*

Proof. Since the proof is similar to that of Lemma 2.5, we omit it here. □

3. Proofs of Theorems 1.3-1.5

In this section, we prove Theorems 1.3-1.5.

Proofs of Theorems 1.3. Setting $\rho_1\to\infty$ and $\rho_2=-q^s$ in Lemma 2.3, and then utilizing Lemma 2.4, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-q^s;q^2)_n q^{n^2+(2m-s+1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}&=\frac{(-q^s;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{m-s}\frac{(-1)^nq^{2n^2+(2m-s+2)n}A^{(1)}_{k,m,s}}{1+q^{2n+s+2k}}\\&\quad+\frac{(-q^s;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+(2m-s+2)n}B^{(1)}_{m,s}}{1-xq^{2n+m}}.\end{aligned}\tag{3.1}$$

Next, we consider the following two cases for even s and odd s.

(1) First, replacing s by $2s$ in (3.1), and then multiplying $(-q^2;q^2)_{s-1}$ on both sides, we derive that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_{n+s-1}q^{n^2+(2m-2s+1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}&=\frac{1}{J_{2,4}}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{m-2s}\frac{(-1)^nq^{2n^2+(2m-2s+2)n}A^{(1)}_{k,m,2s}}{1+q^{2n+2s+2k}}\\&\quad+\frac{B^{(1)}_{m,2s}}{J_{2,4}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+(2m-2s+2)n}}{1-xq^{2n+m}}.\end{aligned}\tag{3.2}$$

If $m \equiv s \pmod 2$, then we set $n \to n-(m-s)/2$ on the right-hand side of (3.2) to obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_{n+s-1}q^{n^2+(2m-2s+1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}\\
&=\frac{1}{J_{2,4}}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{m-2s}\frac{(-1)^{n-(m-s)/2}q^{2n^2+2n-(m-s)(m-s+2)/2}A_{k,m,2s}^{(1)}}{1+q^{2n-m+3s+2k}}\\
&\quad+\frac{B_{m,2s}^{(1)}}{J_{2,4}}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n-(m-s)/2}q^{2n^2+2n-(m-s)(m-s+2)/2}}{1-xq^{2n+s}}\\
&=\sum_{k=0}^{m-2s}(-1)^{\frac{m-s}{2}}q^{-\frac{(m-s)(m-s+2)}{2}+m-3s-2k}A_{k,m,2s}^{(1)}m(q^{2m-6s-4k+2},q^4,-q^{-m+3s+2k})\\
&\quad+(-1)^{\frac{m-s+2}{2}}x^{-1}q^{-\frac{(m-s)(m-s+2)}{2}-s}B_{m,2s}^{(1)}m(x^{-2}q^{2-2s},q^4,xq^s)\\
&=\sum_{k=0}^{m-2s}(-1)^{\frac{m-s}{2}}q^{-\frac{(m-s)(m-s+2)}{2}-m+3s+2k-2}A_{k,m,2s}^{(1)}m(q^{-2m+6s+4k-2},q^4,-q^{m-3s-2k})\\
&\quad+(-1)^{\frac{m-s+2}{2}}xq^{-\frac{(m-s)(m-s+2)}{2}+s-2}B_{m,2s}^{(1)}m(x^2q^{2s-2},q^4,x^{-1}q^{-s}),
\end{aligned}\tag{3.3}$$

where the penultimate step follows from (1.3), and we use (1.5) to obtain the last step.

Next, if $m \equiv s+1 \pmod 2$, then letting $n \to n-(m-s+1)/2$ on the right-hand side of (3.2), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_{n+s-1}q^{n^2+(2m-2s+1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}\\
&=\frac{1}{J_{2,4}}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{m-2s}\frac{(-1)^{n-(m-s+1)/2}q^{2n^2-(m-s+1)^2/2}A_{k,m,2s}^{(1)}}{1+q^{2n-m+3s+2k-1}}\\
&\quad+\frac{B_{m,2s}^{(1)}}{J_{2,4}}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n-(m-s+1)/2}q^{2n^2-(m-s+1)^2/2}}{1-xq^{2n+s-1}}.
\end{aligned}\tag{3.4}$$

By changing $n \to -n$, $x \to x^{-1}$, and $q \to q^2$ in (1.3), we have

$$\frac{1}{J_{2,4}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2}}{1-xq^{2n}}=m(x^2q^2,q^4,x^{-1}).\tag{3.5}$$

Then substituting (3.5) into (3.4) yields that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_{n+s-1}q^{n^2+(2m-2s+1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}\\
&=\sum_{k=0}^{m-2s}(-1)^{\frac{m-s+1}{2}}q^{-\frac{(m-s+1)^2}{2}}A_{k,m,2s}^{(1)}m(q^{-2m+6s+4k},q^4,-q^{m-3s-2k+1})\\
&\quad+(-1)^{\frac{m-s+1}{2}}q^{-\frac{(m-s+1)^2}{2}}B_{m,2s}^{(1)}m(x^2q^{2s},q^4,x^{-1}q^{1-s}).
\end{aligned}\tag{3.6}$$

Hence, combining (3.3) and (3.6), we prove (1.8).

(2) Setting $s\to 2s+1$ in (3.1), and then multiplying $(-q;q^2)_s$ on both sides, we derive that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+s}q^{n^2+(2m-2s)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}\\
&=\frac{1}{J_{1,4}}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{m-2s-1}\frac{(-1)^nq^{2n^2+(2m-2s+1)n}(1-q^{2n+2s+2k+1})A^{(1)}_{k,m,2s+1}}{1-q^{4n+4s+4k+2}}\\
&\quad+\frac{B^{(1)}_{m,2s+1}}{J_{1,4}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+(2m-2s+1)n}(1+xq^{2n+m})}{1-x^2q^{4n+2m}}\\
&=-\frac{1}{J_{1,4}}\sum_{k=0}^{m-2s-1}q^{-2m+2s+1}j(q^{2m-2s-1};q^4)A^{(1)}_{k,m,2s+1}m(q^{-2m+6s+4k+3},q^4,q^{2m-2s-1})\\
&\quad+\frac{1}{J_{1,4}}\sum_{k=0}^{m-2s-1}q^{-2m+4s+2k}j(q^{2m-2s+1};q^4)A^{(1)}_{k,m,2s+1}m(q^{-2m+6s+4k+1},q^4,q^{2m-2s+1})\\
&\quad-\frac{1}{J_{1,4}}q^{-2m+2s+1}j(q^{2m-2s-1};q^4)B^{(1)}_{m,2s+1}m(x^2q^{2s+1},q^4,q^{2m-2s-1})\\
&\quad-\frac{1}{J_{1,4}}xq^{-m+2s-1}j(q^{2m-2s+1};q^4)B^{(1)}_{m,2s+1}m(x^2q^{2s-1},q^4,q^{2m-2s+1}),
\end{aligned}
\tag{3.7}
$$

where the last step follows from (1.2).

If $m\equiv s \pmod 2$, then using (2.1) and (2.2), we have

$$
\begin{aligned}
j(q^{2m-2s-1};q^4)&=j(q^3\cdot q^{2m-2s-4};q^4)=(-1)^{\frac{m-s}{2}-1}q^{\frac{-(m-s)^2+3(m-s)-2}{2}}J_{1,4},\\
j(q^{2m-2s+1};q^4)&=j(q\cdot q^{2m-2s};q^4)=(-1)^{\frac{m-s}{2}}q^{\frac{-(m-s)^2+(m-s)}{2}}J_{1,4}.
\end{aligned}
$$

Then using (1.4) and the above two identities, we rewrite (3.7) as

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+s}q^{n^2+(2m-2s)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}\\
&=\sum_{k=0}^{m-2s-1}(-1)^{\frac{m-s}{2}}q^{\frac{-(m-s)^2-(m-s)}{2}}A^{(1)}_{k,m,2s+1}m(q^{-2m+6s+4k+3},q^4,q^3)\\
&\quad+\sum_{k=0}^{m-2s-1}(-1)^{\frac{m-s}{2}}q^{\frac{-(m-s)^2-3m+7s+4k}{2}}A^{(1)}_{k,m,2s+1}m(q^{-2m+6s+4k+1},q^4,q)\\
&\quad+(-1)^{\frac{m-s}{2}}q^{\frac{-(m-s)^2-(m-s)}{2}}B^{(1)}_{m,2s+1}m(x^2q^{2s+1},q^4,q^3)\\
&\quad+(-1)^{\frac{m-s+2}{2}}xq^{\frac{-(m-s)^2-m+3s-2}{2}}B^{(1)}_{m,2s+1}m(x^2q^{2s-1},q^4,q).
\end{aligned}
\tag{3.8}
$$

If $m\equiv s+1 \pmod 2$, then in view of (2.1) and (2.2), we have

$$
\begin{aligned}
j(q^{2m-2s-1};q^4)&=j(q\cdot q^{2m-2s-2};q^4)=(-1)^{\frac{m-s-1}{2}}q^{\frac{-(m-s)^2+3(m-s)-2}{2}}J_{1,4},\\
j(q^{2m-2s+1};q^4)&=j(q^3\cdot q^{2m-2s-2};q^4)=(-1)^{\frac{m-s-1}{2}}q^{\frac{-(m-s)^2+(m-s)}{2}}J_{1,4}.
\end{aligned}
$$

Utilizing (1.4) and the above two identities, we rewrite (3.7) as

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+s}q^{n^2+(2m-2s)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}}\\
&=\sum_{k=0}^{m-2s-1}(-1)^{\frac{m-s+1}{2}}q^{\frac{-(m-s)^2-(m-s)}{2}}A^{(1)}_{k,m,2s+1}m(q^{-2m+6s+4k+3},q^4,q)\\
&\quad+\sum_{k=0}^{m-2s-1}(-1)^{\frac{m-s-1}{2}}q^{\frac{-(m-s)^2-3m+7s+4k}{2}}A^{(1)}_{k,m,2s+1}m(q^{-2m+6s+4k+1},q^4,q^3)\\
&\quad+(-1)^{\frac{m-s+1}{2}}q^{\frac{-(m-s)^2-(m-s)}{2}}B^{(1)}_{m,2s+1}m(x^2q^{2s+1},q^4,q)\\
&\quad+(-1)^{\frac{m-s+1}{2}}xq^{\frac{-(m-s)^2-m+3s-2}{2}}B^{(1)}_{m,2s+1}m(x^2q^{2s-1},q^4,q^3).
\end{aligned}\tag{3.9}$$

Therefore, according to (3.8) and (3.9), we prove (1.9). □

Proof of Theorem 1.4. Setting $(\rho_1,\rho_2)\to(-q^{2s+1},q^{2t+1})$ in Lemma 2.3, multiplying $(-q;q^2)_s(q;q^2)_t/(-q^2;q^2)_{m-s-t-1}$ on both sides, and then applying Lemma 2.5, we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+s}(q;q^2)_{n+t}(-1)^nq^{(2m-2s-2t)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}(-q^2;q^2)_{m-s-t-1}}\\
&=\frac{2}{\overline{J}_{0,2}}\Bigg(\sum_{k=1}^{m-2s}\sum_{n=-\infty}^{\infty}\frac{q^{n^2+(2m-2s-2t+1)n}A^{(2)}_{k,m,s,t}}{1+q^{2n+2s+2k-1}}+\sum_{k=1}^{m-2t}\sum_{n=-\infty}^{\infty}\frac{q^{n^2+(2m-2s-2t+1)n}B^{(2)}_{k,m,s,t}}{1-q^{2n+2t+2k-1}}\\
&\quad+\sum_{n=-\infty}^{\infty}\frac{q^{n^2+(2m-2s-2t+1)n}C^{(2)}_{m,s,t}}{1-xq^{2n+m}}\Bigg)\\
&=\frac{2}{\overline{J}_{0,2}}\Bigg(\sum_{k=1}^{m-2s}q^{-2m+2s+2t}j(-q^{2m-2s-2t};q^2)A^{(2)}_{k,m,s,t}m(q^{-2m+4s+2t+2k-1},q^2,-q^{2m-2s-2t})\\
&\quad+\sum_{k=1}^{m-2t}q^{-2m+2s+2t}j(-q^{2m-2s-2t};q^2)B^{(2)}_{k,m,s,t}m(-q^{-2m+2s+4t+2k-1},q^2,-q^{2m-2s-2t})\\
&\quad+q^{-2m+2s+2t}j(-q^{2m-2s-2t};q^2)C^{(2)}_{m,s,t}m(-xq^{-m+2s+2t},q^2,-q^{2m-2s-2t})\Bigg).
\end{aligned}\tag{3.10}$$

Notice that except for the condition for m in Lemma 2.5, we add the further condition $2m-2s-2t\geq 1$ to ensure the convergence of (3.10).

Then according to (2.2), we derive that

$$j(-q^{2m-2s-2t};q^2)=q^{-(m-s-t)^2+(m-s-t)}\overline{J}_{0,2}.\tag{3.11}$$

So, combining (1.4), (3.10), and (3.11) yields that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+s}(q;q^2)_{n+t}(-1)^nq^{(2m-2s-2t)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}(-q^2;q^2)_{m-s-t-1}}\\
&=2q^{-(m-s-t)^2-(m-s-t)}\Bigg(\sum_{k=1}^{m-2s}A^{(2)}_{k,m,s,t}m(q^{-2m+4s+2t+2k-1},q^2,-1)
\end{aligned}$$

$$+\sum_{k=1}^{m-2t} B^{(2)}_{k,m,s,t} m(-q^{-2m+2s+4t+2k-1},q^2,-1)+C^{(2)}_{m,s,t} m(-xq^{-m+2s+2t},q^2,-1)\Bigg).$$

Hence, we complete the proof. □

Proof of Theorem 1.5. Setting $(\rho_1,\rho_2)\to(-q^{2s},-q^{2t+1})$ in Lemma 2.3, multiplying $(-q^2;q^2)_{s-1}(-q;q^2)_t/(q;q^2)_{m-s-t}$, and then applying Lemma 2.6, we deduce that

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_{n+s-1}(-q;q^2)_{n+t}q^{(2m-2s-2t+1)n}}{(xq^m,x^{-1}q^m;q^2)_{n+1}(q;q^2)_{m-s-t}}$$
$$=\frac{1}{J_{1,2}}\left(\sum_{k=0}^{m-2s}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+(2m-2s-2t+2)n}A^{(3)}_{k,m,s,t}}{1+q^{2n+2s+2k}}\right.$$
$$\left.+\sum_{k=1}^{m-2t}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+(2m-2s-2t+2)n}B^{(3)}_{k,m,s,t}}{1+q^{2n+2t+2k-1}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+(2m-2s-2t+2)n}C^{(3)}_{m,s,t}}{1-xq^{2n+m}}\right)$$
$$=(-1)^{m-s-t+1}q^{-(m-s-t)^2-2(m-s-t)-1}\left(\sum_{k=0}^{m-2s}A^{(3)}_{k,m,s,t}m(-q^{-2m+4s+2t+2k-1},q^2,q)\right.$$
$$\left.+\sum_{k=1}^{m-2t}B^{(3)}_{k,m,s,t}m(-q^{-2m+2s+4t+2k-2},q^2,q)+C^{(3)}_{m,s,t}m(xq^{-m+2s+2t-1},q^2,q)\right),$$

where we obtain the last step by utilizing (1.2), (1.4), and (2.2). Here we add another condition $2m-2s-2t\geq 0$ due to convergence problems. Therefore, we complete the proof. □

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(S.-P. Cui) School of Mathematics and Statistics, Qinghai Normal University, Xining, Qinghai 810008, P.R. China; Academy of Plateau Science and Sustainability, Xining, Qinghai 810008, P.R. China

E-mail address: jiayoucui@163.com

(N.S.S. Gu) Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P.R. China

E-mail address: gu@nankai.edu.cn

(C.-Y. Su) College of Mathematical Science, Tianjin Normal University, Tianjin 300387, P.R. China

E-mail address: cyangsu@163.com

(M.H.Y. Xie) College of Science, Tianjin University of Technology, Tianjin 300384, P.R. China

E-mail address: xie@email.tjut.edu.cn