Two-disjoint-cycle-cover edge/vertex bipancyclicity of star graphs *

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Abstract

A bipartite graph G is two-disjoint-cycle-cover edge $[r_1, r_2]$-bipancyclic if, for any vertex-disjoint edges uv and xy in G and any even integer ℓ satisfying $r_1 \leqslant \ell \leqslant r_2$, there exist vertex-disjoint cycles C_1 and C_2 such that $|V(C_1)| = \ell$, $|V(C_2)| = |V(G)| - \ell$, $uv \in E(C_1)$ and $xy \in E(C_2)$. In this paper, we prove that the n-star graph S_n is two-disjoint-cycle-cover edge $[6, \frac{n!}{2}]$-bipancyclic for $n \geqslant 5$, and thus it is two-disjoint-cycle-cover vertex $[6, \frac{n!}{2}]$-bipancyclic for $n \geqslant 5$. Additionally, it is examined that S_n is two-disjoint-cycle-cover $[6, \frac{n!}{2}]$-bipancyclic for $n \geqslant 4$.

Key words: Star graph, vertex-disjoint cycles, edge bipancyclicity, vertex bipancyclicity.

1 Introduction

The underlying topology of an interconnection network is usually modeled by a connected simple graph. Cycles are one class of fundamental network topologies, which are suitable for designing simple algorithms with low communication costs. Many efficient parallel algorithms designed on cycles can be used as data structures for distributed computing in those networks that can embed cycles so that the algorithms designed on cycles can be simulated on the embedded cycles. In addition, more cycles of various lengths can be embedded in a network, and more simulated processors can be adjusted to increase the elasticity of demand. Thus, the problem of embedding cycles of various possible lengths into a graph is an important factor for network simulation and merits special attention. This problem has been considered for various special network topologies, for example, hypercubes [10], balanced hypercubes [30], data center networks [11], (n, k)-bubble-sort networks [28], star graphs [12, 29], and so on.

A cycle (or path) having ℓ edges is called an ℓ-cycle (or path), and we say it has length ℓ. For a graph G, denote $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. A graph G is said to be pancyclic [3] if it contains ℓ-cycles for each integer ℓ satisfying $3 \leqslant \ell \leqslant |V(G)|$. The concept of pancyclicity has been extended to edge-pancyclicity and vertex-pancyclicity [4]. A graph G is vertex-pancyclic (resp. edge-pancyclic) if every vertex (resp. edge) lies on ℓ-cycles for each integer ℓ satisfying $3 \leqslant \ell \leqslant |V(G)|$. Note that an edge-pancyclic graph is certainly vertex-pancyclic. Since a bipartite graph has no cycle of odd length, it was proposed in [20] that a bipartite graph G is called bipancyclic if it contains ℓ-cycles for each even integer ℓ satisfying $4 \leqslant \ell \leqslant |V(G)|$. This concept has been extended to vertex-bipancyclicity [21] and edge-bipancyclicity [18].

In [13, 14], Kung et al. investigated the problem of embedding disjoint cycles which cover all vertices of a graph, and proposed the concepts of two-disjoint-cycle-cover (2-DCC in short) pancyclicity, 2-DCC $[r_1, r_2]$-pancyclicity, 2-DCC vertex $[r_1, r_2]$-pancyclicity and 2-DCC edge $[r_1, r_2]$-pancyclicity. Following [14], the concepts of 2-DCC $[r_1, r_2]$-bipancyclicity and 2-DCC vertex $[r_1, r_2]$-bipancyclicity were introduced in [27] and [23, 24] for bipartite graphs, respectively.

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Definition 1.1 ([27]) *A bipartite graph G is* 2-DCC $[r_1, r_2]$*-bipancyclic if, for any even integer ℓ with $r_1 \leqslant \ell \leqslant r_2$, there are vertex-disjoint cycles C_1 and C_2 that cover all vertices of G, one of them has length ℓ.*

Definition 1.2 ([24]) *A bipartite graph G is* 2-DCC *vertex* $[r_1, r_2]$*-bipancyclic if, for any distinct vertices u and v in G and any even integer ℓ satisfying $r_1 \leqslant \ell \leqslant r_2$, there exist vertex-disjoint ℓ-cycle C_1 and $(|V(G)| - \ell)$-cycle C_2 such that $u \in V(C_1)$, and $v \in V(C_2)$.*

Analogously, 2-DCC edge $[r_1, r_2]$-bipancyclic of a bipartite graph is defined as follows.

Definition 1.3 *A bipartite graph G is* 2-DCC *edge* $[r_1, r_2]$*-bipancyclic if, for any vertex-disjoint edges uv and xy in G and any even integer ℓ satisfying $r_1 \leqslant \ell \leqslant r_2$, there exist vertex-disjoint ℓ-cycle C_1 and $(|V(G)| - \ell)$-cycle C_2 such that $uv \in E(C_1)$, and $xy \in E(C_2)$.*

Remark 1.4 *From the definitions, a* 2-DCC *edge* $[r_1, r_2]$*-bipancyclic bipartite graph is certainly* 2-DCC *vertex* $[r_1, r_2]$*-bipancyclic, also is* 2-DCC $[r_1, r_2]$*-bipancyclic, and has an even number of vertices. In addition, if a graph is* 2-DCC *edge/vertex* $[r_1, r_2]$*-(bi)pancyclic then it is also* 2-DCC *edge/vertex* $[|V(G)| - r_2, |V(G)| - r_1]$*-(bi)pancyclic. Thus, as observed in [14, 23], it is reasonable to choose* $r_2 \leqslant \frac{|V(G)|}{2}$.

Two-disjoint-cycle-cover pancyclicity and its various extensions have been wildly studied in the recent years for many popular networks, for example, 2-DCC pancyclicity for alternating group graphs [5], crossed cubes [13] and locally twisted cubes [14], 2-DCC bipancyclicity for balanced hypercubes [27] and bubble-sort star graphs [34], 2-DCC vertex pancyclicity for augmented cubes [26] and locally twisted cubes [14], 2-DCC vertex bipancyclicity for bipartite generalized hypercubes [23] and bipartite hypercube-like networks [24] and 2-DCC edge pancyclicity for locally twisted cubes [14].

The star graphs are Cayley graphs and have been recognized as an attractive alternative to the hypercubes [1, 2]. This class of graphs has been widely investigated in various aspects, such as path routing [15, 25], connectivity and diagnosability [6, 19], broadcasting [8, 22], and embedding problems [9, 17, 29, 31–33], and so on. It is proved by Li [16] that the cycles of even lengths from 6 to $n!$ can be embedded into the n-star graph if the number of edge faults in the graph does not exceed $n - 3$. This paper aims to examine the 2-DCC edge/vertex bipancyclicity and the 2-DCC bipancyclicity of star graphs.

Theorem 1.5 *The n-star graph S_n is two-disjoint-cycle-cover edge $[6, \frac{n!}{2}]$-bipancyclic for $n \geqslant 5$.*

Recall that a 2-DCC edge $[r_1, r_2]$-bipancyclic bipartite graph is 2-DCC vertex $[r_1, r_2]$-bipancyclic, so also is 2-DCC $[r_1, r_2]$-bipancyclic.

Corollary 1.6 *The n-star graph S_n is two-disjoint-cycle-cover vertex $[6, \frac{n!}{2}]$-bipancyclic for $n \geqslant 5$.*

We remark that the conclusions in Theorem 1.5 and Corollary 1.6 do not hold for $n = 4$, see Lemma 2.7. However, Lemma 2.7 says that S_4 is 2-DCC $[6, 12]$-bipancyclic. Combining Theorem 1.5 (or Corollary 1.6), we conclude the following result.

Theorem 1.7 *The n-star graph S_n is two-disjoint-cycle-cover $[6, \frac{n!}{2}]$-bipancyclic for $n \geqslant 4$.*

2 Preliminaries

A bijection of $[n] := \{1, 2, \ldots, n\}$ onto itself is called a *permutation* of $[n]$. Denote by $\mathcal{S}_n$ the set of permutations of $[n]$. Under composition of mappings, $\mathcal{S}_n$ forms a group of order $n!$, called the *symmetric group* on $[n]$. We always write permutations on the left and compose from right to left, for example, $(x{\cdot}y)(i) = x(y(i))$.

For distinct i, $j \in [n]$, we use $t_{i,j}$ to denote the transposition interchanging i and j. Put

$$\mathcal{T} := \{t_{1,i} \mid i \in [n] \setminus \{1\}\}.$$

Then $\mathcal{T}$ is a set of generators of the symmetric group $\mathcal{S}_n$. Thus we have a connected Cayley graph $S_n := \text{Cay}(\mathcal{S}_n, \mathcal{T})$, called the n-star graph, which has vertex set $\mathcal{S}_n$ such that two vertices $x, y \in \mathcal{S}_n$ are adjacent if and only if $x^{-1} \cdot y \in \mathcal{T}$. Clearly, the graph S_n is $(n-1)$-regular.

Recall that a permutation is said to be even or odd if it is a product of an even or odd number of transpositions, respectively. Denote by $\mathcal{E}_n$ and $\mathcal{O}_n$ the sets of even permutations and odd permutations of $[n]$, respectively. Then $\mathcal{S}_n = \mathcal{E}_n \cup \mathcal{O}_n$, $|\mathcal{E}_n| = \frac{n!}{2} = |\mathcal{O}_n|$ and S_n has bipartition $(\mathcal{E}_n, \mathcal{O}_n)$. For convenience, the vertices in $\mathcal{E}_n$ and $\mathcal{O}_n$ are call *even* and *odd* vertices, respectively.

The following two lemmas are proved in [31] and [33], respectively.

Lemma 2.1 ([31]) *Let $n \geqslant 4$ and u, v be vertices with opposite parity in S_n. Then for any edge $e \in E(S_n)$ with $e \neq uv$, there is a Hamilton path containing e between u and v in S_n.*

Especially, let e_1 and e_2 be distinct edges of S_n. If e_1 has ends u and v, then Lemma 2.1 implies that there exists a Hamilton path containing e_2 between u and v. Thus the following result holds.

Corollary 2.2 *Let $n \geqslant 4$. If e_1 and e_2 are distinct edges of S_n, then S_n has a Hamilton cycle containing e_1 and e_2.*

Lemma 2.3 ([33]) *Let $n \geqslant 4$ and M be a matching of size m of S_n, where $m \leqslant n-4$. If u and v are vertices with opposite parity of $S_n - V(M)$, then $S_n - V(M)$ has a Hamilton path between u and v.*

For convenience, the permutation $\begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$ is written as $p_1 p_2 \cdots p_n$. An edge xy of S_n is called an *i-edge* if $x^{-1} \cdot y = t_{1,i}$, while x (or y) is called an *i-neighbor* of y (or x). In particular, denote by $\bar{x}$ the n-neighbor of x in S_n, i.e., $\bar{x} = x \cdot t_{1,n}$.

Fix $i \in [n]$, define

$$\begin{aligned} \mathcal{S}_n^i &:= \{x \in \mathcal{S}_n \mid x(n) = i\}, \\ \mathcal{E}_n^i &:= \{x \in \mathcal{E}_n \mid x(n) = i\}, \\ \mathcal{O}_n^i &:= \{x \in \mathcal{O}_n \mid x(n) = i\}. \end{aligned}$$

Then we have a partition $\{\mathcal{S}_n^i \mid i \in [n]\}$ of the symmetric group $\mathcal{S}_n$. Denote by S_n^i the subgraph of S_n induced by $\mathcal{S}_n^i$. Then S_n^i is a bipartite graph with bipartition $(\mathcal{E}_n^i, \mathcal{O}_n^i)$.

For $i, j \in [n]$ with $j \neq i$, denote by $E^{i,j}$ the set of edges between S_n^i and S_n^j, and put

$$E := \bigcup_{i,j \in [n], i \neq j} E^{i,j} = E(S_n) \setminus \bigcup_{i \in [n]} E(S_n^i).$$

The following two lemmas collect some elementary properties of the n-star graph S_n.

Lemma 2.4 ([7]) *Let $i, j \in [n]$ with $n \geqslant 3$ and $i \neq j$. Then*

(1) *S_n^i is a subgraph of S_n and is isomorphic to the $(n-1)$-star graph S_{n-1}; and*

(2) *E indices a perfect matching of S_n; and*

(3) *$|\mathcal{S}_n^i \cap \mathcal{E}_n| = \frac{(n-1)!}{2} = |\mathcal{S}_n^i \cap \mathcal{O}_n|$, and $|V(E^{i,j}) \cap \mathcal{E}_n^i| = \frac{(n-2)!}{2} = |V(E^{i,j}) \cap \mathcal{O}_n^i|$.*

Pick $x \in \mathcal{S}_n^i$ for $i \in [n]$. Let $y = x \cdot t_{1,i_1}$ and $z = x \cdot t_{1,i_2}$ for distinct $i_1, i_2 \in [n] \setminus \{1, n\}$. Then $\bar{x}(n) = x(1)$, $\bar{y}(n) = x(i_1)$ and $\bar{z}(n) = x(i_2)$. In other words, $\bar{x} \in \mathcal{S}_n^{x(1)}$, $\bar{y} \in \mathcal{S}_n^{x(i_1)}$ and $\bar{z} \in \mathcal{S}_n^{x(i_2)}$. Therefore we have the following observation.

Table 1: Hamilton paths in $S_4 - \{1234, 3214\}$ between u and v

v	$u = 2134$	
2314		2134,3124,1324,4321,2341,3241,4231,2431,3421,1423,4123,2143,1243,4213,2413,3412,1432,4132,3142,1342,4312,2314
1342		2134,3124,1324,2314,4312,3412,1432,4132,3142,2143,4123,1423,2413,4213,1243,3241,4231,2431,3421,4321,2341,1342
4132		2134,3124,4123,1423,2413,4213,1243,2143,3142,1342,2341,3241,4231,2431,3421,4321,1324,2314,4312,3412,1432,4132
4321		2134,3124,1324,2314,4312,1342,3142,4132,1432,3412,2413,4213,1243,2143,4123,1423,3421,2431,4231,3241,2341,4321
3241		2134,4132,1432,3412,4312,2314,1324,3124,4123,1423,2413,4213,1243,2143,3142,1342,2341,4321,3421,2431,4231,3241
3412		2134,3124,1324,2314,4312,1342,3142,4132,1432,2431,4231,3241,2341,4321,3421,1423,4123,2143,1243,4213,2413,3412
3124		2134,4132,3142,1342,2341,3241,4231,2431,1432,3412,4312,2314,1324,4321,3421,1423,2413,4213,1243,2143,4123,3124
2431		2134,3124,1324,2314,4312,1342,3142,4132,1432,3412,2413,4213,1243,2143,4123,1423,3421,4321,2341,3241,4231,2431
2143		2134,3124,1324,2314,4312,3412,1432,4132,3142,1342,2341,4321,3421,2431,4231,3241,1243,4213,2413,1423,4123,2143
4213		2134,3124,1324,2314,4312,3412,1432,4132,3142,1342,2341,4321,3421,2431,4231,3241,1243,2143,4123,1423,2413,4213
1423		2134,4132,1432,3412,4312,2314,1324,3124,4123,2143,3142,1342,2341,4321,3421,2431,4231,3241,1243,4213,2413,1423
	$u = 1324$	
2314		1324,4321,3421,1423,2413,4213,1243,2143,4123,3124,2134,4132,3142,1342,2341,3241,4231,2431,1432,3412,4312,2314
1342		1324,2314,4312,3412,1432,2431,4231,3241,2341,4321,3421,1423,2413,4213,1243,2143,4123,3124,2134,4132,3142,1342
4132		1324,2314,4312,1342,3142,2143,1243,4213,2413,3412,1432,2431,4231,3241,2341,4321,3421,1423,4123,3124,2134,4132
4321		1324,2314,4312,1342,2341,3241,4231,2431,1432,3412,2413,4213,1243,2143,3142,4132,2134,3124,4123,1423,3421,4321
3241		1324,2314,4312,3412,1432,4132,2134,3124,4123,1423,2413,4213,1243,2143,3142,1342,2341,4321,3421,2431,4231,3241
3412		1324,2314,4312,1342,3142,4132,2134,3124,4123,2143,1243,4213,2413,1423,3421,4321,2341,3241,4231,2431,1432,3412
3124		exception
2431		1324,2314,4312,1342,3142,2143,1243,4213,2413,3412,1432,4132,2134,3124,4123,1423,3421,4321,2341,3241,4231,2431
2143		1324,2314,4312,3412,1432,4132,2134,3124,4123,1423,2413,4213,1243,3241,4231,2431,3421,4321,2341,1342,3142,2143
4213		1324,2314,4312,3412,2413,1423,3421,4321,2341,1342,3142,2143,4123,3124,2134,4132,1432,2431,4231,3241,1243,4213
1423		1324,2314,4312,3412,2413,4213,1243,3241,4231,2431,1432,4132,2134,3124,4123,2143,3142,1342,2341,4321,3421,1423
	$u = 4312$	
2314		4312,3412,1432,4132,2134,3124,4123,1423,2413,4213,1243,2143,3142,1342,2341,3241,4231,2431,3421,4321,1324,2314
1342		4312,2314,1324,4321,2341,3241,4231,2431,3421,1423,4123,3124,2134,4132,1432,3412,2413,4213,1243,2143,3142,1342
4132		exception
4321		4312,2314,1324,3124,2134,4132,3142,1342,2341,3241,4231,2431,1432,3412,2413,4213,1243,2143,4123,1423,3421,4321
3241		4312,2314,1324,4321,2341,1342,3142,2143,1243,4213,2413,3412,1432,4132,2134,3124,4123,1423,3421,2431,4231,3241
3412		4312,2314,1324,4321,3421,1423,4123,3124,2134,4132,1432,2431,4231,3241,2341,1342,3142,2143,1243,4213,2413,3412
3124		4312,2314,1324,4321,3421,1423,4123,2143,1243,4213,2413,3412,1432,2431,4231,3241,2341,1342,3142,4132,2134,3124
2431		4312,2314,1324,4321,3421,1423,4123,3124,2134,4132,1432,3412,2413,4213,1243,2143,3142,1342,2341,3241,4231,2431
2143		4312,2314,1324,4321,3421,1423,4123,3124,2134,4132,3142,1342,2341,3241,4231,2431,1432,3412,2413,4213,1243,2143
4213		4312,2314,1324,3124,2134,4132,1432,3412,2413,1423,4123,2143,3142,1342,2341,4321,3421,2431,4231,3241,1243,4213
1423		4312,2314,1324,3124,2134,4132,1432,3412,2413,4213,1243,3241,4231,2431,3421,4321,2341,1342,3142,2143,4123,1423
	$u = 3142$	
2314		3142,2143,4123,1423,2413,4213,1243,3241,4231,2431,3421,4321,2341,1342,4312,3412,1432,4132,2134,3124,1324,2314
1342		3142,2143,1243,4213,2413,3412,1432,4132,2134,3124,4123,1423,3421,2431,4231,3241,2341,4321,1324,2314,4312,1342
4132		3142,2143,1243,4213,2413,3412,1432,2431,4231,3241,2341,1342,4312,2314,1324,4321,3421,1423,4123,3124,2134,4132
4321		3142,4132,2134,3124,1324,2314,4312,1342,2341,3241,4231,2431,1432,3412,2413,4213,1243,2143,4123,1423,3421,4321
3241		3142,4132,2134,3124,1324,2314,4312,1342,2341,4321,3421,1423,4123,2143,1243,4213,2413,3412,1432,2431,4231,3241
3412		3142,1342,2341,3241,4231,2431,1432,4132,2134,3124,4123,2143,1243,4213,2413,1423,3421,4321,1324,2314,4312,3412
3124		3142,1342,4312,2314,1324,4321,2341,3241,4231,2431,3421,1423,4123,2143,1243,4213,2413,3412,1432,4132,2134,3124
2431		3142,2143,4123,1423,3421,4321,2341,1342,4312,2314,1324,3124,2134,4132,1432,3412,2413,4213,1243,3241,4231,2431
2143		3142,1342,2341,3241,4231,2431,3421,4321,1324,2314,4312,3412,1432,4132,2134,3124,4123,1423,2413,4213,1243,2143
4213		3142,2143,4123,1423,2413,3412,1432,4132,2134,3124,1324,2314,4312,1342,2341,4321,3421,2431,4231,3241,1243,4213
1423		3142,4132,2134,3124,4123,2143,1243,4213,2413,3412,1432,2431,4231,3241,2341,1342,4312,2314,1324,4321,3421,1423
	$u = 1432$	
2314		1432,2431,4231,3241,2341,4321,3421,1423,4123,2143,1243,4213,2413,3412,4312,1342,3142,4132,2134,3124,1324,2314
1342		1432,3412,2413,4213,1243,2143,3142,4132,2134,3124,4123,1423,3421,2431,4231,3241,2341,4321,1324,2314,4312,1342
4132		1432,2431,4231,3241,2341,1342,3142,2143,1243,4213,2413,3412,4312,2314,1324,4321,3421,1423,4123,3124,2134,4132
4321		1432,2431,4231,3241,2341,1342,3142,4132,2134,3124,1324,2314,4312,3412,2413,4213,1243,2143,4123,1423,3421,4321
3241		1432,3412,4312,2314,1324,4321,2341,1342,3142,4132,2134,3124,4123,2143,1243,4213,2413,1423,3421,2431,4231,3241
3412		1432,2431,4231,3241,2341,1342,3142,4132,2134,3124,4123,2143,1243,4213,2413,1423,3421,4321,1324,2314,4312,3412
3124		1432,3412,2413,4213,1243,2143,4123,1423,3421,2431,4231,3241,2341,4321,1324,2314,4312,1342,3142,4132,2134,3124
2431		1432,3412,4312,2314,1324,4321,3421,1423,2413,4213,1243,2143,4123,3124,2134,4132,3142,1342,2341,3241,4231,2431
2143		1432,3412,4312,2314,1324,3124,2134,4132,3142,1342,2341,4321,3421,2431,4231,3241,1243,4213,2413,1423,4123,2143
4213		1432,3412,4312,2314,1324,3124,2134,4132,3142,1342,2341,4321,3421,2431,4231,3241,1243,2143,4123,1423,2413,4213
1423		1432,2431,4231,3241,2341,1342,3142,4132,2134,3124,4123,2143,1243,4213,2413,3412,4312,2314,1324,4321,3421,1423
	$u = 2341$	
2314		2341,4321,3421,2431,4231,3241,1243,4213,2413,1423,4123,2143,3142,1342,4312,3412,1432,4132,2134,3124,1324,2314
1342		2341,3241,4231,2431,1432,3412,2413,4213,1243,2143,3142,4132,2134,3124,4123,1423,3421,4321,1324,2314,4312,1342
4132		2341,3241,4231,2431,1432,3412,2413,4213,1243,2143,3142,1342,4312,2314,1324,4321,3421,1423,4123,3124,2134,4132
4321		2341,3241,4231,2431,3421,1423,4123,3124,2134,4132,1432,3412,2413,4213,1243,2143,3142,1342,4312,2314,1324,4321
3241		2341,1342,3142,2143,1243,4213,2413,1423,4123,3124,2134,4132,1432,3412,4312,2314,1324,4321,3421,2431,4231,3241
3412		2341,4321,3421,2431,4231,3241,1243,4213,2413,1423,4123,2143,3142,1342,4312,2314,1324,3124,2134,4132,1432,3412
3124		2341,4321,1324,2314,4312,1342,3142,2143,4123,1423,3421,2431,4231,3241,1243,4213,2413,3412,1432,4132,2134,3124
2431		2341,4321,3421,1423,4123,2143,3142,1342,4312,2314,1324,3124,2134,4132,1432,3412,2413,4213,1243,3241,4231,2431
2143		exception
4213		2341,1342,4312,2314,1324,4321,3421,1423,4123,3124,2134,4132,3142,2143,1243,3241,4231,2431,1432,3412,2413,4213
1423		2341,4321,3421,2431,4231,3241,1243,4213,2413,3412,1432,4132,2134,3124,1324,2314,4312,1342,3142,2143,4123,1423

Table 2: Some pairs of vertex disjoint cycles in S_4.

ℓ	$24-\ell$	$\lvert C_1\rvert = \ell$ and $\lvert C_2\rvert = 24-\ell$
6	18	C_1: 1234,2134,3124,1324,2314,3214,1234
		C_2: 4231,3241,1243,4213,2413,1423,4123,2143,3142,4132,1432,3412,4312,1342,2341,4321,3421,2431,4231
6	18	C_1: 1234,2134,4132,1432,2431,4231,1234
		C_2: 3214,2314,1324,3124,4123,1423,3421,4321,2341,3241,1243,2143,3142,1342,4312,3412,2413,4213,3214
8	16	C_1: 1234,2134,3124,1324,4321,3421,2431,4231,1234
		C_2: 3214,2314,4312,3412,1432,4132,3142,1342,2341,3241,1243,2143,4123,1423,2413,4213,3214
8	16	C_1: 1234,2134,4132,1432,3412,4312,2314,3214,1234
		C_2: 3124,1324,4321,3421,2431,4231,3241,2341,1342,3142,2143,1243,4213,2413,1423,4123,3124
8	16	C_1: 1234,2134,3124,4123,2143,1243,4213,3214,1234
		C_2: 2314,4312,1342,3142,4132,1432,3412,2413,1423,3421,2431,4231,3241,2341,4321,1324,2314
10	14	C_1: 1234,2134,3124,4123,1423,3421,4321,1324,2314,3214,1234
		C_2: 4231,3241,2341,1342,4312,3412,2413,4213,1243,2143,3142,4132,1432,2431,4231
10	14	C_1: 1234,2134,4132,3142,2143,4123,3124,1324,2314,3214,1234
		C_2: 4231,3241,1243,4213,2413,1423,3421,4321,2341,1342,4312,3412,1432,2431,4231
10	14	C_1: 1234,2134,3124,1324,2314,3214,4213,1243,3241,4231,1234
		C_2: 2431,3421,4321,2341,1342,4312,3412,2413,1423,4123,2143,3142,4132,1432,2431
10	14	C_1: 1234,2134,4132,1432,2431,3421,4321,2341,3241,4231,1234
		C_2: 3214,2314,1324,3124,4123,1423,2413,3412,4312,1342,3142,2143,1243,4213,3214

Lemma 2.5 *Let $i \in [n]$ and $x \in \mathcal{S}_n^i$. Suppose that y, z are distinct neighbors of x in S_n^i. Then the n-neighbors of x, y and z lie in 3 distinct subgraphs S_n^j of S_n, where $j \in [n] \setminus \{i\}$. In particular, the n-neighbor of x and the n-neighbors of the other $n-2$ neighbors of x in S_n^i are scattered into the distinct $n-1$ subgraphs S_n^j, where j runs over $[n] \setminus \{i\}$.*

A graph G is called the vertex (resp., edge)-transitive if, for any two vertices $u, v \in V(G)$ (resp., edges $e, f \in E(G)$) there exists an automorphism ϕ of G such that $\phi(u) = v$ (resp., $\phi(e) = f$). By [2], the star graph S_n is both vertex-transitive and edge-transitive.

Lemma 2.6 *Let xy be an edge of S_4, and $u, v \in \mathcal{S}_4 \setminus \{x, y\}$ with opposite parity. Then either there exists a Hamilton path of $S_4 - \{x, y\}$ between u and v, or $\{u, v\}$ is one of six possible exceptions.*

Proof. Noting that S_4 is edge-transitive, without loss of generality, we choose $x = 1234 \in \mathcal{E}_4$ and $y = x{\cdot}t_{1,3} = 3214 \in \mathcal{O}_4$. Define $\pi : \mathcal{S}_4 \to \mathcal{S}_4, a \mapsto t_{2,4} \cdot a \cdot t_{2,4}$. It is easily shown that π is an automorphism of S_4, which has order 2 and fixes both x and y. Let $\mathcal{O} = \{2134, 1324, 4312, 1432, 3142, 2341\}$ and $\mathcal{E} = \{2314, 3124, 3412, 1342, 4132, 3241, 2431, 4321, 4213, 1423, 2143\}$. Then $\mathcal{O}_4 = \mathcal{O} \cup \pi(\mathcal{O}) \cup \{y\}$, and $\mathcal{E}_4 = \mathcal{E} \cup \{x\}$. Without loss of generality, we let $u \in \mathcal{O} \cup \pi(\mathcal{O})$ and $v \in \mathcal{E}$. For each $u \in \mathcal{O}$ and each $v \in \mathcal{E}$, a Hamilton path of $S_4 - \{x, y\}$ between u and v is described as in Table 1, unless three possible exceptions: $u = 1324$, $v = 3124$; $u = 4312$, $v = 4132$; and $u = 2341$, $v = 2143$. Note that there exists a Hamilton path of $S_4 - \{x, y\}$ between u and v if and only if there exists a Hamilton path of $S_4 - \{x, y\}$ between $\pi(u)$ and $\pi(v)$. When u runs over $\pi(\mathcal{O})$, we get the other three possible exceptions for $\{u, v\}$, which are the images of the above exceptions under π. Then the lemma follows. □

The following lemma will be used in the proof of Theorem 1.7.

Lemma 2.7 *The graph S_4 is 2-DCC $[6, 12]$-bipancyclic but not 2-DCC vertex $[6, 12]$-bipancyclic. More precisely, the following hold.*

1. *Let uv and xy be vertex-disjoint edges of S_4. If $\ell \in \{6, 8, 10\}$, then there exist vertex-disjoint ℓ-cycle C_1 and $(24-\ell)$-cycle C_2 in S_4 such that $uv \in E(C_1)$ and $xy \in E(C_2)$;*

2. *S_4 has exactly three pairs $\{C_1, C_2\}$ of vertex-disjoint 12-cycles, and if C_1 and C_2 are vertex-disjoint 12-cycles then the cycle containing 1234 must contain both 1342 and 1423.*

Proof. By the edge-transitivity of S_4, fix uv and traverse all $xy \in E(S_4 \setminus \{u, v\})$. Without loss of generality, we let $u = 1234$, $v = u \cdot t_{1,2} = 2134$. It is easily shown that the conjugation of $t_{3,4}$ on $\mathcal{S}_4$ gives an automorphism of S_4, say $\pi : \mathcal{S}_4 \to \mathcal{S}_4, a \mapsto t_{3,4} \cdot a \cdot t_{3,4}$. Then π has order 2 and fixes both u

and v. Write each edge ab as $\{a,b\}$, and put

$$E = \left\{ \begin{array}{l} \{2314, 1324\}, \{3241, 4231\}, \{4231, 2431\}, \{2431, 3421\}, \{3412, 4312\}, \{4312, 1342\}, \\ \{1342, 3142\}, \{3142, 4132\}, \{4132, 1432\}, \{1432, 3412\}, \{3412, 2413\}, \{2413, 4213\}, \\ \{4213, 1243\}, \{1243, 2143\}, \{2143, 4123\}, \{1423, 2413\}. \end{array} \right\}$$

and $E^* = \{\{1234, 2134\}, \{1234, 3214\}, \{1234, 4231\}, \{2134, 3124\}, \{2134, 4132\}\}$. Then $E(S_4) = E \cup \pi(E) \cup E^*$. By the choice of uv and xy, we have $\{x, y\} \notin E^*$. In addition, the edge xy lies on an ℓ-cycle if and only if so does $\pi(xy)$. To prove part (1), without loss of generality, we may choose $\{x, y\}$ from E. For such edges uv and xy, Table 2 illustrates the existence of cycles desired as in part (1).

Computation with GAP shows that there are exactly three pairs $\{C_1, C_2\}$ of vertex-disjoint 12-cycles, which are listed as follows:

C_1 : 1234, 2134, 3124, 4123, 1423, 2413, 3412, 4312, 1342, 2341, 3241, 4231, 1234;
C_2 : 1243, 2143, 3142, 4132, 1432, 2431, 3421, 4321, 1324, 2314, 3214, 4213, 1243.
C_1 : 1234, 2134, 4132, 3142, 1342, 2341, 4321, 3421, 1423, 2413, 4213, 3214, 1234;
C_2 : 1243, 2143, 4123, 3124, 1324, 2314, 4312, 3412, 1432, 2431, 4231, 3241, 1243.
C_1 : 1234, 3214, 2314, 4312, 1342, 3142, 2143, 4123, 1423, 3421, 2431, 4231, 1234;
C_2 : 1243, 4213, 2413, 3412, 1432, 4132, 2134, 3124, 1324, 4321, 2341, 3241, 1243.

It is easy to check that each cycle C_1 contains the vertices 1234, 1342 and 1423, and then part (2) of the lemma follows. □

Lemma 2.8 *Let C be an h-cycle in S_n^i, where $n \geqslant 5$ and $i \in [n]$.*

(1) *If $h = 6$, then $|\{\bar{x}(n) \mid x \in V(C)\}| = 3$.*

(2) *If $h \geqslant 8$, then there exist vertex-disjoint edges uv and wz in C such that $|\{\bar{u}(n), \bar{v}(n), \bar{w}(n), \bar{z}(n)\}| = 4$.*

Proof. It is easily shown that $x \mapsto t_{i,n} \cdot x$ is an automorphism of S_n which maps S_n^i to S_n^n. Thus, without loss of generality, we let $i = n$. Pick $x \in V(C)$, and assume the edges of C are i_1-edge, i_2-edge, ..., i_{h-1}-edge, i_h-edge in clockwise order around C from x. Then $i_1, i_2, \ldots, i_h \in [n] \setminus \{1, n\}$, $t_{1,i_1} \cdot t_{1,i_2} \cdot \cdots \cdot t_{1,i_h} = 123 \cdots n$, and $t_{1,i_s} \cdot t_{1,i_{s+1}} \cdot \cdots \cdot t_{1,i_t} \neq 123 \cdots n$ for any $1 \leqslant s < t \leqslant h$ with $t - s < h$. After a simple calculation, we conclude that $h = 6$ if and only if $i_s = i_t$ whenever $t - s$ is even.

Let $x_s = x \cdot t_{1,i_1} \cdot t_{1,i_2} \cdot \cdots \cdot t_{1,i_s}$ for $1 \leqslant s \leqslant h - 1$. Then $V(C) = \{x, x_1, \cdots, x_{h-1}\}$. Assume that $h = 6$. Then $i_1 = i_3 = i_5$ and $i_2 = i_4 = i_6$, We have $\bar{x}(n) = \bar{x}_3(n) = x(1)$, $\bar{x}_1(n) = \bar{x}_4(n) = x(i_1)$ and $\bar{x}_2(n) = \bar{x}_5(n) = x(i_2)$, and thus $|\{\bar{x}(n) \mid x \in V(C)\}| = 3$, desired as in part (1) of this lemma. Now let $h \geqslant 8$. Then there exist s and t with $t - s = 2$ and $i_s \neq i_t$; in particular, i_s, i_{s+1} and i_{s+2} are pairwise distinct. Choosing $u = x_{s-1}$, $v = x_s$, $w = x_{s+1}$ and $z = x_{s+2}$, we have $\bar{u}(n) = x_{s-1}(1)$, $\bar{v}(n) = x_{s-1}(i_s)$, $\bar{w}(n) = x_{s-1}(i_{s+1})$, and $\bar{z}(n) = x_{s-1}(i_{s+2})$, and then part (2) follows. □

For a nonempty subset $I \subseteq [n]$, denote by S_n^I the subgraph of S_n induced by $\bigcup_{i \in I} \mathcal{S}_n^i$.

Lemma 2.9 *Let $I \subseteq [n]$ with $|I| \geqslant 2$, where $n \geqslant 5$. Suppose P is either the null graph or a path of S_n^I, and for all $k \in I$ either $V(P) \cap \mathcal{S}_n^k = \emptyset$ or $V(P) \cap \mathcal{S}_n^k$ is the set of ends of some edge on P. Let $e \in E(S_n^i)$ for $i \in I$ with $V(P) \cap \mathcal{S}_n^i = \emptyset$. Then for any distinct $j_1, j_2 \in I$, $u \in \mathcal{O}_n^{j_1} \setminus V(P)$ and $v \in \mathcal{E}_n^{j_2} \setminus V(P)$, there is a Hamilton path of $S_n^I - P$ that contains e between u and v.*

Proof. Let $|I| = s$. Without loss of generality, we let $j_1 = 1$, $j_2 = s$, $I = [s]$. Put $u_1 = u$ and $v_s = v$. For each $1 \leqslant k \leqslant s - 1$, we have $|E_n^{k,k+1} \cap E(P)| \leqslant 1$ and $|V(E_n^{k,k+1}) \cap V(P)| \leqslant 2$ by the assumption. Then Lemma 2.4 allows us choose $v_k u_{k+1} \in E_n^{k,k+1}$ with $v_k \in \mathcal{E}_n^k \setminus V(P)$ and $u_{k+1} \in \mathcal{O}_n^{k+1} \setminus V(P)$. Note that each $S_n^k - P$ is either S_n^k or obtained by deleting the endpoints of some edge on P.

Assume first that $n > 5$. Applying Lemma 2.3 to each $S_n^k - P$, there exists a Hamilton path of $S_n^k - P$, say P_k, between u_k and v_k, where $1 \leqslant k \leqslant s$ and $k \neq i$. By Lemma 2.1, there is a Hamilton path that contains e of S_n^i, say P_i, between u_i and v_i. Then we have a Hamilton path that contains e of $S_n^I - P$ between u and v, say $P_1 + v_1u_2 + P_2 + v_2u_3 + \cdots + v_{s-1}u_s + P_k$.

Now let $n = 5$. Then $|\mathcal{O}_5^k| = 12 = |\mathcal{E}_5^k|$ for $1 \leqslant k \leqslant s$, and $|\mathcal{O}_5^k \cap V(E_5^{k,k+1})| = 3 = |\mathcal{E}_5^k \cap V(E_5^{k,k+1})|$ for $1 \leqslant k \leqslant s-1$. Then, by Lemma 2.6, we may choose u_k and v_k such that there exists a Hamilton path of $S_5^k - P$, say P_k, between u_k and v_k, where $1 \leqslant k \leqslant s$ and $k \neq i$. By Lemma 2.1, there is a Hamilton path that contains e of S_n^i, say P_i, between u_i and v_i. Thus we have a Hamilton path $P_1 + v_1u_2 + P_2 + v_2u_3 + \cdots + v_{k-1}u_k + P_k$ that contains e of $S_5^I - P$ between u and v. This completes the proof. $\square$

Lemma 2.10 *Let $I \subseteq [n]$ with $|I| \geqslant 3$, where $n \geqslant 5$. Suppose P is either the null graph or a path of S_n^I and for all $k \in I$ either $V(P) \cap \mathcal{S}_n^k = \emptyset$ or $V(P) \cap \mathcal{S}_n^k$ is the set of ends of some edge on P. Let $e \in E(S_n^i)$ for $i \in I$ with $V(P) \cap \mathcal{S}_n^i = \emptyset$. Then there is a Hamilton cycle of $S_n^I - P$ that contains e.*

Proof. Pick distinct $i_1, i_2 \in I \setminus \{i\}$. Then S_n^i has exactly $(n-2)!$ vertices x with $x(1) = i_1$, and these vertices x can be partitioned into $n-2$ classes $U_j := \{x \in \mathcal{S}_n^i \mid x(1) = i_1, x(j) = i_2\}$, $j \in [n] \setminus \{1, n\}$, each has length $(n-3)!$. Let $x \in U_j$. Then $\bar{x}(n) = x \cdot t_{1,n}(n) = i_1$, i.e., the n-neighbor $\bar{x}$ of x is contained in $S_n^{i_1}$. Let $U := \bigcup_{j \in [n]\setminus\{1,n\}} U_j$ and $U^* := \{\bar{x} \mid x \in U\}$. Then $U^* \subseteq \mathcal{S}_n^{i_1}$.

Fix distinct $x_1, x_2 \in U$. If $x_1x_2 \in E(S_n^i)$ then $x_2 = x_1 \cdot t_{1,k_1}$ for some $k_1 \in [n] \setminus \{1, n\}$, and so $x_1(1) = i_1 = x_2(1) = x_1 \cdot t_{1,k_1}(1) = x_1(k_1)$, yielding $k_1 = 1$, a contradiction. If $\bar{x}_1\bar{x}_2 \in E(S_n^{i_1})$ then $\bar{x}_2 = \bar{x}_1 \cdot t_{1,k_2}$ for some $k_2 \in [n] \setminus \{1, n\}$, and so $\bar{x}_1(1) = i = \bar{x}_2(1) = \bar{x}_1 \cdot t_{1,k_2}(1) = \bar{x}_1(k_2)$, implying $k_2 = 1$, a contradiction. Therefore U is an independent set of S_n^i and U^* is an independent set of $S_n^{i_1}$.

By the assumption, since U^* is an independent set, we have $|V(P) \cap U^*| \leqslant 1$. It follows that there exist at least $j_1, j_2 \in [n] \setminus \{1, n\}$ such that the n-neighbor $\bar{x}$ of any $x \in U_{j_1} \cup U_{j_2}$ is not contained in $V(P)$. Pick distinct $x_{1_j}, x_{2_j} \in U_j$, put $y_{1_j} = x_{1_j} \cdot t_{1,j}$ and $y_{2_j} = x_{2_j} \cdot t_{1,j}$, where $j \in \{j_1, j_2\}$. Then $\bar{y}_{1_j}, \bar{y}_{2_j} \in \mathcal{S}_n^{i_2}$. Recalling that $x_{1_j}x_{2_j} \notin E(S_n^i)$ from the discussion above, we have $y_{1_j}y_{2_j} \notin E(S_n^i)$, and so $\bar{y}_{1_j}\bar{y}_{2_j} \notin E(S_n^{i_2})$. Then the assumption implies that one of $\bar{y}_{1_j}$ and $\bar{y}_{2_j}$ is not contained in $V(P)$. Thus, for each $j \in \{j_1, j_2\}$, we have an edge $xy \in E(S_n^i)$ such that $\bar{x}, \bar{y} \notin V(P)$, and $\bar{x}(n), \bar{y}(n) \in I_0 := I \setminus \{i\}$. Pick such an edge $e^* = xy$ with $e^* \neq e$. Corollary 2.2 implies that there is a Hamilton cycle C that contains e and e^* in S_n^i. By Lemma 2.9, there is a Hamilton path P^* in $S_n^{I_0} - P$ between $\bar{x}$ and $\bar{y}$. Therefore, $S_n^I - P$ has a Hamilton cycle that contains e, say $C - xy + P^* + x\bar{x} + y\bar{y}$. This completes the proof. $\square$

3 Constructions of cycles

This section aims to construct several cycles from any given cycle in S_n^n. The proof of Theorem 1.5 is based on these constructions. In the following, assume that X is an h-cycle in S_n^n. Fix an edge wz of X, and write $z = w \cdot t_{1,i}$. Clearly, $i \neq n$. Let $[1, n-1] = [n] \setminus \{n\}$.

$\mathcal{D}$**1**: Construction of $(h+4)$-cycle:

We expand the edge wz to a 6-cycle C_1 by adding two i-edges and three n-edges, which has vertices w, z, $\bar{z} := z \cdot t_{1,n}$, $z_1 := \bar{z} \cdot t_{1,i}$, $\bar{z}_1 := z_1 \cdot t_{1,n}$ and $\bar{w} := w \cdot t_{1,n} = \bar{z}_1 \cdot t_{1,i}$. Calculation shows that $\bar{z}(n) = z(1) = z_1(n)$ and $\bar{w}(n) = w(1) = z(i) = \bar{z}_1(n)$. Since $i \neq n$ and $z(n) = n$, we have $z(1) \neq n$ and $z(i) \neq n$, and so $\bar{z}, z_1, \bar{z}_1, \bar{w} \notin \mathcal{S}_n^n$. In particular, $\bar{z}, z_1, \bar{z}_1, \bar{w} \notin V(X)$. Thus we have a cycle $X_1 := (X - wz) + (C_1 - wz)$ of length $h+4$. In addition, $V(X_1) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w(1)} \cup \mathcal{S}_n^{z(1)}$.

$\mathcal{D}$**2**: Construction of $(h+6)$-cycles:

Let $j \in [n] \setminus \{1, i, n\}$. Then the edge wz can be expanded to an 8-cycle C_2 by adding one i-edge, two j-edges and four n-edges, which has vertices w, z, $\bar{z} := z \cdot t_{1,n}$, $z_1 := \bar{z} \cdot t_{1,j}$, $\bar{z}_1 := z_1 \cdot t_{1,n}$, $z_2 := \bar{z}_1 \cdot t_{1,i}$, $\bar{z}_2 := z_2 \cdot t_{1,n}$ and $\bar{w} := w \cdot t_{1,n} = \bar{z}_2 \cdot t_{1,j}$. Considering the images of n under these permutations, we have $\bar{w}(n) = w(1) = \bar{z}_2(n)$, $w(n) = n = z(n)$, $\bar{z}(n) = z(1) = w(i) = z_1(n)$ and

$\bar{z}_1(n) = w(j) = z_2(n)$. By the choices of j and X, we have $\bar{z}, z_1, \bar{z}_1, z_2, \bar{z}_2, \bar{w} \notin V(X)$. Thus we have a cycle $X_2 := (X - wz) + (C_2 - wz)$ of length $h+6$. In addition, $V(X_2) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w(1)} \cup \mathcal{S}_n^{z(1)} \cup \mathcal{S}_n^{w(j)}$. When j runs over $[n] \setminus \{1, i, n\}$, we get $n-3$ cycles of length $h+6$, which contain a common path $X - wz + w\bar{w} + z\bar{z}$.

$\mathcal{D}$**3**: Construction of $(h + (n-1)! + 2)$-cycles:

First, we have a 6-cycle C_1 given as in $\mathcal{D}_1$. Noting that $\bar{w}(n) = w(1) = z(i) = \bar{z}_1(n)$ and $\bar{z}(n) = z(1) = z_1(n)$, by Lemma 2.1, we may pick a Hamilton path P in $S_n^{w(1)}$ between $\bar{w}$ and $\bar{z}_1$ and a Hamilton path P' in $S_n^{z(1)}$ between $\bar{z}$ and z_1. Then we have two $((n-1)!+4)$-cycles $C_3 := C_1 - \bar{w}\bar{z}_1 + P$ and $C_3' := C_1 - \bar{z}z_1 + P'$. Thus we have two $(h + (n-1)! + 2)$-cycles $X_3 := (X - wz) + (C_3 - wz)$ and $X_3' := (X - wz) + (C_3' - wz)$, which contain a common path $X - wz + w\bar{w} + z\bar{z}$. In addition, $\mathcal{S}_n^{w(1)} \subseteq V(X_3) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w(1)} \cup \mathcal{S}_n^{z(1)}$ and $\mathcal{S}_n^{z(1)} \subseteq V(X_3') \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w(1)} \cup \mathcal{S}_n^{z(1)}$.

$\mathcal{D}$**4**. Construction of $(h + k(n-1)! + 2)$-cycles for $2 \leqslant k \leqslant n-2$:

Let $j \in [n] \setminus \{1, i, n\}$, and C_2 be the 8-cycle constructed as in $\mathcal{D}_2$. Recall that $\bar{w}(n) = w(1) = \bar{z}_2(n)$, $w(n) = n = z(n)$, $\bar{z}(n) = z(1) = w(i) = z_1(n)$ and $\bar{z}_1(n) = w(j) = z_2(n)$. By the choices of j and wz, we know that $w(1)$, $w(i)$, $w(j)$ and n are pairwise distinct. Pick $I \subseteq [n] \setminus \{z(1), n\}$ with $w(1), w(j) \in I$. By Lemma 2.9, it is easily shown that S_n^I contains a Hamilton path P between $\bar{w}$ and $\bar{z}_1$. Set $|I| = k$. Then $2 \leqslant k \leqslant n-2$, and we get a cycle C_4 of length $k(n-1)!+4$, say $C_2 - \bar{w}\bar{z}_2 - \bar{z}_2 z_2 - z_2\bar{z}_1 + P$. It is easy to see that $V(C_4) \cap V(X) = \{w, z\}$. Thus we have a cycle $X_4 := (X - wz) + (C_4 - wz)$ of length $h + k(n-1)! + 2$. In addition, $\mathcal{S}_n^I \subseteq V(X_4) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^I \cup \mathcal{S}_n^{z(1)}$ and $\mathcal{S}_n^{z(1)} \nsubseteq V(X_4) \cap V(S_n^{[1,n-1]})$. When j runs over $[n] \setminus \{1, i, n\}$, we get $n-3$ cycles of length $h + k(n-1)! + 2$, which contain a common path $X - wz + w\bar{w} + z\bar{z}$.

$\mathcal{D}$**5**. Construction of $(h + (n-1)! + 4)$-cycles:

Let $j \in [n] \setminus \{1, i, n\}$, and C_2 be the 8-cycle provided as in $\mathcal{D}_2$. Then $\bar{w}, \bar{z}_2 \in \mathcal{S}_n^{w(1)}$, $z_2, \bar{z}_1 \in \mathcal{S}_n^{w(j)}$ and $\bar{z}, z_1 \in \mathcal{S}_n^{z(1)}$. By Lemma 2.1, we may choose a Hamilton path P in $S_n^{w(1)}$ between $\bar{w}$ and $\bar{z}_2$, a Hamilton path P' in $S_n^{w(j)}$ between z_2 and $\bar{z}_1$ and a Hamilton path P'' in $S_n^{z(1)}$ between $\bar{z}$ and z_1. Then we have three $((n-1)!+6)$-cycles $C_5 := C_2 - \bar{w}\bar{z}_2 + P$, $C_5' := C_2 - z_2\bar{z}_1 + P'$ and $C_5'' := C_2 - \bar{z}z_1 + P''$. Thus we get three $(h + (n-1)! + 4)$-cycles $X_5 := (X - wz) + (C_5 - wz)$, $X_5' := (X - wz) + (C_5' - wz)$ and $X_5'' := (X - wz) + (C_5'' - wz)$. In addition, $\mathcal{S}_n^{w(1)} \subseteq V(X_5) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w(1)} \cup \mathcal{S}_n^{z(1)} \cup \mathcal{S}_n^{w(j)}$, $\mathcal{S}_n^{w(j)} \subseteq V(X_5') \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w(1)} \cup \mathcal{S}_n^{z(1)} \cup \mathcal{S}_n^{w(j)}$ and $\mathcal{S}_n^{z(1)} \subseteq V(X_5'') \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w(1)} \cup \mathcal{S}_n^{z(1)} \cup \mathcal{S}_n^{w(j)}$. Letting j run over $[n] \setminus \{1, i, n\}$, we get $3(n-3)$ cycles of length $h + (n-1)! + 4$, which contain a common path $X - wz + w\bar{w} + z\bar{z}$.

4 The proof the Theorem 1.5

We shall process by induction on n. Thus we assume that either $n = 5$, or $n \geqslant 6$ and S_{n-1} is 2-DCC edge $[6, \frac{(n-1)!}{2}]$-bipancyclic. Recall that $S_n^n \cong S_{n-1}$. By Remark 1.4, if $n > 5$ then the graph S_{n-1} is also 2-DCC edge $[\frac{(n-1)!}{2}, (n-1)! - 6]$-bipancyclic. Thus

(I) S_n^n is 2-DCC edge $[6, (n-1)! - 6]$-bipancyclic, where $n > 5$.

Similarly, due to Lemma 2.7,

(II) S_5^5 is 2-DCC $[6, 18]$-bipancyclic, 2-DCC edge $[6, 10]$-bipancyclic and 2-DCC edge $[14, 18]$-bipancyclic.

Let uv and xy be vertex-disjoint edges in S_n. It suffices to prove the following (†) holds for all even integers ℓ with $6 \leqslant \ell \leqslant \frac{n!}{2}$.

(†) S_n has vertex-disjoint ℓ-cycle C and $(n! - \ell)$-cycle C^* such that $uv \in E(C)$ and $xy \in E(C^*)$.

By the edge-transitivity of of S_n, without loss of generality, we let $u = 12\ldots n$, the identity of $\mathcal{S}_n$, and let $v = t_{1,2}$. Pick $m \in [n] \setminus \{1, 2, n\}$, and consider the conjugation of $t_{m,n}$ on $\mathcal{S}_n$. Then we have an automorphism of the star graph S_n, say, $\phi : \mathcal{S}_n \to \mathcal{S}_n$, $a \mapsto t_{m,n} \cdot a \cdot t_{m,n}$. It is straightforward to checked that $\phi(u) = u$ and $\phi(v) = v$. If $y = x \cdot t_{1,n}$ then $\phi(y) = \phi(x \cdot t_{1,n}) = \phi(x) \cdot t_{1,m}$. Thus, replacing xy by $\phi(xy)$ if necessary, we may choose xy as an m-edge for some $m \in [n] \setminus \{1, n\}$. Then $y(n) = x \cdot t_{1,m}(n) = x(n)$. Putting $i_0 = x(n)$, we have $xy \in E(S_n^{i_0})$. Therefore, our task is to prove the above (†) holds for the chosen edges uv and xy when ℓ runs over the even integers from 6 to $\frac{n!}{2}$. Thus, in the following, we always assume that

$$u = 123\ldots(n-1)n,\ v = 213\ldots(n-1)n, \text{ and } xy \in E(S_n^{i_0}) \text{ for some } i_0 \in [n].$$

Recall that $\bar{w} = w \cdot t_{1,n}$ for every $w \in V(S_n)$; in particular, $\bar{w}(n) = w(1)$. For integers $r_1 \leqslant r_2$, denote $[r_1, r_2]$ the set of integers from r_1 to r_2.

Lemma 4.1 *Assume that $k \in [1, \lceil \frac{n}{2} \rceil]$ and $i_0 \neq n$. Then (†) holds for all even integers ℓ in $[(k-1)(n-1)! + 6, k(n-1)! - 6]$.*

Proof. We discuss in three cases according to $\ell \in [6, (n-1)! - 6]$, $\ell = (k-1)(n-1)! + 6$ and $\ell \in [(k-1)(n-1)! + 8, k(n-1)! - 6]$ for $k \in [2, \lceil \frac{n}{2} \rceil]$, respectively. Since $i_0 \neq n$, we have $x, y \notin V(S_n^n)$.

Case 1. Assume that $\ell \in [6, (n-1)! - 6]$. By (I) and (II), we may choose vertex-disjoint ℓ-cycle C and $((n-1)! - \ell)$-cycle C_1 in S_n^n with $uv \in E(C)$. Pick an edge $wz \in E(C_1)$. Since $\bar{w}(n) = w(1) \neq w(n) = n$ and $\bar{z}(n) = z(1) \neq z(n) = n$, we have $\bar{w}, \bar{z} \notin V(S_n^n)$. By Lemma 2.9, $S_n^{[1,n-1]}$ has a Hamilton path P containing xy from $\bar{w}$ to $\bar{z}$, and we have an $(n! - \ell)$-cycle $C^* := (C_1 - wz) + w\bar{w} + z\bar{z} + P$ which contains the edge xy. Then C and C^* are vertex-disjoint, desired as in (†).

Case 2. Assume that $\ell = (k-1)(n-1)! + 6$, where $k \in [2, \lceil \frac{n}{2} \rceil]$.

Subcase 2.1. First consider $\ell = (n-1)! + 6$. Putting $h = (n-1)!$, we have $\ell = h + 6$. We may choose an edge wz in $S_n^n \setminus \{uv\}$ such that $i_0 \notin \{w(1), z(1)\}$. By Corollary 2.2, S_n^n has an h-cycle, say C_1, such that both uv and wz lie on C_1. Write $z = w \cdot t_{1,i}$, and pick $j \in [n] \setminus [1, i, n]$ with $w(j) \neq i_0$. Applying Construction $\mathcal{D}\mathbf{2}$ to C_1, wz and j, we get a cycle of length $h+6$, say C, which contains the path $(C_1 - wz) + w\bar{w} + z\bar{z}$, and $V(C) \cap V(S_n^{[1,n-1]}) \subset \mathcal{S}_n^{w(1)} \cup \mathcal{S}_n^{z(1)} \cup \mathcal{S}_n^{w(j)}$. In particular, $\mathcal{S}_n^{i_0} \cap V(C) = \emptyset$. Since $xy \in E(S_n^{i_0})$, according to Lemma 2.10, $S_n - C$ possesses a Hamilton cycle C^* that contains xy. Then C and C^* are vertex-disjoint cycles desired as in (†).

Subcase 2.2. Now deal with $\ell = (k-1)(n-1)! + 6$, where $k \in [3, \lceil \frac{n}{2} \rceil]$. Putting $h = (n-1)!$ yields that $\ell = h + (k-2)(n-1)! + 6$. Similarly as in Subcase 2.1, we may choose $wz \in E(S_n^n \setminus \{uv\})$ with $i_0 \notin \{w(1), z(1)\}$, and pick an h-cycle C_1 containing uv and wz in S_n^n by Corollary 2.2. Then applying Construction $\mathcal{D}\mathbf{3}$ to C_1 and the edge wz, we obatin an $(h + (n-1)! + 2)$-cycle, write C_0, such that $\mathcal{S}_n^{w(1)} \subseteq V(C_0) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w(1)} \cup \mathcal{S}_n^{z(1)}$ if $k = 3$. Otherwise an $(h + (k-2)(n-1)! + 2)$-cycle C_0 is obtained from C_1 by Construction $\mathcal{D}\mathbf{4}$, and satisfies $\mathcal{S}_n^I \subseteq V(C_0) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^I \cup \mathcal{S}_n^{z(1)}$ for some $I \subseteq [n] \setminus \{n, i_0, z(1)\}$ with $w(1) \in I$ and $|I| = k - 2 \geqslant 2$. Thus for both cases, we have $\mathcal{S}_n^{i_0} \cap V(C_0) = \emptyset$.

Noting that $\bar{z} \in S_n^{z(1)}$ and by Construction $\mathcal{D}\mathbf{3}$ and $\mathcal{D}\mathbf{4}$, we know that C_0 and $S_n^{z(1)}$ intersect some edge on C_0, write $\bar{z}z_1$. By choosing $w_2z_2 \in E(S_n^{z(1)} \setminus \{\bar{z}, z_1\})$ with $w_2(1) = i_0$ and $z_2(1) \notin I \cup \{n\}$ and applying (I) and (II), we can guarantee that S_n^n has vertex-disjoint 6-cycle C_2 and $((n-1)! - 6)$-cycle C_3 such that $\bar{z}z_1 \in E(C_2)$ and $w_2z_2 \in E(C_3)$. Now we get a $((k-1)(n-1)! + 6)$-cycle, says $C := (C_0 - \bar{z}z_1) + (C_2 - \bar{z}z_1)$. Then Lemma 2.9 implies that $S_n - C - C_3$ has a Hamilton path P that contains xy from $\bar{w}_2$ to $\bar{z}_2$. Thus $S_n - C$ has a Hamilton cycle $C^* := (C_3 - w_2z_2) + w_2\bar{w}_2 + z_2\bar{z}_2 + P$ containing xy. Therefore, C and C^* are vertex-disjoint, desired as in (†).

Case 3. Assume that $\ell \in [(k-1)(n-1)! + 8, k(n-1)! - 6]$, where $k \in [2, \lceil \frac{n}{2} \rceil]$.

Subcase 3.1. Suppose first that $\ell \in [(n-1)! + 8, 2(n-1)! - 6]$. Put

$$h = \begin{cases} \ell - (n-1)! - 2, & \text{if } \ell \neq (n-1)! + 14; \\ \ell - (n-1)! - 4, & \text{if } \ell = (n-1)! + 14. \end{cases}$$

Thus, $h \in [6, (n-1)!-8] \setminus \{12\}$ if $\ell \neq (n-1)!+14$, and $h = 10$ otherwise. Note that

$$\ell = \begin{cases} h + (n-1)! + 2, & \text{if } \ell \neq (n-1)! + 14; \\ h + (n-1)! + 4, & \text{if } \ell = (n-1)! + 14. \end{cases}$$

Choosing an edge wz from $S_n^n \setminus \{u, v\}$ with $w(1) = i_0$, and by (I) and (II), we have vertex-disjoint h-cycle C_1 and $((n-1)!-h)$-cycle C_2 in S_n^n with $uv \in E(C_1)$ and $wz \in E(C_2)$. Due to Lemma 2.8, we may choose an edge w_1z_1 from $C_1 \setminus \{uv\}$ such that $i_0 \notin \{w_1(1), z_1(1)\}$. Then since $w_1(1) \neq z_1(1)$, we may let $w_1(1) \neq z(1)$.

If $\ell \neq (n-1)!+14$, then applying Construction $\mathcal{D}\mathbf{3}$ to C_1 and the edge w_1z_1, we get an $(h + (n-1)!+2)$-cycle, say C, such that $\mathcal{S}_n^{w_1(1)} \subseteq V(C) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w_1(1)} \cup \mathcal{S}_n^{z_1(1)}$. Otherwise, $\ell = (n-1)!+14$. Write $z_1 = w_1 \cdot t_{1,i}$, and take $j \in [n] \setminus [1, i, n]$ with $w_1(j) \neq i_0$. We construct an $(h+(n-1)!+4)$-cycle C from C_1 by using Construction $\mathcal{D}\mathbf{5}$ and the edge w_1z_1, which satisfies $\mathcal{S}_n^{w_1(1)} \subseteq V(C) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w_1(1)} \cup \mathcal{S}_n^{z_1(1)} \cup \mathcal{S}_n^{w_1(j)}$. Noting that $i_0 \notin \{w_1(1), z_1(1), w_1(j)\}$ and $\bar{z} \notin \mathcal{S}_n^{w_1(1)}$, for both case, we get $\mathcal{S}_n^{i_0} \cap V(C) = \emptyset$ and $\bar{z} \notin V(C)$. Now it follows from Lemma 2.9 that $S_n - C - C_2$ has a Hamilton path P containing xy from $\bar{z}$ to $\bar{w}$. Thus $S_n - C$ has a Hamilton cycle $C^* := (C_2 - wz) + z\bar{z} + w\bar{w} + P$ that contains xy. Therefore, C and C^* are vertex-disjoint cycles required as in (†).

Subcase 3.2. Suppose that $\ell \neq (k-1)(n-1)!+14$, where $k \in [3, \lceil \frac{n}{2} \rceil]$. Putting $h \in [6, (n-1)!-8] \setminus \{12\}$ shows that $\ell = h + (k-1)(n-1)! + 2$. Choosing $wz \in E(S_n^n \setminus \{u, v\})$ with $w(1) = i_0$, similarly in Subcase 3.1, we have vertex-disjoint h-cycle C_1 and $((n-1)!-h)$-cycle C_2 with $uv \in E(C_1)$ and $wz \in E(C_2)$ in S_n^n by (I) and (II). Lemma 2.8 implies that we may choose $w_1z_1 \in E(C_1 \setminus \{uv\})$ with $i_0 \notin \{w_1(1), z_1(1)\}$ and $w_1(1) \neq z(1)$.

Assume first that $n = 5$, which implies that $k = 3$. By picking another neighbor of w in C_2, write z', and noting that $\{n, i_0\} \cap \{w_1(1), z_1(1), z(1), z'(1)\} = \emptyset$, we see $|\{w_1(1), z_1(1), z(1), z'(1)\}| \leqslant 3$. Without loss of generality, let $z(1) = z_1(1)$. Let $z_1 = w_1 \cdot t_{1,i}$, and choose $j \in [n] \setminus [1, i, n]$ with $w_1(j) \neq i_0$. Applying Construction $\mathcal{D}\mathbf{4}$ to C_1, w_1z_1 and j, we get an $(h + 2(n-1)! + 2)$-cycle C such that $\mathcal{S}_n^{w_1(1)} \cup \mathcal{S}_n^{w_1(j)} \subseteq V(C) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w_1(1)} \cup \mathcal{S}_n^{z_1(1)} \cup \mathcal{S}_n^{w_1(j)}$. Assume now that $n \geqslant 6$. Regardless of whether $z(1) = z_1(1)$. Now applying Construction $\mathcal{D}\mathbf{4}$ to C_1 and the edge w_1z_1, we have an $(h + (k-1)(n-1)! + 2)$-cycle C such that $\mathcal{S}_n^I \subseteq V(C) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^I \cup \mathcal{S}_n^{z_1(1)}$ for some $I \subseteq [n] \setminus \{n, i_0, z(1), z_1(1)\}$ with $w(1) \in I$ and $|I| = k-1 \geqslant 2$. Recalling that $i_0 \notin I$ and $\bar{z} \notin \mathcal{S}_n^I$, for both cases, we have $\mathcal{S}_n^{i_0} \cap V(C) = \emptyset$ and $\bar{z} \notin V(C)$. Then it follows from Lemma 2.9 that there exists a Hamilton path P that contains xy from $\bar{z}$ to $\bar{w}$ in $S_n - C - C_2$. Thus $S_n - C$ has a Hamilton cycle $C^* := (C_2 - wz) + z\bar{z} + w\bar{w} + P$ which contains the edge xy. Therefore, C and C^* are vertex-disjoint cycles desired as in (†).

Subcase 3.3. Suppose now that $\ell = (k-1)(n-1)!+14$, where $k \in [3, \lceil \frac{n}{2} \rceil]$, and let $h = (n-1)!$. Then $\ell = h + (k-2)(n-1)! + 14$. Analogously to the argument in Subcase 2.2, only instead of vertex-disjoint 6-cycle C_3 and $((n-1)!-6)$-cycle C_4 we shall use vertex-disjoint 14-cycle C_3 and $((n-1)!-14)$-cycle C_4. Then we obtain vertex-disjoint cycles C and C^* such that $uv \in E(C)$ and $xy \in E(C^*)$. □

Lemma 4.2 *Assume that $k \in [1, \lceil \frac{n}{2} \rceil]$ and $i_0 = n$. Then (†) holds for all even integers ℓ in $[(k-1)(n-1)!+6, k(n-1)!-6]$.*

Proof. We shall distinguish three cases according to the value of ℓ. Noting that $i_0 = n$, we have $x, y \in V(S_n^n)$.

Case 1. Assume that $\ell \in [6, (n-1)!-6]$. First, if $\ell \in [6, (n-1)!-6] \setminus \{12\}$, then put $h = \ell$. Otherwise, $\ell = 12$. Putting $h = 8$, we have $\ell = h + 4$. By (I) and (II), there exist vertex-disjoint h-cycle C_1 and $((n-1)!-h)$-cycle C_2 with $uv \in E(C_1)$ and $xy \in E(C_2)$ in S_n^n. Choosing two edges $w_1z_1 \in E(C_1 \setminus \{uv\})$ and $w_2z_2 \in E(C_2 \setminus \{xy\})$, we have that $\{\bar{w}_1, \bar{z}_1, \bar{w}_2, \bar{z}_2\}$ are pairwise different. Let C be an $(h+4)$-cycle obtained by applying construction $\mathcal{D}\mathbf{1}$ to C_1 and the edge w_1z_1 if $\ell = 12$. We note that $V(C)\cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w_1(1)} \cup \mathcal{S}_n^{z_1(1)}$ and $\bar{w}_2, \bar{z}_2 \notin V(C)$. Otherwise, $\ell \in [6,(n-1)!-6]\setminus\{12\}$, and pick $C := C_1$. Then Lemma 2.9 yields that $S_n - C - C_2$ has a Hamilton path P from $\bar{w}_2$ to $\bar{z}_2$. Thus $S_n - C$ has a Hamilton cycle $C^* := (C_2 - w_2z_2) + z_2\bar{z}_2 + w_2\bar{w}_2 + P$ containing xy. Then C and C^* are vertex-disjoint cycles desired as in (†).

Case 2. Assume that $\ell = (k-1)(n-1)! + 6$ for $k \in [2, \lceil \frac{n}{2} \rceil]$.

Subcase 2.1. Suppose first that $\ell = (n-1)! + 6$. Putting $h = (n-1)! - 6$, we have $\ell = h + 12$. According to (I) and (II), S_n^n has vertex-disjoint h-cycle C_1 and $((n-1)! - h)$-cycle C_2 with $uv \in E(C_1)$ and $xy \in E(C_2)$. Since $|E(C_2)| = 6$, we can pick first $wz \in E(C_2 \setminus \{xy\})$, and then pick $w_1z_1 \in E(C_1 \setminus \{uv\})$ such that $z_1(1) = z(1)$ and $|\{w_1(1), z(1), w(1)\}| = 3$. Then an $(h+4)$-cycle, say C_0, we get by applying construction $\mathcal{D}\mathbf{1}$ to C_1 and the edge w_1z_1. By noting that C_0 and $S_n^{z(1)}$ intersect some edge on C_0, write $\bar{z}_1z_1'$, and taking a neighbor of $\bar{z}$ in $S_n^{z(1)} \setminus \{\bar{z}_1, z_1'\}$, says z_2, with $z_2(1) \notin \{n, w(1)\}$, we obtain vertex-disjoint 10-cycle C_3 and $((n-1)! - 10)$-cycle C_4 with $\bar{z}_1z_1' \in E(C_3)$ and $\bar{z}z_2 \in E(C_4)$ in $S_n^{z(1)}$ using (I) and (II). Now an $((n-1)!+6)$-cycle is obtained, write $C := (C_0 - \bar{z}_1z_1') + (C_3 - \bar{z}_1z_1'))$. Recall from Lemma 2.9 that $S_n - C - C_2 - C_4$ has a Hamilton path P from $\bar{z}_2$ to $\bar{w}$. Thus $S_n - C$ has a Hamilton cycle $C^* := (C_2 - wz) + (C_4 - \bar{z}z_2) + w\bar{w} + z\bar{z} + z_2\bar{z}_2 + P$ which contains the edge xy. Then C and C^* are vertex-disjoint, desired as in (†).

Subcase 2.2. Suppose now that $\ell = (k-1)(n-1)! + 6$ for $k \in [3, \lceil \frac{n}{2} \rceil]$. Putting $h = (n-1)! - 8$ yields that $\ell = h + (k-2)(n-1)! + 14$. By (I) and (II), S_n^n, there exist vertex-disjoint h-cycle C_1 and $((n-1)! - h)$-cycle C_2 with $uv \in E(C_1)$ and $xy \in E(C_2)$. Similarly as in Subcase 2.1, by Lemma 2.4 and 2.5, we choose $wz \in E(C_2 \setminus \{xy\})$ and $w_1z_1 \in E(C_1 \setminus \{uv\})$ with $z_1(1) = z(1)$ and $|\{w_1(1), z(1), w(1)\}| = 3$ since $|E(C_2)| = 8$.

Let $k = 3$. Then by applying Construction $\mathcal{D}\mathbf{3}$ to C_1 and the edge w_1z_1, we get an $(h+(n-1)!+2)$-cycle C_0 such that $\mathcal{S}_n^{w_1(1)} \subseteq V(C_0) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w_1(1)} \cup \mathcal{S}_n^{z_1(1)}$. However, for $k \geqslant 4$, applying Construction $\mathcal{D}\mathbf{4}$ to C_1 and the edge w_1z_1, we have an $(h + (k-2)(n-1)! + 2)$-cycle C_0 such that $\mathcal{S}_n^I \subseteq V(C_0) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^I \cup \mathcal{S}_n^{z_1(1)}$ for some $I \subseteq [n] \setminus \{n, w(1), z(1)\}$ with $w_1(1) \in I$ and $|I| = k-2 \geqslant 2$. Noting that $\bar{z}_1 \in S_n^{z(1)}$ and by Construction $\mathcal{D}\mathbf{3}$ and $\mathcal{D}\mathbf{4}$, we have C_0 and $S_n^{z(1)}$ intersect some edge on C_0, write $\bar{z}_1z_1'$. Then taking a neighbor of $\bar{z}$ in $S_n^{z(1)} \setminus \{\bar{z}_1, z_1'\}$, says z_2, with $z_2(1) \notin \{n, w(1)\}$, we obtain vertex-disjoint 14-cycle C_3 and $((n-1)! - 14)$-cycle C_4 with $\bar{z}_1z_1' \in E(C_3)$ and $\bar{z}z_2 \in E(C_4)$ in $S_n^{z(1)}$ by (I) and (II). Now we get a $((k-1)(n-1)!+6)$-cycle, says $C := (C_0 - \bar{z}_1z_1') + (C_3 - \bar{z}_1z_1'))$. Then Lemma 2.9 yields that there is a Hamilton path P from $\bar{w}$ to $\bar{z}_2$ in $S_n - C - C_2 - C_4$. Thus $S_n - C$ has a Hamilton cycle $C^* := (C_2 - wz) + (C_4 - \bar{z}z_2) + w\bar{w} + z\bar{z} + z_2\bar{z}_2 + P$ that contains xy. Therefore, C and C^* are vertex-disjoint, required as in (†).

Case 3. Assume that $\ell \in [(k-1)(n-1)! + 8, k(n-1)! - 6]$ for $k \in [2, \lceil \frac{n}{2} \rceil]$.

Subcase 3.1. Suppose first that $\ell \in [(n-1)! + 8, 2(n-1)! - 6]$. Take

$$h = \begin{cases} \ell - (n-1)! - 2, & \text{if } \ell \neq (n-1)! + 14; \\ \ell - (n-1)! - 4, & \text{if } \ell = (n-1)! + 14. \end{cases}$$

Thus, $h \in [6, (n-1)! - 8] \setminus \{12\}$ if $\ell \neq (n-1)! + 14$, and $h = 10$ otherwise. Note that

$$\ell = \begin{cases} h + (n-1)! + 2, & \text{if } \ell \neq (n-1)! + 14; \\ h + (n-1)! + 4, & \text{if } \ell = (n-1)! + 14. \end{cases}$$

Recall from (I) and (II) that there exist vertex-disjoint h-cycle C_1 and $((n-1)! - h)$-cycle C_2 with $uv \in E(C_1)$ and $xy \in E(C_2)$ in S_n^n. By Lemma 2.8, we may choose two edges $w_1z_1 \in E(C_1 \setminus \{uv\})$ and $wz \in E(C_2 \setminus \{xy\})$ such that $|\{w_1(1), z_1(1), w(1), z(1)\}| \geqslant 3$. Without loss of generality, let $w_1(1) \notin \{z(1), w(1)\}$.

If $\ell \neq (n-1)! + 14$, then an $(h + (n-1)! + 2)$-cycle, say C, is obtained by applying Construction $\mathcal{D}\mathbf{3}$ to C_1 and the edge w_1z_1, and satisfies $\mathcal{S}_n^{w_1(1)} \subseteq V(C) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w_1(1)} \cup \mathcal{S}_n^{z_1(1)}$. Otherwise, $\ell = (n-1)! + 14$. Write $z_1 = w_1 \cdot t_{1,i}$, and pick $j \in [n] \setminus [1, i, n]$. Applying Construction $\mathcal{D}\mathbf{5}$ to

C_1, w_1z_1 and j, we get an $(h+(n-1)!+4)$-cycle, say C, such that $\mathcal{S}_n^{w_1(1)} \subseteq V(C)\cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w_1(1)} \cup \mathcal{S}_n^{z_1(1)} \cup \mathcal{S}_n^{w_1(j)}$. Noting that $\bar{w}, \bar{z} \notin \mathcal{S}_n^{w_1(1)}$, for both cases, we have $\bar{w}, \bar{z} \notin V(C)$. Then Lemma 2.9 yields that $S_n - C - C_2$ has a Hamilton path P from $\bar{z}$ to $\bar{w}$. Thus $S_n - C$ has a Hamilton cycle $C^* := (C_2 - wz) + z\bar{z} + w\bar{w} + P$ that contains xy. Therefore, C and C^* are vertex-disjoint cycles desired as in (†).

Subcase 3.2. Suppose that $\ell \in [(k-1)(n-1)!+8, k(n-1)!-6] \setminus \{(k-1)(n-1)!+14\}$ for $k \in [3, \lceil \frac{n}{2} \rceil]$. Putting $h \in [6, (n-1)!-8] \setminus \{12\}$, we get $\ell = h + (k-1)(n-1)! + 2$. By (I) and (II), we have vertex-disjoint h-cycle C_1 and $((n-1)!-h)$-cycle C_2 with $uv \in E(C_1)$ and $xy \in E(C_2)$ in S_n^n. And Lemma 2.8 shows that we may choose two edges $w_1z_1 \in E(C_1 \setminus \{uv\})$ and $wz \in E(C_2 \setminus \{xy\})$ such that $|\{w_1(1), z_1(1), w(1), z(1)\}| \geqslant 3$. Without loss of generality, let $w_1(1) \notin \{z(1), w(1)\}$.

If $n=5$, then $k=3$. Let z_1' be another neighbor of w_1 in C_1. Since $\{n, w_1(1)\} \cap \{z_1(1), z_1'(1), w(1), z(1)\} = \emptyset$, we see $|\{z_1(1), z_1'(1), w(1), z(1)\}| \leqslant 3$. Without loss of generality, let $z(1) = z_1(1)$. Write $z_1 = w_1 \cdot t_{1,i}$, and pick $j \in [n] \setminus [1, i, n]$ with $w_1(j) \neq w(1)$. Applying Construction $\mathcal{D}\mathbf{4}$ to C_1, w_1z_1 and j, we obtain an $(h+2(n-1)!+2)$-cycle, say C, such that $\mathcal{S}_n^{w_1(1)} \cup \mathcal{S}_n^{w_1(j)} \subseteq V(C) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w_1(1)} \cup \mathcal{S}_n^{z_1(1)} \cup \mathcal{S}_n^{w_1(j)}$. Otherwise, $n \geqslant 6$. We constructe an $(h+(k-1)(n-1)!+2)$-cycle, say C, by applying Construction $\mathcal{D}\mathbf{4}$ to C_1 and the edge w_1z_1, which satisfies $\mathcal{S}_n^I \subseteq V(C) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^I \cup \mathcal{S}_n^{z_1(1)}$ for some $I \subseteq [n] \setminus \{n, w(1), z(1), z_1(1)\}$ with $w_1(1) \in I$ and $|I| = k-1 \geqslant 2$. Thus for both cases, we have $\bar{w}, \bar{z} \notin V(C)$. Then from Lemma 2.9, we find that $S_n - C - C_2$ has a Hamilton path P from $\bar{z}$ to $\bar{w}$. Thus $S_n - C$ has a Hamilton cycle $C^* := (C_2 - wz) + z\bar{z} + w\bar{w} + P$ which contains the edge xy. Then C and C^* are vertex-disjoint cycles desired as in (†).

Subcase 3.3. Suppose now that $\ell = (k-1)(n-1)! + 14$ for $k \in [3, \lceil \frac{n}{2} \rceil]$. Putting $h=6$ shows that $\ell = h + (k-1)(n-1)! + 8$. Recall from (I) and (II) that S_n^n has vertex-disjoint h-cycle C_1 and $((n-1)!-h)$-cycle C_2 with $uv \in E(C_1)$ and $xy \in E(C_2)$. Noting that $|E(C_1)| = 6$ and using Lemma 2.8, we choose first $w_1z_1 \in E(C_1 \setminus \{uv\})$, and then choose $wz \in E(C_2 \setminus \{xy\})$ with $z_1(1) = z(1)$ and $|\{w_1(1), z(1), w(1)\}| = 3$.

Now applying Construction $\mathcal{D}\mathbf{4}$ to C_1 and the edge w_1z_1, we obtain an $(h+(k-1)(n-1)!+2)$-cycle C_0 such that $\mathcal{S}_n^I \subseteq V(C_0) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^I \cup \mathcal{S}_n^{z_1(1)}$ for some $I \subseteq [n] \setminus \{n, w(1), z(1)\}$ with $w_1(1) \in I$ and $|I| = k-1 \geqslant 2$. Noting that $\bar{z}_1 \in S_n^{z(1)}$ and by Construction $\mathcal{D}\mathbf{4}$, we have C_0 and $S_n^{z(1)}$ intersect some edge on C_0, write $\bar{z}_1z_1'$. Consider the case that $n=5$, which implies that $k=3$. Taking a neighbor of $\bar{z}$ in $S_n^{z(1)} \setminus \{\bar{z}_1, z_1'\}$, says z_2, with $z_2(1) = w(1)$. However, for $n \geqslant 6$, we choose a neighbor z_2 of $\bar{z}$ in $S_n^{z(1)} \setminus \{\bar{z}_1, z_1'\}$ such that $z_2(1) \notin \{n, w(1)\}$. According to (I) and (II), we obtain vertex-disjoint 8-cycle C_3 and $((n-1)!-8)$-cycle C_4 with $\bar{z}_1z_1' \in E(C_3)$ and $\bar{z}z_2 \in E(C_4)$ in $S_n^{z(1)}$. Then clearly an $(h+(k-1)(n-1)!+8)$-cycle, which says $C := (C_0 - \bar{z}_1z_1') + (C_3 - \bar{z}_1z_1'))$ is determined. Then Lemma 2.9 yields that $S_n - C - C_2 - C_4$ has a Hamilton path P from $\bar{z}_2$ to $\bar{w}$. Thus $S_n - C$ has a Hamilton cycle $C^* := (C_2 - wz) + (C_4 - \bar{z}z_2) + w\bar{w} + z\bar{z} + z_2\bar{z}_2 + P$ that contains xy. Then C and C^* are vertex-disjoint, desired as in (†). □

Lemma 4.3 *Assume that $k \in [1, \lceil \frac{n}{2} \rceil]$. Then (†) holds for all even integers $\ell = k(n-1)! - 4$.*

Proof. Put

$$h = \begin{cases} \ell - 4, & \text{if } \ell = (n-1)! - 4; \\ \ell - (n-1)! - 2, & \text{if } \ell = 2(n-1)! - 4; \\ \ell - (k-1)(n-1)! - 2, & \text{if } \ell = k(n-1)! - 4, \text{ where } k \in [3, \lceil \frac{n}{2} \rceil]. \end{cases}$$

Thus, $h = (n-1)! - 8$ if $\ell = (n-1)! - 4$, and $h = (n-1)! - 6$ otherwise. Note that

$$\ell = \begin{cases} h + 4, & \text{if } \ell = (n-1)! - 4; \\ h + (n-1)! + 2, & \text{if } \ell = 2(n-1)! - 4; \\ h + (k-1)(n-1)! + 2, & \text{if } \ell = k(n-1)! - 4, \text{ where } k \in [3, \lceil \frac{n}{2} \rceil]. \end{cases}$$

Assume that $i_0 \neq n$. Pick an edge wz of $S_n^n \setminus \{u, v\}$ such that $w(1) = i_0$. It follows from (I) and (II) that S_n^n has vertex-disjoint h-cycle C_1 and $((n-1)! - h)$-cycle C_2 with $uv \in E(C_1)$ and $wz \in E(C_2)$. Since $|E(C_2)| = 6$ or 8 and by Lemma 2.8, we may choose an edge $w_1z_1 \in E(C_1 \setminus \{uv\})$ such that $z_1(1) = z(1)$ and $w_1(1) \neq i_0$.

Assume that $i_0 = n$. Recall from (I) and (II) that S_n^n has vertex-disjoint h-cycle C_1 and $((n-1)! - h)$-cycle C_2 with $uv \in E(C_1)$ and $xy \in E(C_2)$. Since $|E(C_2)| = 6$ or 8 and by Lemma 2.8, we choose first an edge $wz \in E(C_2 \setminus \{xy\})$, and then choose an edge $w_1z_1 \in E(C_1 \setminus \{uv\})$ with $z_1(1) = z(1)$ and $|\{w_1(1), z(1), w(1)\}| = 3$.

Suppose first that $\ell = (n-1)! - 4$. We get an $(h+4)$-cycle C by applying Construction $\mathcal{D}\mathbf{1}$ to C_1 and the edge w_1z_1. Suppose that $\ell = 2(n-1)! - 4$. Applying Construction $\mathcal{D}\mathbf{3}$ to C_1 and the edge w_1z_1, we find an $(h+(n-1)!+2)$-cycle, say C, such that $\mathcal{S}_n^{w_1(1)} \subseteq V(C) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w_1(1)} \cup \mathcal{S}_n^{z_1(1)}$. Suppose now that $\ell = k(n-1)! - 4$, where $k \in [3, \lceil \frac{n}{2} \rceil]$. Then an $(h+(k-1)(n-1)!+2)$-cycle, say C, is obtained by applying Construction $\mathcal{D}\mathbf{4}$ to C_1 and the edge w_1z_1, and satisfies $\mathcal{S}_n^I \subseteq V(C) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^I \cup \mathcal{S}_n^{z_1(1)}$ for some $I \subseteq [n] \setminus \{n, w(1), z(1)\}$ with $w_1(1) \in I$ and $|I| = k - 1 \geqslant 2$. Thus for the above, we have $\mathcal{S}_n^{i_0} \cap V(C) = \emptyset$ and $\bar{z} \notin V(C)$ if $i_0 \neq n$. Otherwise $\bar{w}, \bar{z} \notin V(C)$. Then applying Lemma 2.9 to $S_n - C - C_2$, we have either a Hamilton path P containing xy from $\bar{w}$ to $\bar{z}$ ($i_0 \neq n$) or a Hamilton path P from $\bar{w}$ to $\bar{z}$ ($i_0 = n$). Thus $S_n - C$ has a Hamilton cycle $C^* := (C_2 - wz) + z\bar{z} + w\bar{w} + P$ that contains xy. Then C and C^* are vertex-disjoint cycles desired as in (†). □

Lemma 4.4 *Assume that $k \in [1, \lceil \frac{n}{2} \rceil]$. Then (†) holds for all even integers $\ell = k(n-1)! - 2$.*

Proof. Put

$$h = \begin{cases} \ell - 4, & \text{if } \ell = (n-1)! - 2; \\ \ell - (n-1)! - 6, & \text{if } \ell = 2(n-1)! - 2; \\ \ell - (k-1)(n-1)! - 6, & \text{if } \ell = k(n-1)! - 2, \text{ where } k \in [3, \lceil \frac{n}{2} \rceil]. \end{cases}$$

Thus, $h = (n-1)! - 6$ if $\ell = (n-1)! - 2$, and $h = (n-1)! - 8$ otherwise. Note that

$$\ell = \begin{cases} h + 4, & \text{if } \ell = (n-1)! - 2; \\ h + (n-1)! + 6, & \text{if } \ell = 2(n-1)! - 2; \\ h + (k-1)(n-1)! + 6, & \text{if } \ell = k(n-1)! - 2, \text{ where } k \in [3, \lceil \frac{n}{2} \rceil]. \end{cases}$$

Assume that $i_0 \neq n$. Let us first choose an edge wz of $S_n^n \setminus \{u, v\}$ such that $w(1) = i_0$. By (I) and (II), we have vertex-disjoint h-cycle C_1 and $((n-1)! - h)$-cycle C_2 with $uv \in E(C_1)$ and $wz \in E(C_2)$ in S_n^n. According to Lemma 2.8, we may choose an edge $w_1z_1 \in E(C_1 \setminus \{uv\})$ such that $z_1(1) = z(1)$ and $w_1(1) \neq i_0$ since $|E(C_2)| = 6$ or 8.

Assume that $i_0 = n$. Applying (I) and (II) to S_n^n, we get vertex-disjoint h-cycle C_1 and $((n-1)! - h)$-cycle C_2 with $uv \in E(C_1)$ and $xy \in E(C_2)$. Recalling that $|E(C_2)| = 6$ or 8 and using Lemma 2.8, we choose first an edge $wz \in E(C_2 \setminus \{xy\})$, and then choose an edge $w_1z_1 \in E(C_1 \setminus \{uv\})$ with $z_1(1) = z(1)$ and $|\{w_1(1), z(1), w(1)\}| = 3$.

The case that $\ell = (n-1)! - 2$ can proceed by applying Construction $\mathcal{D}\mathbf{1}$ to C_1 and the edge w_1z_1, we get an $(h+4)$-cycle, written C. Obviously, we have $\mathcal{S}_n^{i_0} \cap V(C) = \emptyset$ and $\bar{z} \notin V(C)$ if $i_0 \neq n$. Otherwise $\bar{w}, \bar{z} \notin V(C)$. Then Lemma 2.9 yields that $S_n - C - C_2$ has either a Hamilton path P containing xy from $\bar{w}$ to $\bar{z}$ ($i_0 \neq n$) or a Hamilton path P from $\bar{w}$ to $\bar{z}$ ($i_0 = n$). Thus $S_n - C$ has a Hamilton cycle $C^* := (C_2 - wz) + z\bar{z} + w\bar{w} + P$ that contains xy. Then C and C^* are vertex-disjoint cycles desired as in (†).

Consider the case that $\ell = 2(n-1)! - 2$. We get an $(h + (n-1)! + 2)$-cycle, written C_0, by applying Construction $\mathcal{D}\mathbf{3}$ to C_1 and the edge w_1z_1, which satisifies $\mathcal{S}_n^{w_1(1)} \subseteq V(C_0) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^{w_1(1)} \cup \mathcal{S}_n^{z_1(1)}$. Noting that $\bar{z}_1 \in S_n^{z(1)}$ and using Construction $\mathcal{D}\mathbf{3}$, we have that C_0 and $S_n^{z(1)}$ intersect some edge on C_0, write $\bar{z}_1z_1'$. Then pick a neighbor z_2 of $\bar{z}$ in $S_n^{z(1)} \setminus \{\bar{z}_1, z_1'\}$ such that $z_2(1) \notin \{n, w(1)\}$.

Now deal with the case that $\ell = k(n-1)!-2$, where $k \in [3, \lceil \frac{n}{2} \rceil]$. Applying Construction $\mathcal{D}\mathbf{4}$ to C_1 and the edge w_1z_1, we obtain an $(h+(k-1)(n-1)!+2)$-cycle C_0 such that $\mathcal{S}_n^I \subseteq V(C_0) \cap V(S_n^{[1,n-1]}) \subseteq \mathcal{S}_n^I \cup \mathcal{S}_n^{z_1(1)}$ for some $I \subseteq [n] \setminus \{n, w(1), z(1)\}$ with $w_1(1) \in I$ and $|I| = k-1 \geqslant 2$. Thus for the above, we have $\mathcal{S}_n^{i_0} \cap V(C) = \emptyset$ and $\bar{z} \notin V(C)$ if $i_0 \neq n$. Otherwise $\bar{w}, \bar{z} \notin V(C)$. Recalling that $\bar{z}_1 \in S_n^{z(1)}$ and by Construction $\mathcal{D}\mathbf{4}$, we see that C_0 and $S_n^{z(1)}$ intersect some edge on C_0, write $\bar{z}_1 z_1'$. Suppose that $n = 5$ and thus $k = 3$. Choosing a neighbor of $\bar{z}$ in $S_n^{z(1)} \setminus \{\bar{z}_1, z_1'\}$, says z_2, with $z_2(1) = w(1)$. Suppose that $n \geqslant 6$. We pick a neighbor z_2 of $\bar{z}$ in $S_n^{z(1)} \setminus \{\bar{z}_1, z_1'\}$ such that $z_2(1) \notin \{n, w(1)\}$.

By (I) and (II), for both cases, we get vertex-disjoint 6-cycle C_3 and $((n-1)!-6)$-cycle C_4 with $\bar{z}_1 z_1' \in E(C_3)$ and $\bar{z}z_2 \in E(C_4)$. Then a $(k(n-1)!-2)$-cycle, says $C := (C_0 - \bar{z}_1 z_1') + (C_3 - \bar{z}_1 z_1')$ is established. Again recall that by Lemma 2.9, we have either a Hamilton path P that contains xy from $\bar{w}$ to $\bar{z}_2$ ($i_0 \neq n$) or a Hamilton path P from $\bar{w}$ to $\bar{z}_2$ ($i_0 = n$) in $S_n - C - C_2 - C_4$. Thus $S_n - C$ has a Hamilton cycle $C^* := (C_2 - wz) + (C_4 - \bar{z}z_2) + w\bar{w} + z\bar{z} + z_2\bar{z}_2 + P$ which contains the edge xy. Therefore, C and C^* are vertex-disjoint, desired as in (†). □

Lemma 4.5 *Assume that $k \in [1, \lceil \frac{n}{2} \rceil]$. Then (†) holds for all even integers $\ell = k(n-1)!$.*

Proof. We shall distinguish two cases according to $\ell = (n-1)!$ and $\ell = k(n-1)!$, where $k \in [2, \lceil \frac{n}{2} \rceil]$.

Case 1. Assume that $\ell = (n-1)!$. Suppose that first $i_0 \neq n$. Corollary 2.2 implies that S_n^n has an $(n-1)!$-cycle C that containing uv. According to Lemma 2.10, $S_n - C$ has a Hamilton cycle C^* that contains xy. Therefore, C and C^* are vertex-disjoint, desired as in (†). Suppose now that $i_0 = n$. Putting $h = (n-1)! - 6$, we have $\ell = h + 6$. By (I) and (II), S_n^n has vertex-disjoint h-cycle C_1 and $((n-1)!-h)$-cycle C_2 with $uv \in E(C_1)$ and $xy \in E(C_2)$. Since $|E(C_2)| = 6$ or 8 and by Lemma 2.8, we pick first an edge $wz \in E(C_2 \setminus \{xy\})$, and then pick an edge $w_1z_1 \in E(C_1 \setminus \{uv\})$ with $z_1(1) = z(1)$ and $|\{w_1(1), z(1), w(1)\}| = 3$. This case can finish by applying Construction $\mathcal{D}\mathbf{2}$ to C_1 and the edge w_1z_1, we get an $(h+6)$-cycle, written C. Obviously, we have $\bar{w}, \bar{z} \notin V(C)$. Then Lemma 2.9 yields that there exist a Hamilton path P from $\bar{z}$ to $\bar{w}$ in $S_n - C - C_2$. Thus $S_n - C$ has a Hamilton cycle $C^* := (C_2 - wz) + z\bar{z} + w\bar{w} + P$ that contains xy. Then C and C^* are vertex-disjoint cycles desired as in (†).

Case 2. Assume that $\ell = k(n-1)!$, where $k \in [2, \lceil \frac{n}{2} \rceil]$. Putting $h = (n-1)! - 6$ yields that $\ell = h + (k-1)(n-1)! + 6$. Analogously to the argument in Lemma 4.4, only instead of $h = (n-1)! - 8$ we shall use $h = (n-1)! - 6$. Then we obtain vertex-disjoint cycles C and C^* such that $uv \in E(C)$ and $xy \in E(C^*)$. □

Lemma 4.6 *Assume that $k \in [1, \lceil \frac{n}{2} \rceil]$. Then (†) holds for all even integers $\ell = k(n-1)! + 2$.*

Proof. We shall distinguish two cases according to $\ell = (n-1)! + 2$ and $\ell = k(n-1)! + 2$, where $k \in [2, \lceil \frac{n}{2} \rceil]$.

Case 1. Assume that $\ell = (n-1)! + 2$. Putting $h = (n-1)! - 6$ implies that $\ell = h + 8$. If $i_0 \neq n$, then choose an edge wz of $S_n^n \setminus \{u, v\}$ such that $w(1) = i_0$. It follows from (I) and (II) that S_n^n has vertex-disjoint h-cycle C_1 and $((n-1)!-h)$-cycle C_2 with $uv \in E(C_1)$ and $wz \in E(C_2)$. Since $|E(C_2)| = 6$ or 8, we may choose an edge $w_1z_1 \in E(C_1 \setminus \{uv\})$ such that $z_1(1) = z(1)$ and $w_1(1) \neq i_0$ by Lemma 2.8. Otherwise, $i_0 = n$. By (I) and (II), there exist vertex-disjoint h-cycle C_1 and $((n-1)!-h)$-cycle C_2 with $uv \in E(C_1)$ and $xy \in E(C_2)$ in S_n^n. Considering $|E(C_2)| = 6$ or 8 and using Lemma 2.8, we pick first an edge $wz \in E(C_2 \setminus \{xy\})$, and then choose an edge $w_1z_1 \in E(C_1 \setminus \{uv\})$ with $z_1(1) = z(1)$ and $|\{w_1(1), z(1), w(1)\}| = 3$.

For both cases, we obtain an $(h+4)$-cycle, written C_0, by applying Construction $\mathcal{D}\mathbf{1}$ to C_1 and the edge w_1z_1. By noting that C_0 and $S_n^{z(1)}$ intersect some edge on C_0, write $\bar{z}_1 z_1'$, and taking a neighbor of $\bar{z}$ in $S_n^{z(1)} \setminus \{\bar{z}_1, z_1'\}$, says z_2, with $z_2(1) \notin \{n, w(1)\}$, we obtain vertex-disjoint 6-cycle C_3 and $((n-1)!-6)$-cycle C_4 with $\bar{z}_1 z_1' \in E(C_3)$ and $\bar{z}z_2 \in E(C_4)$ in $S_n^{z(1)}$ by (I) and (II). Then we get an $((n-1)!+2)$-cycle, says $C := (C_0 - \bar{z}_1 z_1') + (C_3 - \bar{z}_1 z_1')$. According to Lemma 2.9, $S_n - C - C_2 - C_4$

has either a Hamilton path P that contains xy from $\bar{w}$ to $\bar{z}_2$ ($i_0 \neq n$) or a Hamilton path P from $\bar{w}$ to $\bar{z}_2$ ($i_0 = n$). Thus $S_n - C$ has a Hamilton cycle $C^* := (C_2 - wz) + (C_4 - \bar{z}z_2) + w\bar{w} + z\bar{z} + z_2\bar{z}_2 + P$ which contains the edge xy. Then C and C^* are vertex-disjoint, desired as in (†).

Case 2. Assume that $\ell = k(n-1)! + 2$, where $k \in [2, \lceil \frac{n}{2} \rceil]$. Putting $h = (n-1)! - 6$ yields that $\ell = h + (k-1)(n-1)! + 8$. Analogously to the argument in Lemma 4.4, not only instead of $h = (n-1)! - 8$ we shall use $h = (n-1)! - 6$, but also instead of vertex-disjoint 6-cycle C_3 and $((n-1)! - 6)$-cycle C_4 we shall use vertex-disjoint 8-cycle C_3 and $((n-1)! - 8)$-cycle C_4. Then we obtain vertex-disjoint cycles C and C^* such that $uv \in E(C)$ and $xy \in E(C^*)$. □

Lemma 4.7 *Assume that $k \in [1, \lceil \frac{n}{2} \rceil]$. Then (†) holds for all even integers $\ell = k(n-1)! + 4$.*

Proof. We shall distinguish two cases according to $\ell = (n-1)! + 4$ and $\ell = k(n-1)! + 4$, where $k \in [2, \lceil \frac{n}{2} \rceil]$.

Case 1. Assume that $\ell = (n-1)! + 4$. Suppose that $i_0 \neq n$. Putting $h = (n-1)!$ shows that $\ell = h + 4$. Choosing $wz \in E(S_n^n \setminus \{uv\})$ with $i_0 \notin \{w(1), z(1)\}$, we get that S_n^n has an h-cycle C_1 containing both uv and wz by Corollary 2.2. Applying Construction $\mathcal{D}\mathbf{1}$ to C_1 and the edge wz, we obtain an $(h+4)$-cycle, written C, such that $\mathcal{S}_n^{i_0} \cap V(C) = \emptyset$. Since $xy \in \mathcal{S}_n^{i_0}$, Lemma 2.10 yields that $S_n - C$ has a Hamilton cycle C^* that contains xy. Therefore, C and C^* are vertex-disjoint cycles desired as in (†). Suppose that $i_0 = n$. Putting $h = (n-1)! - 6$, we have $\ell = h + 10$. Analogously to the argument in Case 1 of Lemma 4.6, only instead of vertex-disjoint 6-cycle C_3 and $((n-1)! - 6)$-cycle C_4 we shall use vertex-disjoint 8-cycle C_3 and $((n-1)! - 8)$-cycle C_4. Then we obtain vertex-disjoint cycles C and C^* such that $uv \in E(C)$ and $xy \in E(C^*)$.

Case 2. Assume that $\ell = k(n-1)! + 4$, where $k \in [2, \lceil \frac{n}{2} \rceil]$. Putting $h = (n-1)! - 6$ implies that $\ell = h + (k-1)(n-1)! + 10$. Analogously to the argument in Lemma 4.4, not only instead of $h = (n-1)! - 8$ we shall use $h = (n-1)! - 6$, but also instead of vertex-disjoint 6-cycle C_3 and $((n-1)! - 6)$-cycle C_4 we shall use vertex-disjoint 10-cycle C_3 and $((n-1)! - 10)$-cycle C_4. Then we obtain vertex-disjoint cycles C and C^* such that $uv \in E(C)$ and $xy \in E(C^*)$.

This completes the proof of the theorem. □

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