Extremal numbers for disjoint copies of a clique

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Abstract

The Turán number $ex(n, H)$ of H is the maximum number of edges of an n-vertex simple graph containing no copy of H as a subgraph. Denote $EX(n, H)$ as the set of graphs that have no copy of H as a subgraph and with size $ex(n, H)$. In this paper, utilizing a celebrated theorem of Hajnal and Szemerédi together with some results of Chen, Lih, and Wu, and of Kierstead and Kostochka, we determine $ex(n, 3K_{p+1})$ and $ex(t(p+1), tK_{p+1})$, and characterize all extremal graphs $EX(t(p+1), tK_{p+1})$ for all positive integers t, n, and p with $p \geq 2$.

Keywords: Turán numbers; Extremal graphs; Disjoint cliques

1 Introduction

Our notations in this paper are standard (see, e.g. [18]). All graphs considered in this paper are simple. Let $G = (V(G), E(G))$ be a simple graph of size $e(G)$. The complement $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$, two vertices being adjacent in $\overline{G}$ if and only if they are not adjacent in G. For a set S, by $|S|$ we denote the cardinality of S. Let G and H be two disjoint graphs, denote by $G \cup H$ the disjoint union of G and H and by tG the disjoint union of t copies of a graph G. For a subgraph H of G, by $G - H$ we mean a graph obtained from G by deleting all vertices of H and all incident edges. Denote by $G + H$ the join of graphs G and H, that is the graph obtained from $G \cup H$ by joining each vertex of G with each vertex of H.

A graph is said to be *equitably t-colourable* if its vertex set can be partitioned into t independent subsets $V_1, V_2, \ldots, V_t$ such that $\big||V_i| - |V_j|\big| \leq 1$ for any $i, j \in [t]$. The smallest integer t for which G is equitably t-colourable is called the *equitable chromatic number* of G, denoted by $\chi_=(G)$.

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Let H be a graph. We say the graph G is *H-free*, if G contains no copy of H as a subgraph. Here G is $t\overline{K}_{p+1}$-free means that G contains no t disjoint independent sets with size $p+1$. The *Turán number* $ex(n,H)$ of H is the maximum number of edges of an n-vertex H-free graph. Let $EX(n,H) = \{G : G \text{ is } H\text{-free with } |V(G)| = n \text{ and } e(G) = ex(n,H)\}$.

It is widely considered that the starting point of extremal graph theory is the Mantel's theorem [13] in 1907, which determines the maximum number of edges in a triangle-free graph on n vertices. In 1941, this theorem was strengthened by Turán [17] who determined $ex(n, K_{p+1})$ and proved $EX(n, K_{p+1}) = \{T_{n,p}\}$, where $T_{n,p}$ is a complete p-partite graph on n vertices with as equal parts as possible with $p \geq 1$ and $n \geq p+1$. Here $T_{n,1} = \overline{K}_n$, which is an empty graph on n vertices. In 1959, Erdős and Gallai [4] determined $ex(n, tK_2)$ for any positive integers n and t.

Theorem 1.1 (Erdős et al. [4]) *Let $n \geq 2t$. Then*

$$ex(n, tK_2) = \max\left\{\binom{2t-1}{2}, \binom{n}{2} - \binom{n-t+1}{2}\right\}.$$

Some years later, Erdős [5] proved $ex(n, (t+1)K_3) = e(K_t + T_{n-t,2})$, provided that $n > 400t^2$, and this was improved to a linear bound that $n > \frac{9t}{2} + 4$ by Moon [15]. For general cases, Simonovits [16] showed that $K_{t-1} + T_{n-t+1,p}$ is the unique extremal graph containing no tK_{p+1} for sufficiently large n, and some special cases were appeared also in [15]. For the generalized Turán numbers about cliques, see [6, 12, 19].

Notice that determining all values of $ex(n, tK_{p+1})$ is still an open problem, in views of the difficulty of obtaining the whole picture without asking n sufficiently large. Recently, Chen, Lu, and Yuan [3] determined $ex(n, 2K_{p+1})$ for all positive integers n and p.

Theorem 1.2 (Chen et al. [3]) *Let $n \geq 2(p+1)$ and $p \geq 2$. Then*

$$ex(n, 2K_{p+1}) = \begin{cases} \binom{n}{2} - 3(n-2p-1), & n \leq 3p+1; \\ (n-1) + t_{n-1,p}, & n \geq 3p+2. \end{cases}$$

In this paper, we determine $ex(n, 3K_{p+1})$ and $ex(t(p+1), tK_{p+1})$ for all positive integers n, t, and p, with $p \geq 2$ and $n \geq 3(p+1)$. In addition, we provide a unified proof to determine $ex(n, 2K_{p+1})$ and $ex(n, 3K_{p+1})$. Our results are as follows.

Theorem 1.3 *Let $p \geq 2$, $t \geq 1$, and $n = t(p+1)$. Then*

$$ex(n, tK_{p+1}) = \begin{cases} \binom{n}{2} - \binom{t+1}{2}, & t \leq 2p; \\ \binom{n}{2} - (n-p), & t \geq 2p+1. \end{cases}$$

Let $\overline{H}$ be an extremal graph for tK_{p+1} with $|V(H)| = t(p+1)$. Then

(1) for $t \leq 2p-1$, $H \in \{K_{t+1} \cup \overline{K}_{n-(t+1)}\}$;

(2) for $t=2p$, $H\in\{K_{t+1}\cup\overline{K}_{n-(t+1)}\}$ or
$H\in\{K_{1,x}\cup(n-x-p)K_2\cup(2p+x-1-n)K_1 : n-2p+1\leq x\leq n-p\}$;

(3) for $t\geq 2p+1$, $H\in\{K_{1,x}\cup(n-x-p)K_2\cup(2p+x-1-n)K_1 : n-2p+1\leq x\leq n-p\}$.

Theorem 1.4 *Let $p\geq 2$ and $T^k_{n-k,p}=K_k+T_{n-k,p}$ for $k\geq 1$. Then*

$$ex(n,2K_{p+1})=\begin{cases}\binom{n}{2}-3(n-2p-1), & 2p+2\leq n\leq 3p+1;\\ e(T^1_{n-1,p}), & n\geq 3p+2,\end{cases}$$

and

$$ex(n,3K_{p+1})=\begin{cases}\binom{n}{2}-6, & n=3p+3;\\ \binom{n}{2}-5(n-3p-2), & 3p+4\leq n\leq 5p+2;\\ e(T^2_{n-2,p}), & n\geq 5p+3.\end{cases}$$

When $n=t(p+1)$, a graph H is tK_{p+1}-free if and only if $\overline{H}$ is not equitably t-colourable. Meyer [14] introduced the notion of equitably colourable and conjectured that $\chi_{=}(G)\leq\Delta(G)$ for any connected graph G, which is neither a complete graph nor an odd cycle. Lih and Wu [11] confirmed Mayer's conjecture for bipartite graphs. Later, Chen and Lih [2] determined the formula of equitable chromatic numbers of trees. This line of research prompted Chen, Lih, and Wu [1] to put forth the following conjecture.

Conjecture 1.5 (Chen et al. [1]) *Every connected graph G, different from K_m, C_{2m+1}, and $K_{2m+1,2m+1}$ for $m\geq 1$, is equitably $\Delta(G)$-colourable.*

In the same paper, Chen, Lih, and Wu confirmed the conjecture for $\Delta\leq 3$. Later, Kierstead and Kostochka [10] confirmed the conjecture for $\Delta=4$.

Theorem 1.6 (Chen et al. [1] and Kierstead et al. [10]) *Every connected graph G with $\Delta(G)\leq 4$, different from K_m, C_{2m+1}, and $K_{2m+1,2m+1}$ for $m\geq 1$, is equitably $\Delta(G)$-colourable.*

The rest of this paper is organized as follows. In Section 2, some basic lemmas are provided, which will be used frequently in this paper. The proofs of Theorems 1.3 and 1.4 are presented in Sections 3 and 4, respectively.

2 Preliminaries

In 1970, one well-known result of Hajnal and Szemerédi [7] implied the following theorem, whereas a shorter proof appeared in [9].

Theorem 2.1 (Hajnal et al. [7] and Kierstead et al. [9]) *Every graph G is equitably k-colourable for all $k \geq \Delta(G)+1$.*

We say P is a *perfect matching* of G if $P \subseteq E(G)$ and $|P| = \frac{|V(G)|}{2}$ such that no two edges of P are adjacent.

Theorem 2.2 (Hall [8]) *Let $G = G[X, Y]$ be a bipartite graph. Then G contains a matching that saturates every vertex in X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.*

Inspired by Theorem 1.1 and Lemma 2.2 in the paper of Chen, Lu, and Yuan [3], we obtain the following two lemmas.

Lemma 2.3 *Let G be a tK_{p+1}-free graph on $n \geq t(p+1)$ vertices with $p \geq 1$ and $t \geq 2$. Then $\Delta(\overline{G}) \geq \lfloor \frac{n-t}{p} \rfloor \geq t$ and $\delta(G) \leq n-1-\lfloor \frac{n-t}{p} \rfloor$.*

Proof. By contradiction, we may assume that $\Delta(\overline{G}) \leq \lfloor \frac{n-t}{p} \rfloor - 1$. By Theorem 2.1, the graph $\overline{G}$ is equitably $\lfloor \frac{n-t}{p} \rfloor$-colourable. Let $\ell = \lfloor \frac{n-t}{p} \rfloor$. We can see $\ell \geq t$ and $\bigcup_{i\in[\ell]} A_i = V(\overline{G})$, where all of A_i are disjoint independent sets with $|A_1| \geq |A_2| \geq \cdots \geq |A_\ell|$ and $|A_1| - |A_\ell| \leq 1$. Since G is tK_{p+1}-free, we have $|A_t| \leq p$, which follows that $|A_i| \leq p+1$ for any $i \in [t-1]$ and $|A_j| \leq |A_t| \leq p$ for any $t \leq j \leq \ell$. Thus

$$\begin{aligned} n = |V(G)| &= \Big|\bigcup_{i\in[\ell]} A_i\Big| \\ &\leq (t-1)\cdot(p+1) + (\ell - t + 1)\cdot p \\ &= p\cdot \ell + t - 1 \\ &\leq p \cdot \frac{n-t}{p} + t - 1 \\ &= n-1, \end{aligned}$$

a contradiction. Thus $\Delta(\overline{G}) \geq \lfloor \frac{n-t}{p} \rfloor \geq t$. We see $\delta(G) = n-1-\Delta(\overline{G}) \leq n-1-\lfloor \frac{n-t}{p} \rfloor$. This proves Lemma 2.3. ■

Lemma 2.4 *Let $H_{p+1}(n)$ be an extremal graph for tK_{p+1}, $T^{(t-1)}_{n-(t-1),p} = K_{t-1} + T_{n-(t-1),p}$, and $n_0 \geq t(p+1)$, where $p \geq 1$ and $t \geq 2$. If $ex(n_0, tK_{p+1}) = e(T^{(t-1)}_{n_0-(t-1),p})$, then $ex(n, tK_{p+1}) = e(T^{(t-1)}_{n-(t-1),p})$ for any $n \geq n_0$.*

Proof. Let $n \geq t(p+1)$. Since $H_{p+1}(n)$ is tK_{p+1}-free, we have $H_{p+1}(n) - \{v\}$ is tK_{p+1}-free, where $v \in V(H_{p+1}(n))$ and $d(v) = \delta(H_{p+1}(n))$. By Lemma 2.3, it follows that

$$e(H_{p+1}(n-1)) \geq e(H_{p+1}(n)) - d(v) \geq e(H_{p+1}(n)) - (n-1-\lfloor \frac{n-t}{p} \rfloor),$$

that is,

$$\begin{aligned} e(H_{p+1}(n)) - e(H_{p+1}(n-1)) &\le n - 1 - \lfloor \frac{n-t}{p} \rfloor \\ &= n - \lceil \frac{n-(t-1)}{p} \rceil \\ &= \delta(T^{(t-1)}_{n-(t-1),p}) \\ &= e(T^{(t-1)}_{n-(t-1),p}) - e(T^{(t-1)}_{n-1-(t-1),p}). \end{aligned}$$

Thus

$$e(H_{p+1}(n)) - e(T^{(t-1)}_{n-(t-1),p}) \le e(H_{p+1}(n-1)) - e(T^{(t-1)}_{n-1-(t-1),p}),$$

implying the sequence $\{e(H_{p+1}(n)) - e(T^{(t-1)}_{n-(t-1),p})\}$ is nonincreasing about n. Note that the graph $T^{(t-1)}_{n-(t-1),p}$ is tK_{p+1}-free and so $e(H_{p+1}(n)) - e(T^{(t-1)}_{n-(t-1),p}) \ge 0$. If $e(H_{p+1}(n)) - e(T^{(t-1)}_{n-(t-1),p}) = 0$ when $n = n_0$, then $e(H_{p+1}(n)) - e(T^{(t-1)}_{n-(t-1),p}) = 0$ and so $e(H_{p+1}(n)) = e(T^{(t-1)}_{n-(t-1),p})$ for any $n \ge n_0$. That is, if $ex(n_0, tK_{p+1}) = e(T^{(t-1)}_{n_0-(t-1),p})$, then $ex(n, tK_{p+1}) = e(T^{(t-1)}_{n-(t-1),p})$ for any $n \ge n_0$. ■

3 The proof of Theorem 1.3

Proof of Theorem 1.3. Let $\overline{H}$ be an extremal graph for tK_{p+1} on n vertices with $n = t(p+1)$. Then the graph H is $t\overline{K}_{p+1}$-free with minimum edges. Let $G_1 = K_{t+1} \cup \overline{K}_{n-t-1}$ and $G_2 = K_{1,n-p} \cup \overline{K}_{p-1}$. Notice that both G_1 and G_2 are $t\overline{K}_{p+1}$-free. We have

$$e(H) \le e(G_1) = \binom{t+1}{2} \tag{1}$$

and

$$e(H) \le e(G_2) = n - p. \tag{2}$$

Thus

$$e(H) \le \begin{cases} \binom{t+1}{2}, & t \le 2p; \\ n - p, & t \ge 2p+1. \end{cases} \tag{3}$$

Next we need to prove

$$e(H) \ge \begin{cases} \binom{t+1}{2}, & t \le 2p; \\ n - p, & t \ge 2p+1, \end{cases} \tag{4}$$

and characterize the extremal graph when equalities hold. We will consider the following two cases to finish the proof.

Case 1. For any vertex v in H with $d_H(v) = \Delta(H)$, there is no $\overline{K}_{p+1}$ containing v.

In this case, there is no $\overline{K}_p$ in H_1, where $H_1 = H - N[v]$. By (2), $e(H) \le n - p$, we have $\Delta(H) \le n - p$. Here we want to provide a lower bound for $e(H_1)$, that is an upper bound for $e(\overline{H}_1)$. Note that H_1 is $\overline{K}_p$-free, we have $\overline{H}_1$ is K_p-free. By Turán's theorem,

$$e(\overline{H}_1) \le ex(|V(H_1)|, K_p) = e(T_{|V(H_1)|,p-1}).$$

Observe that there are at most $p-1$ components in the complement of $T_{|V(H_1)|,p-1}$. Thus

$$e(H_1) \ge \binom{|V(H_1)|}{2} - e(T_{|V(H_1)|,p-1}) \ge |V(H_1)| - p + 1,$$

the equalities hold only when each component of H_1 has at most two vertices and $H_1 = e(H_1)K_2 \cup (|V(H_1)| - 2e(H_1))K_1 = (|V(H_1)| - p + 1)K_2 \cup (2p - 2 - |V(H_1)|)K_1$. It yields that $|V(H_1)| \le 2p - 2$. We can see

$$e(H) \ge e(H_1) + d_H(v) \ge |V(H_1)| - p + 1 + d_H(v) = (n - d_H(v) - 1) - p + 1 + d_H(v) = n - p.$$

Then (4) holds. By (3), then $t \ge 2p$ and $e(H) = n - p$. In this case, $|V(H_1)| \le 2(p-1)$ and $H = K_{1,d_H(v)} \cup (n - d_H(v) - p)K_2 \cup (2p + d_H(v) - 1 - n)K_1$, where $d_H(v) = \Delta(H)$ and $n - 2p + 1 \le d_H(v) \le n - p$. This completes the proof of Case 1.

Case 2. There exists some vertex $v \in V(H)$ with $d_H(v) = \Delta(H)$ which is contained in some $\overline{K}_{p+1}$.

In this case, we have H contains a copy of $\overline{K}_{p+1}$, which yields that $t \ge 2$. First we want to prove $e(H) \ge \binom{t+1}{2}$. By contradiction, assume $e(H) < \binom{t+1}{2}$. Among all graphs, we choose the minimum t such that H is $t\overline{K}_{p+1}$-free with $t(p+1)$ vertices, $e(H) < \binom{t+1}{2}$, and there exists some vertex $v \in V(H)$ with $d_H(v) = \Delta(H)$ which is contained in some $\overline{K}_{p+1}$. Let the independent set with size $p+1$ containing v denote as F and $H_1 = H - F$. Then H_1 is $(t-1)\overline{K}_{p+1}$-free with $(t-1)(p+1)$ vertices. By Lemma 2.3, $t \le \Delta(H) = d_H(v) \le e(H[H_1, F]) = e(H) - e(H_1)$, implying that $e(H_1) \le e(H) - t < \binom{t}{2}$. By the choice of t, we have for any vertex $v \in V(H_1)$ with $d_{H_1}(v) = \Delta(H_1)$, there is no $\overline{K}_{p+1}$ containing v. Notice that we obtain the lower bound $e(H) \ge n - p$ in Case 1 only use the property of H is $t(p+1)$ vertices $t\overline{K}_{p+1}$-free. Similar with the proof in Case 1, we have $e(H_1) \ge |V(H_1)| - p = (t-1)(p+1) - p$. Since $(t-1)(p+1) - p \le e(H_1) < \binom{t}{2}$, $t \ge 2p+2$ or $t \le 1$. Recall $t \ge 2$. Thus $t \ge 2p+2$ and so $e(H) \ge e(H_1) + t \ge (t-1)(p+1) - p + 2p + 2 > n - p$, which contradicts to (2). Therefore, $e(H) \ge \binom{t+1}{2}$. By (1) and (2), we have $e(H) = \binom{t+1}{2}$ and $t \le 2p$.

We want to prove $H \in \{K_{t+1} \cup \overline{K}_{n-(t+1)}\}$. By the condition of Case 2, we have $t \ge 2$, $e(H) = \binom{t+1}{2}$, and $t \le 2p$. Suppose there exists some $2 \le t \le 2p$ such that $H \notin \{K_{t+1} \cup \overline{K}_{n-(t+1)}\}$. We choose the minimum such t. Recall that there exists some vertex $v \in V(H)$ with $d_H(v) = \Delta(H)$ which is contained in some $\overline{K}_{p+1}$. Let the independent set with size $p+1$ containing v denote as F and $H_1 = H - F$. Then H_1 is $(t-1)\overline{K}_{p+1}$-free with $(t-1)(p+1)$ vertices. By Lemma 2.3, $t \leq \Delta(H) = d_H(v) \leq e(H[H_1, F]) = e(H) - e(H_1)$, implying $e(H_1) \leq \binom{t}{2}$. If H_1 satisfies the condition of Case 2, then $t-1 \geq 2$ and $e(H_1) \geq \binom{t}{2}$ by the first paragraph of Case 2. That is $e(H_1) = \binom{t}{2}$. By the choice of t is minimum, then $H_1 \in \{K_t \cup \overline{K}_{(t-1)(p+1)-t}\}$. If H_1 satisfies the condition of Case 1, then by the same proof, we have $e(H_1) \geq (t-1)(p+1) - p$. Recall $e(H_1) \leq \binom{t}{2}$. Solving $(t-1)(p+1) - p \leq \binom{t}{2}$, we obtain $t \geq 2p+1$ or $t \leq 2$. By $2 \leq t \leq 2p$, it follows that $t = 2$. In this case, we have $e(H_1) = \binom{t}{2}$ and $H_1 = K_2 \cup \overline{K}_{p-1}$ by Turán's theorem. In a conclusion, we have $H_1 \in \{K_t \cup \overline{K}_{(t-1)(p+1)-t}\}$ and so $d_H(v) = t$. Denote the K_t in H_1 as F_1. If $V(F_1) \cap N(v) = \emptyset$, then there are exactly two nontrivial components in H, that is K_t and $K_{1,t}$. Since each component of H is equitably t-colourable, we have H is equitably t-colourable. Thus H contains a copy of $t\overline{K}_{p+1}$, a contradiction. Next we only need to consider $V(F_1) \cap N(v) \neq \emptyset$. Let $u \in V(F_1) \cap N(v)$. Then $d_H(u) = t$. Clearly, $V(H) \setminus N[u]$ is an independent set with size $tp-1$. Then there is an independent set with size $p+1$ containing u by $t \geq 2$, denoted by F_2. Then $H - F_2$ is $(t-1)\overline{K}_{p+1}$-free and $e(H - F_2) \leq e(H) - d_H(u) = \binom{t}{2}$. Applying $H_1 = H - F_2$, we have $e(H - F_2) = \binom{t}{2}$ and $H - F_2 = K_t \cup \overline{K}_{(t-1)(p+1)-t}$. Thus $V(F_1) = N(v)$ and so $H = K_{t+1} \cup \overline{K}_{n-(t+1)}$. This completes the proof of Case 2.

This completes the proof of Theorem 1.3. ∎

4 The Proof of Theorem 1.4

Lemma 4.1 *Let $\overline{H}$ be an extremal graph for tK_{p+1} on $(2t-1)p+t-1$ vertices with $t \geq 2$. If $e(H) \geq (2t-1)(t-1)p$, then $ex(n, tK_{p+1}) = e(T^{t-1}_{n-(t-1),p})$ for any $n \geq (2t-1)p+t-1$.*

Proof. Let $G = pK_{2t-1} \cup \overline{K}_{t-1}$ with $|V(G)| = (2t-1)p+t-1$. Clearly, G is $t\overline{K}_{p+1}$-free and $e(H) \leq e(G) = (2t-1)(t-1)p$. Note that $e(H) \geq (2t-1)(t-1)p$. Then $e(H) = (2t-1)(t-1)p$. In this case $ex((2t-1)p+t-1, tK_{p+1}) = e(T^{t-1}_{(2t-1)p,p})$. By Lemma 2.4, $ex(n, tK_{p+1}) = e(T^{t-1}_{n-(t-1),p})$ for any $n \geq (2t-1)p+t-1$. This completes the proof of Lemma 4.1. ∎

Lemma 4.2 *Let $\overline{H}$ be an extremal graph for tK_{p+1} on $tp+t-1+s$ vertices with $1 \leq s \leq (t-1)p+1$, where $1 \leq t \leq 3$ and $p \geq 2$. Then $e(H) \geq (2t-1)s$.*

Proof. Clearly, the graph H is $t\overline{K}_{p+1}$-free and H has minimum number of edges. We use induction on t and s. By Turán's theorem [17], $e(H) \geq (2t-1)s$ for $t = 1$ and $1 \leq s \leq (t-1)p+1$. We may assume that $t \geq 2$ and the result holds for $t-1$. Next we show the result holds for t and prove $e(H) \geq (2t-1)s$. We choose p as the smallest integer such that H is $t\overline{K}_{p+1}$-free and $e(H) < (2t-1)s$ with $|V(H)| = tp+t-1+s$, for some s with $1 \le s \le (t-1)p+1$ and $p \ge 2$. If such p does not exist, then $e(H) \ge (2t-1)s$ and we are done. Thus we assume that such p exists.

Claim 1 *Let G be a $t\overline{K}_p$-free graph and G has minimum number of edges with $|V(G)| = t(p-1)+t-1+x$, $p \ge 2$, and $t \ge 2$. For $1 \le x \le (t-1)(p-1)+1$, we have $e(G) \ge (2t-1)x$. Furthermore, $e(G) \ge (2t-1)x$ for any $x \ge 1$.*

Proof. First, we prove $e(G) \ge (2t-1)x$ when $1 \le x \le (t-1)(p-1)+1$. By contradiction, suppose $e(G) < (2t-1)x$ for some $1 \le x \le (t-1)(p-1)+1$. If $p \ge 3$, then the smallest integer such that $e(H) < (2t-1)s$ is $p-1$ rather than p, which contradicts the choice of p. Thus we only need to consider the case $p = 2$. Then $|V(G)| = 2t-1+x$, where $1 \le x \le t$. By Theorem 1.1,

$$\begin{aligned} e(G) &\ge \min\left\{\binom{|V(G)|}{2} - \binom{2t-1}{2}, \binom{|V(G)|-t+1}{2}\right\} \\ &= \min\left\{\frac{1}{2}[(4t-3)x+x^2], \frac{1}{2}[t^2+2tx+x^2-t-x]\right\} \\ &\ge (2t-1)x. \end{aligned}$$

Thus $e(G) \ge (2t-1)x$.

Next we prove the second part of Claim 1, that is $e(G) \ge (2t-1)x$ for any $x \ge 1$. Let $n_x = t(p-1)+t-1+x$ and $n_{x+1} = n_x+1 = t(p-1)+t+x$ with $x \ge 1$. For $x \ge (t-1)(p-1)+1$, Lemma 4.1 implies that

$$e(G) = \binom{n_x}{2} - ex(n_x, tK_p) = \binom{n_x}{2} - e\big(T^{t-1}_{n_x-(t-1),p-1}\big).$$

Note that $e(G) \ge (2t-1)x$ for $x = (t-1)(p-1)+1$. If we can show

$$\binom{n_{x+1}}{2} - e\big(T^{t-1}_{n_{x+1}-(t-1),p-1}\big) - \left[\binom{n_x}{2} - e\big(T^{t-1}_{n_x-(t-1),p-1}\big)\right] \ge 2t-1 \qquad (5)$$

for any $x \ge (t-1)(p-1)+1$, then $e(G) \ge (2t-1)x$ for any $x \ge (t-1)(p-1)+1$. Together with $e(G) \ge (2t-1)x$ for $1 \le x \le (t-1)(p-1)+1$, we have $e(G) \ge (2t-1)x$ for $x \ge 1$. Thus we only need to show (5) holds. We can see

$$\begin{aligned}
&\binom{n_{x+1}}{2} - e\big(T^{t-1}_{n_{x+1}-(t-1),p-1}\big) - \left[\binom{n_x}{2} - e\big(T^{t-1}_{n_x-(t-1),p-1}\big)\right] \\
&= n_x - e\big(T^{t-1}_{n_{x+1}-(t-1),p-1}\big) + e\big(T^{t-1}_{n_x-(t-1),p-1}\big) \\
&= n_x - \delta\big(T^{t-1}_{n_{x+1}-(t-1),p-1}\big) \\
&= n_x - \left[n_{x+1} - \left\lceil \frac{n_{x+1}-t+1}{p-1} \right\rceil\right] \\
&= \left\lceil \frac{n_x-t+2}{p-1} \right\rceil - 1 \\
&\geq 2t-1.
\end{aligned}$$

The last inequality holds because $x \geq (t-1)(p-1)+1$ and $n_x \geq (2t-1)(p-1)+t$. Therefore, $e(G) \geq (2t-1)x$ for any $x \geq 1$. This completes the proof of Claim 1. ■

Next we prove $e(G) \geq (2t-1)s$ for $1 \leq s \leq (t-1)p+1$. By Theorem 1.3 and $1 \leq t \leq 3$, $e(H) \geq (2t-1)s$ for $s=1$. Assume that $s \geq 2$ and the result holds for $s-1$. We assert that if $\Delta(H) \geq 2t-1$, then $e(H) \geq (2t-1)s$. Suppose there is a vertex $u \in V(H)$ with $d_H(u) = \Delta(H) \geq 2t-1$. Then $H - \{u\}$ is $t\overline{K}_{p+1}$-free with $|V(H-\{u\})| = tp+t-1+s-1$, where $2 \leq s \leq (t-1)p+1$. By induction hypothesis on s, $e(H-\{u\}) \geq (2t-1)(s-1)$ and so $e(H) \geq e(H-\{u\}) + d_H(u) \geq (2t-1)s$. Thus we only need to consider the case $\Delta(H) \leq 2t-2$. We will complete our proof by the following two cases.

Case 1. $(t-2)p+1 \leq s \leq (t-1)p+1$**.**

By Lemma 2.3, $\Delta(H) \geq \left\lfloor \frac{|V(H)|-t}{p} \right\rfloor \geq 2t-2$. Thus $\Delta(H) = 2t-2$. We conclude that H is not equitably $(2t-2)$-colourable. By contradiction, suppose that H is equitably $(2t-2)$-colourable. Let $(C_1, \ldots, C_{2t-2})$ be an equitable $(2t-2)$-colouring of H with $|C_1| \geq |C_2| \geq \cdots \geq |C_{2t-2}|$ and $|C_1| - |C_{2t-2}| \leq 1$. Since H is $t\overline{K}_{p+1}$-free, we have $|C_t| \leq p$ and so $|C_i| \leq p+1$ for any $i \in [t-1]$ and $|C_j| \leq p$ for any $j \in [2t-2] \setminus [t-1]$. It implies that

$$|V(H)| = \sum_{i \in [2t-2]} |C_i| \leq (t-1)(p+1) + (t-1)p = (2t-2)p+t-1 < tp+t-1+s = |V(H)|$$

by $s \geq (t-2)p+1$, a contradiction. Thus H is not equitably $(2t-2)$-colourable.

Notice that if each component of H is equitably $(2t-2)$-colourable, then H is equitably $(2t-2)$-colourable. Since H is not equitably $(2t-2)$-colourable, there exists some component of H, say F, that is not equitably $(2t-2)$-colourable. Then $\Delta(F) = \Delta(H) = 2t-2$, else $\Delta(F) < 2t-2$, by Theorem 2.1, F is equitably $(2t-2)$-colourable, a contradiction. By Theorem 1.6, Conjecture 1.5 holds for $\Delta \leq 2t-2 \leq 4$. Note that $2t-2$ is even and $2t-2 \leq 4$. Theorem 1.6 states that for any connected graph with maximum degree 2 that is not equitably 2-colourable, it is isomorphic to an odd cycle; for any connected graph with maximum degree 4 that is not equitably 4-colourable, it is isomorphic to a complete graph K_5. Then F is isomorphic to K_{2t-1} or an odd cycle, denoted by F (here if H is connected, we say some component of H is H, that is $F = H$). Let $H_1 = H - F$. If F is isomorphic to K_{2t-1}, then $|V(H_1)| = t(p-1)+t-1+s-t+1$, where $1 \leq s-t+1 \leq (t-1)(p-1)+1$ by $s \geq 2$ and $(t-2)p+1 \leq s \leq (t-1)p+1$. We assert that H_1 is $t\overline{K}_p$-free. If H_1 contains a copy of $t\overline{K}_p$, let $\{x_1, x_2, \dots, x_t\} \subseteq F$ and $I_1 \cup \dots \cup I_t = V(t\overline{K}_p)$, where I_i is an independent set with size p for any $i \in [t]$, then $I_i \cup \{x_i\}$ is an independent set with size $p+1$ in H, which contradicts the fact that H is $t\overline{K}_{p+1}$-free, as asserted. By Claim 1, $e(H_1) \geq (2t-1)(s-t+1)$ and so $e(H) \geq e(H_1) + \binom{2t-1}{2} \geq (2t-1)s$. If F is isomorphic to an odd cycle, then $t = 2$. We may assume that there are x components of H, each of which is isomorphic to an odd cycle. We delete one vertex from each of these x components, the result graph denoted by H_2. Thus $\Delta(H_2) \leq \Delta(H) = 2$ and there is no component of H_2 that is isomorphic to an odd cycle. Theorem 1.6 implies that H_2 is equitably 2-colourable. If $|V(H_2)| \geq 2p+2$, then there is a copy of $2\overline{K}_{p+1}$ and so H has a copy of $2\overline{K}_{p+1}$, which contradicts the fact that H is $2\overline{K}_{p+1}$-free. We have $|V(H_2)| \leq 2p+1$. Thus there are at least $|V(H)| - 2p - 1 = s$ components of H, each of which is isomorphic to an odd cycle and has at least three edges. It follows that $e(H) \geq 3s = (2t-1)s$.

Case 2. $1 \leq s \leq (t-2)p$**.**

For $t = 2$, we have proved that $e(H) \geq 3s$, where $1 \leq s \leq p+1$. It remains to consider the case $t = 3$ and $1 \leq s \leq (t-2)p= p$. Since H is $3\overline{K}_{p+1}$-free and H has minimum number of edges, H contains a copy of $\overline{K}_{p+1}$.

Claim 2 *For any independent set I with size $p+1$, then $e(H[I, V(H) \setminus I]) \leq 2s-1$.*

Proof. By contradiction, suppose that there exists some independent set I with size $p+1$ such that $e(H[I, V(H) \setminus I]) \geq 2s$. Then $H - I$ is $(t-1)\overline{K}_{p+1}$-free with $|V(H-I)| = 2p+1+s$, where $1 \leq s \leq p+1$. By induction hypothesis on t, $e(H-I) \geq (2t-3)s= 3s$. Thus $e(H) \geq e(H[I, V(H) \setminus I]) + e(H-I) \geq (2t-1)s= 5s$, a contradiction. ■

We choose an independent set I with size $p+1$ such that $e(H[I, V(H) \setminus I])$ is as large as possible. Let $X = \{u \in V(H) \setminus I : N(u) \cap I = \emptyset\}$. Note that

$$2s-1 \geq e(H[I, V(H) \setminus I]) \geq |V(H) \setminus I| - |X| = 2p+1+s-|X|.$$

It follows that $|X| \geq 2p+2-s$. Since $1 \leq s \leq p$, we have $|X| \geq p+2$.

We assert that $\min\{d_H(v) : v \in I\} \geq \max\{d_H(x) : x \in X\}$. Let $v_1 \in I$ such that $d_H(v_1) = \min\{d_H(v) : v \in I\}$ and $x_1 \in X$ such that $d_H(x_1) = \max\{d_H(x) : x \in X\}$. Set $I' = I \cup \{x_1\} \setminus \{v_1\}$, we can see I' is an independent set with $|I'| = p+1$. If $d_H(x_1) \geq d_H(v_1)+1$, then

$$e(H[I', V(H) \setminus I']) = e(H[I, V(H) \setminus I]) - d_H(v_1) + d_H(x_1) \geq e(H[I, V(H) \setminus I]) + 1,$$

which contradicts the maximality of $e(H[I, V(H) \setminus I])$, as asserted. Clearly, $\min\{d_H(v) : v \in I\} \leq 1$, since otherwise $e(H[I, V(H) \setminus I]) \geq 2|I| \geq 2(p+1) \geq 2s$, a contradiction. That is $\max\{d_H(x) : x \in X\} \leq 1$.

Since $\max\{d_H(x) : x \in X\} \leq 1$ and $|X| \geq p+2 \geq 3$, we can randomly choose 3 vertices $\{x_1, x_2, x_3\}$ from $V(H)$ such that $d_H(x_i) \leq 1$ for any $i \in [3]$. Let $H_1 = H - \{x_1, x_2, x_3\}$ and $\{x_{i1}\} = N(x_i)$ if it exists for any $i \in [3]$.

Claim 3 *Then H_1 contains a copy of $3\overline{K}_p$ and $d(x_i) = 1$ for any $i \in [3]$. For any copy of $3\overline{K}_p$ of H_1, $\{x_{11}, x_{21}, x_{31}\}$ belongs to the same $\overline{K}_p$.*

Proof. Note that $|V(H_1)| = 3(p-1) + 3 - 1 + s$ with $1 \leq s \leq p+1$. If H_1 is $3\overline{K}_p$-free, by Claim 1, then $e(H_1) \geq 5s$. Thus $e(H) \geq e(H_1) \geq 5s$, a contradiction. Therefore, H_1 contains a copy of $3\overline{K}_p$ with $V(3\overline{K}_p)$=$Y_1 \cup Y_2 \cup Y_3$, where Y_i is an independent set in H_1 with $|Y_i| = p$ for each $i \in [3]$. Note that H is $3\overline{K}_{p+1}$-free and $d_H(x_i) \leq 1$ for each $i \in [3]$. Let us consider a bipartite graph B with $V(B) = \{x'_1, x'_2, x'_3\} \cup \{y_1, y_2, y_3\}$ and $E(B) = \{x'_i y_j :$ if $N_H(x_i) \cap Y_j = \emptyset$ for any $i, j \in [3]\}$. We can see if there is an edge $x'_i y_j$, then $Y_j \cup \{x_i\}$ is an independent set in H. Thus, if there is a perfect matching in B, then there is a copy of $3\overline{K}_{p+1}$ in H, which contradicts the fact that H is $3\overline{K}_{p+1}$-free. Thus B has no perfect matching. By Theorem 2.2, there exists $\emptyset \neq S \subseteq \{x'_1, x'_2, x'_3\}$ such that $|N_B(S)| < |S|$. Since $d_H(x_i) \leq 1$ for any $i \in [3]$, $d_B(x'_i) \geq 2$. We see $|S| > |N_B(S)| \geq 2$, then $S = \{x'_1, x'_2, x'_3\}$ and $|N_B(S)| = 2$, that is, $d_B(x'_i) = 2$ and $d_H(x_i) = 3 - d_B(x'_i) = 1$ and there is some integer $j \in [3]$ such that $N(x_i) \subseteq Y_j$ for any $i \in [3]$.

■

We may assume that $\{x_{11}, x_{21}, x_{31}\} \subseteq Y_1$ and subject to it, $e(H[Y_1, V(H) \setminus Y_1])$ is as small as possible.

Claim 4 *For each vertex $v \in V(H) \setminus Y_1$, we have $N(v) \cap Y_1 \neq \emptyset$.*

Proof. By contradiction, suppose that there exists some vertex $u \in V(H) \setminus Y_1$ such that $N(u) \cap Y_1 = \emptyset$. Then $u \notin \{x_1, x_2, x_3\}$. If $u \in Y_2$, then $Y_1 \cup \{u\}$, $Y_2 \cup \{x_1, x_2\} \setminus \{u\}$ and $Y_3 \cup \{x_3\}$ are disjoint independent sets with size $p+1$, a contradiction. The case that $u \in Y_3$ is similar. If $u \notin Y_2 \cup Y_3$, then $Y_1 \cup \{u\}$, $Y_k \cup \{x_k\}$ for $2 \leq k \leq 3$ are disjoint independent sets with size $p+1$, a contradiction. Thus Claim 4 holds. ■

By Claim 4, $d_H(v) \geq 1$ for any $v \in V(H) \setminus Y_1$.

Claim 5 *There exists at least one vertex $y \in Y_2$ such that $d_H(y) = 1$ and $|N(y) \cap Y_1| = 1$.*

Proof. By contradiction, suppose $d_H(y) \geq 2$ for each vertex $y \in Y_2$. Let $I = Y_2 \cup \{x_1\}$. We can see I is an independent set with $|I| = p+1$. Thus, $e(H[I, V(H) \setminus I]) \geq 2|Y_2|+1$=$2p+1 >$

$2s$, which contradicts Claim 2. Thus there exists at least one vertex $y \in Y_2$ such that $d_H(y) = 1$. It follows from Claim 4 that Claim 5 holds. ■

Let $W = V(H) \setminus (\{x_1, x_2, x_3\} \cup Y_1 \cup Y_2 \cup Y_3)$. Clearly, $|W| = s - 1$.

Claim 6 *There exists at least one vertex $w \in W$ such that $N(w) \cap Y_2 = \emptyset$.*

Proof. By contradiction, suppose $N(w) \cap Y_2 \neq \emptyset$ for any $w \in W$. Let $I = Y_2 \cup \{x_1\}$. By Claim 4, $e(H[I, V(H) \setminus I]) \geq e(H[I, Y_1]) + e(H[Y_2, W]) \geq p + 1 + s - 1 \geq 2s$, a contradiction. ■

By Claim 5, let $N(y) \cap Y_1 = \{y_1\}$. Then $Y_1' = \{y\} \cup Y_1 \setminus \{y_1\}$ is an independent set in H_1 with $|Y_1'| = p$. By Claim 6, $Y_2' = \{w\} \cup Y_2 \setminus \{y\}$ is an independent set in H_1 with $|Y_2'| = p$. Thus $Y_1' \cup Y_2' \cup Y_3$ consists a copy of $3\overline{K}_p$ in H_1. By Claim 3, $\{x_{11}, x_{21}, x_{31}\} \subseteq \{y\} \cup Y_1 \setminus \{y_1\}$. We assert that for any 1-degree vertex $x \in V(H)$ with $N(x) = \{x'\}$, then $d_H(x') \geq 2$. Suppose $d_H(x') = 1$. By the randomness of $\{x_1, x_2, x_3\}$, we may assume that $\{x, x'\} \subseteq \{x_1, x_2, x_3\}$, which contradicts Claim 3, as asserted. Thus $d_H(y_1) \geq 2$. Therefore,

$$e(H[Y_1', V(H) \setminus Y_1']) = e(H[Y_1, V(H) \setminus Y_1]) - d_H(y_1) + d_H(y) < e(H[Y_1, V(H) \setminus Y_1]),$$

which contradicts the minimality of $e(H[Y_1, V(H) \setminus Y_1])$. Therefore, the minimum p such that $e(H) \leq (2t-1)s - 1$ does not exist, and so $e(H) \geq (2t-1)s$. ■

Proof of Theorem 1.4. Let $\overline{H}$ be an extremal graph for tK_{p+1} on $n' = tp + t - 1 + s$ vertices with $1 \leq s \leq (t-1)p + 1$ and $1 \leq t \leq 3$. Then the graph H is $t\overline{K}_{p+1}$-free and H has minimum number of edges. By Lemma 4.2, $e(H) \geq (2t-1)s$. For $t = 2$, let $G_1 = xK_3 \cup yK_4 \cup \overline{K}_{n'-3x-4y}$, where $x + 2y = s$ and $1 \leq s \leq p + 1$. For $t = 3$, let $G_2 = zK_5 \cup \ell K_6 \cup \overline{K}_{n'-5z-6\ell}$, where $2z + 3\ell = s$ and $2 \leq s \leq 2p + 1$. We can see G_i is $t\overline{K}_{p+1}$-free and $e(G_i) = (2t-1)s$ for any $i \in [2]$. Then $e(H) \leq e(G) = (2t-1)s$. Thus $e(H) = (2t-1)s$ for $t = 2$ with $1 \leq s \leq p+1$, and $t = 3$ with $2 \leq s \leq 2p+1$. Therefore, for $2p + 2 \leq n' \leq 3p + 2$

$$ex(n', 2K_{p+1}) = \binom{n'}{2} - 3(n' - 2p - 1),$$

and for $3p + 4 \leq n' \leq 5p + 3$,

$$ex(n', 3K_{p+1}) = \binom{n'}{2} - 5(n' - 2p - 1).$$

By Theorem 1.3, $ex(3p+3, 3K_{p+1}) = \binom{3p+3}{2} - 6$.

By Lemma 4.1, we have

$$ex(n, 2K_{p+1}) = \begin{cases} \binom{n}{2} - 3(n - 2p - 1), & 2p + 2 \leq n \leq 3p + 1; \\ e(T^1_{n-1,p}), & n \geq 3p + 2, \end{cases}$$

and

$$ex(n, 3K_{p+1}) = \begin{cases} \binom{n}{2} - 6, & n = 3p+3; \\ \binom{n}{2} - 5(n-3p-2), & 3p+4 \le n \le 5p+2; \\ e(T^2_{n-2,p}), & n \ge 5p+3. \end{cases}$$

This completes the proof of Theorem 1.4. ■

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