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GALLAI-RAMSEY NUMBERS FOR RAINBOW TREES AND MONOCHROMATIC COMPLETE BIPARTITE GRAPHS [1]

Luyi Li

Center for Combinatorics
Nankai University
Tianjin 300071, China

e-mail: liluyi@mail.nankai.edu.cn

Xueliang Li

Center for Combinatorics
Nankai University
Tianjin 300071, China

e-mail: lxl@nankai.edu.cn

Yaping Mao

Academy of Plateau Science and Sustainability, and School of Mathematics and Statistics
Qinghai Normal University
Xining, Qinghai 810008, China

e-mail: maoyaping@ymail.com

and

Yuan Si

Center for Combinatorics
Nankai University
Tianjin 300071, China

e-mail: yuan_si@aliyun.com

Abstract

Given two non-empty graphs G, H and a positive integer k, the Gallai-Ramsey number $\mathrm{gr}_k(G : H)$ is defined as the minimum positive integer N such that for all $n \geq N$, every k-edge-colored K_n contains either a rainbow

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subgraph G or a monochromatic subgraph H. In this paper, we get some exact values or bounds of $\mathrm{gr}_k(K_{1,3} : H)$, $\mathrm{gr}_k(P_5 : H)$, and $\mathrm{gr}_k(P_4^+ : H)$ for $k \geq 3$, where H is a complete bipartite graph.

Keywords: Ramsey theory; Gallai-Ramsey number; Complete bipartite graph.

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1. Introduction

In this paper, we consider finite, simple, and undirected graphs. Let $V(G)$ and $E(G)$ denote the vertex and edge sets of a graph G, respectively. A *k-edge-coloring* of G is a function $c : E(G) \to \{1, 2, \dots, k\}$, where $\{1, 2, \dots, k\}$ is a set of colors. An edge-coloring of a graph with a given number of colors is *exact* if each color is used at least once, and we only study exact edge-colorings of graphs in this paper. A *rainbow* graph refers to an edge-colored graph whose edges have distinct colors, while a *monochromatic* graph refers to an edge-colored graph whose edges have the same color. More commonly used notation and terminology in graph theory are not repeated here. For specific notions, we refer to the textbook [2].

1.1. Ramsey numbers

Ramsey theory originated in the 1920s and was first proposed by the British mathematician F.P. Ramsey. Since 1930, Ramsey problems have been hot topics in discrete mathematics. There are many papers on Ramsey theory, including the original paper of Ramsey [16].

For $k \geq 2$, given graphs $G_1, G_2, \dots, G_k$, the *Ramsey number* $R(G_1, G_2, \dots, G_k)$ is defined as the minimum positive integer n such that every k-edge-colored K_n contains a monochromatic subgraph G_i with color i, where $1 \leq i \leq n$. If $G_1 = G_2 = \ldots = G_k = G$, then we simply write the Ramsey number as $R_k(G)$. If $k = 2$ and $G_1 = G_2 = G$, then we write the Ramsey number as $R(G)$. In [3], Burr determined the exact value of $R(K_{2,3})$. In [10], Harborth and Mengersen gave the exact value of $R(K_{1,3}, K_{3,3})$.

Theorem 1. [3, 10] *$R(K_{2,3}) = 10$, $R(K_{1,3}, K_{3,3}) = 8$.*

For more results on Ramsey numbers, we refer to the survey [15].

1.2. Gallai-Ramsey numbers

Gallai's paper [7] was the first to explore the intriguing structure of an edge-colored complete graph without rainbow triangles. Consequently, this type of edge-coloring of a complete graph with no rainbow triangles is known as *Gallai coloring*. Gallai's result was restated in [4, 9]. For the following statement, a nontrivial partition means a partition with at least two parts.

Theorem 2. [4, 7, 9] *If G is an edge-colored complete graph without rainbow triangles, then there exists a nontrivial partition of $V(G)$ such that the number of colors between different parts is at most two, and the edges connecting each pair of parts are all the same color.*

In [5], Faudree, Gould, Jacobson, and Magnant defined *Gallai-Ramsey number* $\mathrm{gr}_k(G : H)$.

Definition 3. [5] Given two non-empty graphs G, H and a positive integer k, define the Gallai-Ramsey number $\mathrm{gr}_k(G : H)$ to be the minimum integer N such that for all $n \geq N$, every k-edge-colored K_n contains either a rainbow subgraph G or a monochromatic subgraph H.

Noticing that Gallai-Ramsey numbers consider only edge-colorings of complete graphs. So, according to the definitions of Ramsey number and Gallai-Ramsey number, we have

$$\mathrm{gr}_k(G : H) \leq R_k(H) < \infty.$$

Additionally, if $2 \leq k \leq |E(G)| - 1$, then it is clear that there is no rainbow subgraph G in any k-edge-colored complete graph. Therefore, in this case, we have

$$\mathrm{gr}_k(G : H) = R_k(H).$$

In the study of k-edge-colorings, in addition to "exact k-edge-coloring", another definition is the so-called "at most k-edge-coloring", which means that the actual number of colors used does not exceed k, and it is allowed to be less than k. In [11], Li, Besse, Magnant, Wang, and Watts gave a conjecture about the Gallai-Ramsey number for rainbow P_5 under the at most k-edge-coloring rule.

Conjecture 4. [11] *For any graph H with no isolated vertices, we have*

$$\mathrm{gr}_k(P_5 : H) = R_3(H).$$

For more recent results about Gallai-Ramsey numbers, we refer to the monograph book [14].

1.3. Structural theorems under rainbow-tree-free colorings

In [18], Thomason and Wagner obtained the following results.

Theorem 5. [18] *For an integer $n \geq 4$, let K_n be an edge-colored complete graph so that it contains no rainbow P_4. Then one of the following statements holds.*

(i) *At most two colors are used;*

(ii) $n = 4$ *and three colors are used, each color forming a perfect matching.*

Thomason and Wagner pointed out in the same paper that when the number of colors $k \geq 4$, the structures of a k-edge-colored complete graph without rainbow P_5 are relatively clear. They gave several coloring structures, of which only one coloring structure (i.e., Theorem 6 (ii)) has more variations. In Theorem 6 (ii), there is a special color, which Thomason and Wagner called the dominant color. The edges incident with each vertex can only have at most one other color besides the dominant color. So in the description of Theorem 6 (ii), we assume that color 1 is the dominant color.

Theorem 6. [18] *For positive integers k and n, if K_n is a k-edge-colored complete graph without rainbow subgraph P_5, then one of the following statements holds.*

(i) $k \leq 3$ *or* $n \leq 4$*;*

(ii) *There exists a partition $(V_2, V_3, \ldots, V_k)$ of $V(K_n)$. For any integer i, $2 \leq i \leq k$, the color of an edge with any two vertices in V_i is either the dominant color (i.e., color 1) or the color i. For any two integers i and j, $2 \leq i < j \leq k$, the color of all edges with one vertex in V_i and the other in V_j have the dominant color (i.e., color 1). This coloring structure is shown in Figure 1;*

(iii) *$K_n - v$ is monochromatic for some vertex v;*

(iv) *There are three vertices a, b, and c such that the edges ab, bc, and ac have color 2, 3, and 4, respectively, some edges incident with a have color 3, and all the other edges have color 1;*

(v) *There are four vertices a, b, c, and d such that the edges ab, ac, ad, bc, and bd have color 2, 3, 4, 4, and 3, respectively, the edge cd has color 1 or 2, and all the other edges have color 1;*

(vi) *$n = 5$, $V(K_n) = \{a, b, c, d, e\}$, the edges ad, ae, and bc have color 1, the edges bd, be, and ac have color 2, the edges cd, ce, and ab have color 3, and the edge de has color 4.*

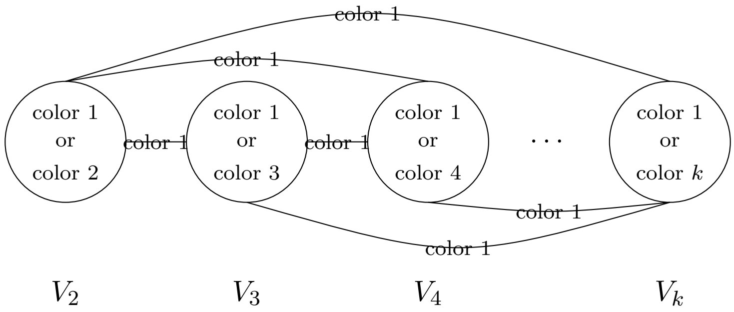

Figure 1. The partition $(V_2, V_3, \ldots, V_k)$ of $V(K_n)$ in Theorem 6 (ii). Each circle in the figure represents a vertex subset. The lines between the circles represent all edges between the induced subgraphs by two vertex subsets. The "color 1" on the line indicates that the edges between the induced subgraphs by these two vertex subsets are all color 1. The "color 1 or color i" inside the vertex subset V_i $(2 \leq i \leq k)$ indicates that the edges of the induced subgraph by V_i are either color 1 or color i.

For an integer $n \geq 4$, let $G_1(n)$ be a 3-edge-colored K_n that satisfies the following conditions: The vertices of K_n are partitioned into three pairwise disjoint sets V_1, V_2, and V_3 such that for $1 \leq i \leq 3$ (with indices modulo 3), all the edges between V_i and V_{i+1} have color i, and all the edges connecting pairs of vertices within V_{i+1} have color i or $i+1$. This coloring structure is shown in Figure 2. Noticing that one of V_1, V_2, and V_3 is allowed to be empty, but at least two of them are non-empty (otherwise at most only two colors can appear).

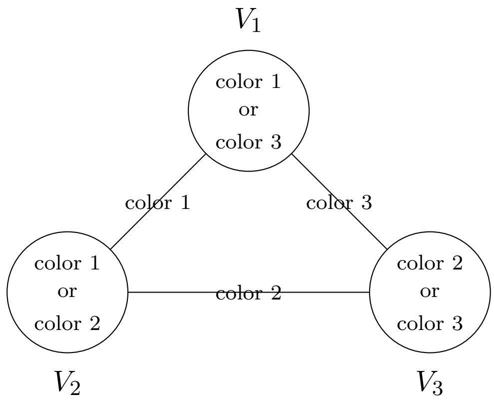

Figure 2. The partition (V_1, V_2, V_3) of $V(K_n)$ in Theorem 7 (ii). The drawing method and its meaning of this figure are the same as Figure 1.

The *local k-coloring* of a graph G refers to the edge coloring of G, satisfying that the colors of the edges incident to each vertex of G are at most k. In [8], Gyárfás, Lehel, Schelp, and Tuza gave the coloring structure of a local 2-colored complete graph K_n with k colors. Using the original notation of [8], let A_{ij} be a vertex subset of complete graph K_n, and each edge of the induced subgraph by A_{ij} has either color i or color j. Then there are only two types of coloring structures of the local 2-colored complete graph K_n with k colors. One structure is $k = 3$ and there exists a partition of $V(K_n)$, denoted as (A_{12}, A_{13}, A_{23}). The other structure is $k \geq 3$ and there exists a dominant color, which may be assumed to be color 1. The vertex set of K_n has a partition, denoted as $(A_{12}, A_{13}, \ldots, A_{1k})$. In [1], Bass, Magnant, Ozeki, and Pyron studied the edge-colored complete graphs without rainbow $K_{1,3}$ from structural perspectives. Among them, the $G_1(n)$ is a local 2-colored K_n. In fact, Theorem 6 (ii) is the other structure of local 2-colored K_n.

Theorem 7. [1, 8] *For positive integers k and n, if K_n is a k-edge-colored complete graph without rainbow subgraph $K_{1,3}$, then one of the following statements holds.*

(i) $k \leq 2$ *or* $n \leq 3$*;*

(ii) $k = 3$ *and* $K_n = G_1(n)$*;*

(iii) $k \geq 4$ *and Item* (ii) *in Theorem 6 holds.*

Next we give two types of edge-colored complete graphs without rainbow P_4^+, where P_4^+ is the tree consisting of a P_4 with one extra pendent edge incident with an inner vertex (the vertex with degree 2) of P_4. In other words, P_4^+ can also be seen as adding one extra pendent edge incident with a leaf vertex (the vertex with degree 1) of $K_{1,3}$.

For an integer $n \geq 4$, let $G_2(n)$ be a 4-edge-colored K_n in which there is exactly one edge, say xy, having color 2. Every edge from x to all the other vertices except y has color 3, and every edge from y to all the other vertices except x has color 4. All the edges not incident to vertices x, y have color 1. This graph contains no rainbow subgraph P_4^+ but contains a rainbow subgraph $K_{1,3}$ and (if $n \geq 5$) a rainbow subgraph P_5.

For an integer $n \geq 4$, let $G_3(n)$ be a 4-edge-colored K_n in which there exists a rainbow subgraph K_3 having colors 1, 2, and 3, say $V(K_3) = \{a, b, c\}$, the edge ab has color 1, the edge bc has color 2 and the edge ac has color 3. Let every edge incident with at most one vertex in the rainbow subgraph K_3 have color 4. This graph contains no rainbow subgraphs P_4^+ and P_5, but contains a rainbow subgraph $K_{1,3}$.

Theorem 8. [1, 17] *For positive integers k and n, if K_n is a k-edge-colored complete graph without rainbow subgraph P_4^+, then one of the following statements holds.*

(i) $k \leq 3$ *or* $n \leq 4$*;*

(ii) $k = 4$ *and* $K_n \in \{G_2(n), G_3(n)\}$*;*

(iii) $k \geq 4$ and K_n contains no rainbow $K_{1,3}$. In particular, Item (ii) in Theorem 6 holds.

In [13], Li, Wang, and Liu got some exact values and bounds for $\mathrm{gr}_k(P_5 : K_t)$, and got the structural theorems for complete bipartite graphs without rainbow subgraphs P_4 and P_5. In [6], Fujita and Magnant obtained the structural theorem for $G = S_3^+$. In [12], Li and Wang studied Gallai-Ramsey numbers for monochromatic stars in the rainbow K_3-free and S_3^+-free colorings. In [20], Zou, Wang, Lai, and Mao derived results for $\mathrm{gr}_k(P_5 : H)$ $(k \geq 3)$, where H is a general or special graph.

In next section, we will give some propositions and lemmas. In Section 3, we determine some exact values or bounds of $\mathrm{gr}_k(K_{1,3} : K_{m,n})$ for $m \in \{1, 2, 3, 4\}$. In Section 4, we determine some exact values of $\mathrm{gr}_k(P_5 : K_{m,n})$ and $\mathrm{gr}_k(P_4^+ : K_{m,n})$ for $m \in \{2, 3, 4\}$. In the last section, some related open problems are proposed.

2. Preliminaries

In 2019, Li, Wang, and Liu, in [13], determined the bound of k such that any k-edge-colored K_n always has a rainbow subgraph P_5. When $k \leq n$, we can construct a k-edge-colored K_n according to Theorem 6 (iii) such that it contains no rainbow subgraph P_5. Therefore, the bound of k is sharp.

Proposition 9. [13] *For integers $n \geq 5$ and $n + 1 \leq k \leq \binom{n}{2}$, there is always a rainbow subgraph P_5 in any k-edge-colored K_n. In addition, the bound of k is sharp.*

We determine the sharp bound of k such that any k-edge-colored K_n always has a rainbow subgraph $K_{1,3}$ or P_4^+.

Proposition 10. *For integers $n \geq 4$ and $\lceil \frac{n+3}{2} \rceil \leq k \leq \binom{n}{2}$, there is always a rainbow subgraph $K_{1,3}$ in any k-edge-colored K_n. In addition, the bound of k is sharp.*

Proof. Suppose that there is a k-edge-colored K_n containing no rainbow subgraph $K_{1,3}$. Since $k \geq \lceil \frac{n+3}{2} \rceil \geq 4$, it follows that (i) and (ii) of Theorem 7 do not hold. Next, we assume that Theorem 7 (iii) holds. Noticing that every color appears, which implies that $|V_i| \geq 2$ for each $i \in \{2, 3, \ldots, k\}$. Hence, $n \geq 2(k - 1)$, that is, $k \leq \lfloor \frac{n+2}{2} \rfloor$, which contradicts the fact that $\lceil \frac{n+3}{2} \rceil \leq k \leq \binom{n}{2}$. Since $\lceil \frac{n+3}{2} \rceil - 1 = \lfloor \frac{n+2}{2} \rfloor$, it follows that the bound of k is sharp. ■

Similar to the proof of Proposition 10, we can give the following proposition directly.

Proposition 11. *For integers $n \geq 6$ and $\lceil \frac{n+3}{2} \rceil \leq k \leq \binom{n}{2}$, there is always a rainbow subgraph P_4^+ in any k-edge-colored K_n. In particular, for an integer $5 \leq k \leq 10$, there is always a rainbow subgraph P_4^+ in any k-edge-colored K_5. In addition, the bound of k is sharp.*

Consider a k-edge-colored K_n. If $k = 2$, then there is obviously no rainbow subgraph K_3 or $K_{1,3}$ in K_n; if $2 \leq k \leq 3$, then there is obviously no rainbow subgraph P_5 or P_4^+ in K_n. Therefore, the following lemma can be given directly.

Lemma 12. *For graphs $G \in \{K_3, K_{1,3}, P_5, P_4^+\}$ and H, we have*

$$\mathrm{gr}_2(G : H) = R(H).$$

For graphs $G \in \{P_5, P_4^+\}$ and H, we have

$$\mathrm{gr}_3(G : H) = R_3(H).$$

In [19], Zhou, Li, Mao, and Wei gave some general results between $\mathrm{gr}_k(K_{1,3} : H)$, $\mathrm{gr}_k(P_5 : H)$ and $\mathrm{gr}_k(P_4^+ : H)$ $(k \geq 4)$.

Lemma 13. [19] $\mathrm{gr}_4(P_5 : H) \geq \mathrm{gr}_4(K_{1,3} : H)$.

Lemma 14. [19] *For integers $k \geq 5$ and $\mathrm{gr}_k(K_{1,3} : H) \geq 5$, we have*

$$\mathrm{gr}_k(P_5 : H) = \begin{cases} \max\left\{|V(H)| + 1, \mathrm{gr}_k(K_{1,3} : H)\right\}, & 5 \leq k \leq |V(H)|; \\ \mathrm{gr}_k(K_{1,3} : H), & k \geq |V(H)| + 1 \geq 5. \end{cases}$$

Lemma 15. [19] *For integers $k \geq 5$ and $\mathrm{gr}_k(K_{1,3} : H) \geq 5$, we have*

$$\mathrm{gr}_k(P_4^+ : H) = \mathrm{gr}_k(K_{1,3} : H).$$

Similarly, we can also get the following result.

Lemma 16. $\mathrm{gr}_4(P_4^+ : H) \geq \mathrm{gr}_4(K_{1,3} : H)$.

Remark 17. We must correct a small flaw in Theorems 14 and 15 given in the original paper [19], which is that the lack of condition $\mathrm{gr}_k(K_{1,3} : H) \geq 5$ can lead to errors. Noticing that if $k \geq 5$ and $\mathrm{gr}_k(K_{1,3} : H) = 4$, then $\mathrm{gr}_k(P_5 : H) > 4$ and $\mathrm{gr}_k(P_4^+ : H) > 4$. This is because for any k-edge-colored K_4 with $5 \leq k \leq 6$, there is no rainbow subgraph P_5 or P_4^+, and also no monochromatic subgraph H (except for the trivial case where $H = K_2$ or $H = 2K_2$).

When the number of colors $k \geq 4$, we know from Theorem 7 (iii) (i.e., Theorem 6 (ii)) that if a k-edge-colored complete graph does not contain a rainbow subgraph $K_{1,3}$, then there is only one coloring structure. Conversely, if the coloring structure of a k-edge-colored complete graph satisfies what is described in

Theorem 7 (iii), then the complete graph does not contain a rainbow subgraph $K_{1,3}$. In order to describe the edge-coloring structure of lower bounds in the following sections more concisely, we construct a family of k-edge-colored complete graphs based on the coloring structure given in Theorem 7 (iii). Therefore, every k-edge-colored complete graph described in Definition 18 does not contain a rainbow subgraph $K_{1,3}$.

Definition 18. Let integer $k \geq 4$ and $[K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}]$ be a k-edge-colored complete graph obtained from $k-1$ vertex-disjoint complete graphs $K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}$ such that all the edges of K_{t_i} are colored by $i+1$ for each $1 \leq i \leq k-1$ and all the edges between K_{t_i} and K_{t_j} are colored by 1 for any two integers $1 \leq i < j \leq k-1$.

3. RESULTS INVOLVING RAINBOW $K_{1,3}$

For a large integer k, the Gallai-Ramsey number $\mathrm{gr}_k(K_{1,3} : K_{m,n})$ is a function that depends only on k.

Theorem 19. *Let integers $n \geq m \geq 1$ and $n \geq 3$. If $k \geq \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil + 1$, then*

$$\mathrm{gr}_k(K_{1,3} : K_{m,n}) = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil.$$

Proof. Let $N_k = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$. For the lower bound, if there is an exact k-edge-coloring of a complete graph K_{N_k-1}, then $k \leq \binom{N_k-1}{2}$, contradicting $N_k = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$. It follows that $\mathrm{gr}_k(K_{1,3} : K_{m,n}) \geq \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$. For any k-edge-colored K_N ($N \geq N_k$), it follows from $n \geq m \geq 1$ and $n \geq 3$ that $k \geq \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil + 1 \geq 4$ and $N_k < 2k-2$ for all $k \geq 4$.

If $N_k \leq N \leq 2k-3$, then it follows from Proposition 10 that there is always a rainbow subgraph $K_{1,3}$, the result thus follows. Next we assume that $N \geq 2k-2$. Suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{m,n}$. It follows from the fact that $k \geq 4$ that Theorem 7 (i) and (ii) do not hold. If Theorem 7 (iii) holds, then $|V_i| \geq 2$ for each $i \in \{2, 3, \ldots, k\}$. Let $A = \bigcup_{i=2}^{\lceil m/2 \rceil+1} V_i$ and $B = \bigcup_{i=\lceil m/2 \rceil+2}^{\lceil m/2 \rceil+\lceil n/2 \rceil+1} V_i$. From Theorem 7 ($iii$), the edges from A and B are colored by the same color. Since $|A| \geq m$ and $|B| \geq n$, it follows that there is a monochromatic subgraph $K_{m,n}$, a contradiction. The result thus follows. ■

Theorem 20. *For integers $k \geq 4$, $m \in \{1, 2\}$ and $n \geq 3$, we have*

$$\mathrm{gr}_k(K_{1,3} : K_{m,n}) = \begin{cases} \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, & 3 \leq n \leq 2k-4; \\ n+a, & a(k-2)+1 \leq n \leq (a+1)(k-2) \text{ where } a \geq 2 \text{ is an integer.} \end{cases}$$

Proof. Assume that $3 \le n \le 2k-4$. Since $\lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil + 1 \le \lceil \frac{2}{2} \rceil + \lceil \frac{2k-4}{2} \rceil + 1 = k$, it follows from Theorem 19 that $\mathrm{gr}_k(K_{1,3} : K_{m,n}) = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$.

Assume that $a(k-2)+1 \le n \le (a+1)(k-2)$ where $a \ge 2$ is an integer. Let $t_1 = n - a(k-3) - 1$ and $t_i = a$ for each $2 \le i \le k-1$. Then $K_{n+a-1} = [K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}]$ is a k-edge-colored complete graph and contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{m,n}$, and so $\mathrm{gr}_k(K_{1,3} : K_{m,n}) \ge n + a$.

Consider any k-edge-colored K_N ($N \ge n+a$) and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{m,n}$. It follows from the fact that $k \ge 4$ that Theorem 7 (i) and (ii) do not hold. If Theorem 7 (iii) holds, then $|V_i| \ge 2$ for each $i \in \{2,3,\ldots,k\}$ and $\sum_{i=2}^{k} |V_i| \ge n + a$. Without loss of generality, set $|V_2| \ge |V_3| \ge \ldots \ge |V_k| \ge 2$.

If $2 \le |V_2| \le a$, then $|V(K_N)| - |V_2| \ge n$ and hence there is a monochromatic subgraph $K_{2,n}$, a contradiction. Next we assume that $|V_2| \ge a+1$. In this case, noticing that $|V_2| \ge a+1 > 2$ and $\sum_{i=3}^{k} |V_i| \ge (a+1)(k-2) \ge n$, there is a monochromatic subgraph $K_{a+1,n}$. Therefore, K_N contains a monochromatic subgraph $K_{2,n}$, a contradiction. ■

Theorem 21. *For integers $k \ge 4$ and $n \ge 3$, we have*

$$\mathrm{gr}_k(K_{1,3} : K_{3,n}) = \begin{cases} \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, & 3 \le n \le 2k-6\, (k \ge 5); \\ 2k-1, & 2k-5 \le n \le 2k-4; \\ n+4, & 2k-3 \le n \le 4k-10; \\ \left(\frac{k-2}{k-3}\right)(n-3-a)+a+3, & n \ge 4k-9 \text{ and } n-3 \equiv a \pmod{k-3} \\ & \text{where } a \in \{0,1,\ldots,k-4\}. \end{cases}$$

Proof. Assume that $3 \le n \le 2k-6\,(k \ge 5)$. Since $\lceil \frac{3}{2} \rceil + \lceil \frac{n}{2} \rceil + 1 \le \lceil \frac{3}{2} \rceil + \lceil \frac{2k-6}{2} \rceil + 1 = k$, it follows from Theorem 19 that $\mathrm{gr}_k(K_{1,3} : K_{3,n}) = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$. Next, we distinguish the following three cases to prove this theorem.

Case 1. $2k-5 \le n \le 2k-4$.

Let $t_i = 2$ for each $1 \le i \le k-1$. Then $K_{2(k-1)} = [K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}]$ is a k-edge-colored complete graph and contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{3,n}$, and so $\mathrm{gr}_k(K_{1,3} : K_{3,n}) \ge 2(k-1)+1 = 2k-1$.

Consider any k-edge-colored K_N ($N \ge 2k-1$) and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{3,n}$. It follows from the fact that $k \ge 4$ that Theorem 7 (i) and (ii) do not hold. If Theorem 7 (iii) holds, then $|V_i| \ge 2$ for each $i \in \{2,3,\ldots,k\}$ and $\sum_{i=2}^{k} |V_i| \ge 2k-1$. Without loss of generality, set $|V_2| \ge |V_3| \ge \ldots \ge |V_k| \ge 2$.

If $|V_2| = 2$, then $|V_2| = |V_3| = \ldots = |V_k| = 2$, and hence $\sum_{i=2}^{k} |V_i| = 2k-2$, which contradicts $\sum_{i=2}^{k} |V_i| \geq 2k-1$. If $|V_2| \geq 3$, then the complete bipartite graph with the bipartition $\left(V_2, \bigcup_{i=3}^{k} V_i\right)$ contains a monochromatic subgraph $K_{3,2k-4}$, a contradiction.

Case 2. $2k-3 \leq n \leq 4k-10$.

Let $t_1 = n-2k+7$ and $t_i = 2$ for each $2 \leq i \leq k-1$. Then $K_{n+3} = [K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}]$ is a k-edge-colored complete graph and contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{3,n}$, and so $\mathrm{gr}_k(K_{1,3} : K_{3,n}) \geq n+4$.

Consider any k-edge-colored K_N $(N \geq n+4)$ and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{3,n}$. It follows from the fact that $k \geq 4$ that Theorem 7 (i) and (ii) do not hold. If Theorem 7 (iii) holds, then $|V_i| \geq 2$ for each $i \in \{2,3,\ldots,k\}$ and $\sum_{i=2}^{k} |V_i| \geq n+4$. Without loss of generality, set $|V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 2$.

If $|V_{k-1}| = 2$, then $|V_{k-1}| = |V_k| = 2$ and hence there is a monochromatic subgraph $K_{4,n}$, a contradiction. If $3 \leq |V_{k-1}| \leq 4$, then $|V(K_N)| - |V_k| \geq n$ and hence there is a monochromatic subgraph $K_{3,n}$, a contradiction. If $|V_{k-1}| \geq n-2(k-4)$, then the complete bipartite graph with the bipartition $\left(V_2 \cup V_k, \bigcup_{i=3}^{k-1} V_i\right)$ contains a monochromatic subgraph $K_{4,n}$, a contradiction. Next we assume that $5 \leq |V_{k-1}| \leq n-2k+7$. Recall that $k \geq 4$ and $2k-3 \leq n \leq 4k-10$. From the above all, we know that $|V_2| \geq |V_3| \geq \ldots \geq |V_{k-1}| \geq 5$ and $|V_k| \geq 2$. Since $\sum_{i=3}^{k} |V_i| \geq 5(k-3)+2 > 4k-10 \geq n$ and $|V_2| \geq 5$, it follows that there is a monochromatic subgraph $K_{5,n}$, a contradiction.

Case 3. $n \geq 4k-9$ and $n-3 \equiv a \pmod{k-3}$ where $a \in \{0,1,\ldots,k-4\}$.

It follows from $n-3 \equiv a \pmod{k-3}$ that $\frac{n-3-a}{k-3}$ is an integer. Let $q = \frac{n-3-a}{k-3}$, $t_1 = q+a$, $t_2 = 2$ and $t_i = q$ for each $3 \leq i \leq k-1$. Then $K_{(k-2)q+a+2} = [K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}]$ is a k-edge-colored complete graph. Next, we only need to verify that this k-edge-colored $K_{(k-2)q+a+2}$ does not contain a monochromatic subgraph $K_{3,n}$.

Let the bipartition of the complete bipartite graph $K_{3,n}$ be (X,Y), where $|X| = 3$ and $|Y| = n$. Obviously, the monochromatic $K_{3,n}$ cannot be inside any of the K_{t_i}, where $1 \leq i \leq k-1$. Noticing that $\frac{n-3-a}{k-3} \geq \frac{4k-12-a}{k-3} \geq \frac{3k-8}{k-3} > 3$. If $X \subseteq V(K_{t_j})$ for some $3 \leq j \leq k-1$, then

$$|V(K_{(k-2)q+a+2})| - |V(K_{t_j})| = (k-2)q+a+2-q = (k-3)q+a+2 = n-1.$$

This means that there is no monochromatic subgraph $K_{3,n}$ in such k-edge-colored $K_{(k-2)q+a+2}$. Similarly, if $X \subseteq V(K_{t_1})$, there is also no monochromatic subgraph $K_{3,n}$, and so $\mathrm{gr}_k(K_{1,3} : K_{3,n}) \geq (k-2)q+a+3$.

Consider any k-edge-colored K_N ($N \geq (k-2)q+a+3$) and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{3,n}$. It follows from the fact that $k \geq 4$ that Theorem 7 (i) and (ii) do not hold. If Theorem 7 (iii) holds, then $|V_i| \geq 2$ for each $i \in \{2,3,\ldots,k\}$ and $\sum_{i=2}^{k}|V_i| \geq (k-2)q+a+3$. Without loss of generality, set $|V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 2$.

If $|V_{k-1}| = 2$, then $|V_{k-1}| = |V_k| = 2$ and for $n \geq 4k-9$,

$$|V(K_N)| - (|V_{k-1}| + |V_k|) \geq (k-2)q + a - 1 \geq n,$$

hence there is a monochromatic subgraph $K_{4,n}$, a contradiction. If $3 \leq |V_{k-1}| \leq (k-2)q+a+3-n$, then $|V(K_N)|-|V_{k-1}| \geq n$, and hence there is a monochromatic subgraph $K_{3,n}$, a contradiction. Next we assume that $|V_{k-1}| \geq (k-2)q+a+4-n$. Since

$$\begin{aligned} |V_2| \geq |V_{k-1}| &\geq (k-2)q+a+4-n \geq \frac{4k-9}{k-3} - \frac{(3+a)(k-2)}{k-3} + a + 4 \\ &= \frac{k-3-a}{k-3} + 4 \geq \frac{1}{k-3} + 4 > 4 \end{aligned}$$

and

$$\begin{aligned} \sum_{i=3}^{k}|V_i| &\geq \sum_{i=3}^{k-1}|V_i| + 2 \geq (k-3)[(k-2)q+a+4-n]+2 \\ &= n-(3+a)(k-2)+(4+a)(k-3)+2 \\ &= n+k-4-a \geq n+a+4-4-a = n, \end{aligned}$$

it follows that there is a monochromatic subgraph $K_{4,n}$ with bipartition $\left(V_2, \bigcup_{i=3}^{k} V_i\right)$, a contradiction. ■

Theorem 22. *For integers $k \geq 4$ and $n \geq 4$, we have*

$$\mathrm{gr}_k(K_{1,3} : K_{4,n}) = \begin{cases} \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, & 4 \leq n \leq 2k-6\,(k \geq 5); \\ n+4, & 2k-5 \leq n \leq 2k-4\ (k \geq 5); \\ n+4, & 2k-3 \leq n \leq 3k-9\,(k \geq 6); \\ 3k-2, & 3k-8 \leq n \leq 3k-7; \\ 3k-1, & n = 3k-6; \\ n+6, & 3k-5 \leq n \leq 6k-16; \\ \left(\frac{k-2}{k-3}\right)(n-3-a)+a+3, & n \geq 6k-15 \text{ and } n-3 \equiv a \pmod{k-3} \\ & \text{where } a \in \{0,1,\ldots,k-4\}. \end{cases}$$

Proof. Assume that $4 \leq n \leq 2k-6\,(k \geq 5)$. Since $\lceil \frac{4}{2} \rceil + \lceil \frac{n}{2} \rceil + 1 \leq \lceil \frac{4}{2} \rceil + \lceil \frac{2k-6}{2} \rceil + 1 = k$, it follows from Theorem 19 that $\mathrm{gr}_k(K_{1,3} : K_{4,n}) = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$. Next, we distinguish the following five cases to prove this theorem.

Case 1. $2k-5 \leq n \leq 2k-4\,(k \geq 5)$ or $2k-3 \leq n \leq 3k-9\,(k \geq 6)$.

Let $t_1 = n-2k+7$ and $t_i = 2$ for each $2 \leq i \leq k-1$. Then $K_{n+3} = [K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}]$ is a k-edge-colored complete graph and contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{4,n}$, and so $\mathrm{gr}_k(K_{1,3} : K_{4,n}) \geq n+4$.

Consider any k-edge-colored K_N $(N \geq n+4)$ and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{4,n}$. It follows from the fact that $k \geq 4$ that Theorem 7 (i) and (ii) do not hold. If Theorem 7 (iii) holds, then $|V_i| \geq 2$ for each $i \in \{2,3,\ldots,k\}$ and $\sum_{i=2}^{k} |V_i| \geq n+4$. Without loss of generality, set $|V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 2$.

If $|V_{k-1}| = 2$, then $|V_{k-1}| = |V_k| = 2$, and hence the complete bipartite graph with the bipartition $\left(V_{k-1} \cup V_k, \bigcup_{i=2}^{k-2} V_i\right)$ contains a monochromatic subgraph $K_{4,n}$, a contradiction. If $|V_{k-1}| \geq 3$, then $|V_2| \geq |V_3| \geq \ldots \geq |V_{k-1}| \geq 3$. Since $\sum_{i=2}^{k-2} |V_i| \geq 3\,(k-3) \geq 2k-4$ $(k \geq 5)$ and $|V_{k-1}|+|V_k| \geq 5$, it follows that there is a monochromatic subgraph $K_{5,n}$, a contradiction.

Case 2. $3k-8 \leq n \leq 3k-7$.

Let $t_i = 3$ for each $1 \leq i \leq k-1$. Then $K_{3(k-1)} = [K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}]$ is a k-edge-colored complete graph and contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{4,n}$, and so $\mathrm{gr}_k(K_{1,3} : K_{4,n}) \geq 3(k-1)+1 = 3k-2$.

Consider any k-edge-colored K_N $(N \geq 3k-2)$ and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{4,n}$. It follows from the fact that $k \geq 4$ that Theorem 7 (i) and (ii) do not hold. If Theorem 7 (iii) holds, then $|V_i| \geq 2$ for each $i \in \{2,3,\ldots,k\}$ and $\sum_{i=2}^{k} |V_i| \geq 3k-2$. Without loss of generality, set $|V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 2$.

If $|V_{k-1}| = 2$, then $|V_{k-1}| = |V_k| = 2$. Since $|V(K_N)| - (|V_{k-1}|+|V_k|) \geq 3k-6$, it follows that there is a monochromatic subgraph $K_{4,3k-6}$, a contradiction. Then $|V_{k-1}| \geq 3$. If $|V_2| \geq 4$, then since $\sum_{t=3}^{k} |V_t| \geq 3(k-3)+2 = 3k-7$, we have that there is a monochromatic subgraph $K_{4,3k-7}$, a contradiction. Hence, $|V_i| = 3$ for all $i \in \{2,3,\ldots,k-1\}$. In this case, $\sum_{i=2}^{k-1} |V_i| = 3(k-2) = 3k-6$ and $2 \leq |V_k| \leq 3$, and hence $\sum_{i=2}^{k} |V_i| \leq 3(k-2)+3 = 3k-3$, which contradicts $\sum_{i=2}^{k} |V_i| \geq 3k-2$.

Case 3. $n = 3k-6$.

Let $t_1 = 5$, $t_2 = 2$ and $t_i = 3$ for each $3 \leq i \leq k-1$. Then $K_{3k-2} = [K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}]$ is a k-edge-colored complete graph and contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{4,3k-6}$, and so $\mathrm{gr}_k(K_{1,3} : K_{4,3k-6}) \geq 3k-1$.

Consider any k-edge-colored K_N ($N \geq 3k-1$) and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{4,3k-6}$. It follows from the fact that $k \geq 4$ that Theorem 7 (i) and (ii) do not hold. If Theorem 7 (iii) holds, then $|V_i| \geq 2$ for each $i \in \{2, 3, \dots, k\}$ and $\sum_{i=2}^{k} |V_i| \geq 3k-1$. Without loss of generality, set $|V_2| \geq |V_3| \geq \dots \geq |V_k| \geq 2$.

If $|V_{k-1}| = 2$, then $|V_{k-1}| = |V_k| = 2$. Since $|V(K_N)| - (|V_{k-1}| + |V_k|) > 3k-6$, it follows that there is a monochromatic subgraph $K_{4,3k-6}$, a contradiction. Thus $|V_{k-1}| \geq 3$.

Claim 23. $|V_2| = 3$.

Proof of Claim 1. Suppose that $|V_2| \geq 4$. If $|V_k| \geq 3$, then $\sum_{t=3}^{k} |V_t| \geq 3k-6 = n$, and hence there is a monochromatic subgraph $K_{4,n}$, a contradiction. If $|V_k| = 2$ and $|V_{k-1}| = 3$, then $|V(K_N)| - (|V_{k-1}| + |V_k|) \geq 3k-6$, and hence there is a monochromatic subgraph $K_{5,3k-6}$, a contradiction. If $|V_k| = 2$ and $|V_{k-1}| \geq 4$, then $|V_2| \geq |V_3| \geq \dots \geq |V_{k-1}| \geq 4$ and $\sum_{i=2}^{k-2} |V_i| + |V_k| \geq 4(k-3) + 2 = 4k - 10 \geq 3k-6$ ($k \geq 4$), and hence there is a monochromatic subgraph $K_{4,3k-6}$, a contradiction. Thus, Claim 1 is proven.

Recall that $3 = |V_2| \geq |V_3| \geq \dots \geq |V_k| \geq 2$ and $|V_{k-1}| \geq 3$. It follows that $|V_2| = |V_3| = \dots = |V_{k-1}| = 3$, which implies that $\sum_{i=2}^{k-1} |V_i| = 3(k-2) = 3k-6$. Noticing that $2 \leq |V_k| \leq 3$, and hence $\sum_{i=2}^{k} |V_i| \leq 3(k-2) + 3 = 3k-3$, which contradicts $\sum_{i=2}^{k} |V_i| \geq 3k-1$.

Case 4. $3k-5 \leq n \leq 6k-16$.

Let $t_1 = n - 3k + 11$ and $t_i = 3$ for each $2 \leq i \leq k-1$. Then $K_{n+5} = [K_{t_1}, K_{t_2}, \dots, K_{t_{k-1}}]$ is a k-edge-colored complete graph and contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{4,n}$, and so $\mathrm{gr}_k(K_{1,3} : K_{4,n}) \geq n+6$.

Consider any k-edge-colored K_N ($N \geq n+6$) and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{4,n}$. It follows from the fact that $k \geq 4$ that Theorem 7 (i) and (ii) do not hold. If Theorem 7 (iii) holds, then $|V_i| \geq 2$ for each $i \in \{2, 3, \dots, k\}$ and $\sum_{i=2}^{k} |V_i| \geq n+6$. Without loss of generality, set $|V_2| \geq |V_3| \geq \dots \geq |V_k| \geq 2$.

If $2 \leq |V_{k-1}| \leq 3$, then $4 \leq |V_{k-1}| + |V_k| \leq 6$. Since $|V(K_N)| - (|V_{k-1}| + |V_k|) \geq n$, it follows that there is a monochromatic subgraph $K_{4,n}$, a contradiction. If $4 \leq |V_{k-1}| \leq 6$, then $|V(K_N)| - |V_{k-1}| \geq n$, and hence there is a monochromatic subgraph $K_{4,n}$, a contradiction. If $|V_{k-1}| \geq 7$, then $|V_2| \geq |V_3| \geq \dots \geq |V_{k-1}| \geq 7$ and $|V_k| \geq 2$. Since $\sum_{i=2}^{k-2} |V_i| + |V_k| \geq 7(k-3) + 2 > 6k-16 \geq n$, it follows that there is a monochromatic subgraph $K_{7,n}$, a contradiction.

Case 5. $n \geq 6k-15$ and $n - 3 \equiv a \pmod{k-3}$ where $a \in \{0, 1, \dots, k-4\}$.

It follows from $n-3 \equiv a \pmod{k-3}$ that $\frac{n-3-a}{k-3}$ is an integer. Let $q = \frac{n-3-a}{k-3}$, $t_1 = q + a$, $t_2 = 2$ and $t_i = q$ for each $3 \leq i \leq k-1$. Then $K_{(k-2)q+a+2} = [K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}]$ is a k-edge-colored complete graph. Next, we only need to verify that this k-edge-colored $K_{(k-2)q+a+2}$ does not contain a monochromatic subgraph $K_{4,n}$.

Let the bipartition of the complete bipartite graph $K_{4,n}$ be (X, Y), where $|X| = 4$ and $|Y| = n$. Obviously, the monochromatic $K_{4,n}$ cannot be inside any of the K_{t_i}, where $1 \leq i \leq k-1$. Noticing that $\frac{n-3-a}{k-3} \geq \frac{6k-18-a}{k-3} \geq \frac{5k-14}{k-3} > 5$. If $X \subseteq V(K_{t_j})$ for some $3 \leq j \leq k-1$, then

$$|V(K_{(k-2)q+a+2})| - |V(K_{t_j})| = (k-2)q + a + 2 - q = (k-3)q + a + 2 = n - 1.$$

This means that there is no monochromatic subgraph $K_{4,n}$ in such k-edge-colored $K_{(k-2)q+a+2}$. Similarly, if $X \subseteq V(K_{t_1})$, there is also no monochromatic subgraph $K_{4,n}$, and so $\mathrm{gr}_k(K_{1,3} : K_{4,n}) \geq (k-2)q + a + 3$.

Consider any k-edge-colored K_N $(N \geq (k-2)q + a + 3)$ and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{4,n}$. It follows from the fact that $k \geq 4$ that Theorem 7 (i) and (ii) do not hold. If Theorem 7 (iii) holds, then $|V_i| \geq 2$ for each $i \in \{2, 3, \ldots, k\}$ and $\sum_{i=2}^{k} |V_i| \geq (k-2)q + a + 3$. Without loss of generality, set $|V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 2$.

If $2 \leq |V_{k-1}| \leq 3$, then $4 \leq |V_{k-1}| + |V_k| \leq 6$ and for $n \geq 6k - 15$,

$$|V(K_N)| - (|V_{k-1}| + |V_k|) \geq (k-2)q + a - 3 \geq n,$$

hence there is a monochromatic subgraph $K_{4,n}$, a contradiction. If $4 \leq |V_{k-1}| \leq (k-2)q+a+3-n$, then $|V(K_N)| - |V_{k-1}| \geq n$, and hence there is a monochromatic subgraph $K_{4,n}$, a contradiction. Next we assume that $|V_{k-1}| \geq (k-2)q+a+4-n$. Since

$$\begin{aligned} |V_2| \geq |V_{k-1}| &\geq (k-2)q + a + 4 - n \geq \frac{6k-15}{k-3} - \frac{(a+3)(k-2)}{k-3} + a + 4 \\ &= \frac{3k-9-a}{k-3} + 4 \geq \frac{2k-5}{k-3} + 4 > 6 \end{aligned}$$

and

$$\begin{aligned} \sum_{i=3}^{k} |V_i| &\geq \sum_{i=3}^{k-1} |V_i| + 2 \geq (k-3)[(k-2)q + a + 4 - n] + 2 \\ &= n - (3+a)(k-2) + (4+a)(k-3) + 2 \\ &= n + k - 4 - a \geq n + a + 4 - 4 - a = n, \end{aligned}$$

it follows that there is a monochromatic subgraph $K_{6,n}$ with bipartition $\left(V_2, \bigcup_{i=3}^{k} V_i\right)$, a contradiction. ■

For $k = 3$, we have the following results.

Lemma 24. *For an integer* $n \geq 3$*, we have*

$$\mathrm{gr}_3(K_{1,3} : K_{n,n}) \geq R(K_{n-1,n}) + 2.$$

Proof. Let G be an edge-colored complete graph of order $R(K_{n-1,n}) - 1$ with two colors 1 and 2 such that no monochromatic subgraph $K_{n-1,n}$ exists. We construct $K_{R(K_{n-1,n})+1}$ from G by adding two vertices x_1 and x_2 such that the edge x_1x_2 is colored by 3 and the edges between x_i and G are colored by i for each $i \in \{1, 2\}$. One can easily check that there is neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{n,n}$ under such a 3-edge-colored $K_{R(K_{n-1,n})+1}$, and so $\mathrm{gr}_3(K_{1,3} : K_{n,n}) \geq R(K_{n-1,n}) + 2$. ■

Theorem 25. $\mathrm{gr}_3(K_{1,3} : K_{3,3}) = 12$.

Proof. By Theorem 1, we have $R(K_{2,3}) = 10$, and it follows from Lemma 24 that $\mathrm{gr}_3(K_{1,3} : K_{3,3}) \geq 12$. Consider any 3-edge-colored K_N ($N \geq 12$) and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{3,3}$. Noticing that the number of colors $k = 3$, and K_N does not contain a rainbow subgraph $K_{1,3}$, so by Theorem 7 (ii), $K_N = G_1(N)$. Recall the definition of $G_1(N)$ with partite sets V_1, V_2, and V_3.

If $|V_i|, |V_j| \geq 3$ for $i, j \in \{1, 2, 3\}$, then there is a monochromatic subgraph $K_{3,3}$, a contradiction. Recall $N \geq 12$, without loss of generality, and we assume that $|V_1| \geq 3$ and $|V_3| \leq |V_2| \leq 2$. Let G_i be the subgraph induced by V_i in K_N for each $i = \{1, 2, 3\}$. If $|V_2| = 2$, then $|V_3| \leq 2$ and $|V_1| \geq 8$. It follows from Theorem 1 ($R(K_{1,3}, K_{3,3}) = 8$) that there is either a monochromatic $K_{1,3}$ with color 1 or a monochromatic $K_{3,3}$ with color 3 in G_1. Noticing that the edges from G_1 to G_2 are colored by 1, and the edges from G_1 to G_3 are colored by 3, there is a monochromatic subgraph $K_{3,3}$, a contradiction. If $|V_2| = 1$, then $|V_3| = 1$ and $|V_1| \geq 10$. Since $R(K_{2,3}) = 10$, there is either a monochromatic $K_{2,3}$ with color 1 or a monochromatic $K_{2,3}$ with color 3 in G_1. Noticing that the edges from G_1 to G_2 are colored by 1, and the edges from G_1 to G_3 are colored by 3, there is a monochromatic subgraph $K_{3,3}$, a contradiction. ■

Theorem 26. *For an integer* $n \geq 3$*, we have*

$$\mathrm{gr}_3(K_{1,3} : K_{1,n}) = 2n.$$

Proof. Let G_1 be a monochromatic copy of K_{n-1} with color 3, and G_2 be a monochromatic copy of K_{n-1} with color 2, and G_3 be a copy of K_1. We construct a 3-edge-colored K_{2n-1} by considering G_1, G_2, and G_3, and adding all the edges between vertices of G_i and G_j for all $i \neq j$. We color these added edges as follows: For G_i and G_{i+1} (with indices modulo 3), we color all the edges with color i. One can easily check that there is neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{1,n}$ under such a 3-edge-colored K_{2n-1}, and so $\mathrm{gr}_3(K_{1,3} : K_{1,n}) \geq 2n$.

Consider any 3-edge-colored K_N ($N \geq 2n$) and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $K_{1,n}$. By Theorem 7 (ii), there is a partition (V_1, V_2, V_3) of $V(K_N)$ such that $K_N = G_1(N)$ when $k = 3$. For each vertex $v \in V_1$, from the coloring structure of $G_1(N)$, the color of all edges connecting v to all vertices in V_2 is color 1. Therefore, to avoid a monochromatic (with color 1) subgraph $K_{1,n}$, the vertex v can have at most $n - |V_2| - 1$ edges of color 1 in the induced subgraph by V_1. Similarly, the color of all edges connecting v to all vertices in V_3 is color 3. Therefore, to avoid a monochromatic (with color 3) subgraph $K_{1,n}$, the vertex v can have at most $n - |V_3| - 1$ edges of color 3 in the induced subgraph by V_1. Noticing that each edge of the induced subgraph by V_1 can only have color 1 or color 3, the degree of v in the induced subgraph by V_1 is at most $n - |V_2| - 1 + n - |V_3| - 1$, which implies $|V_1| - 1 \leq 2n - (|V_2| + |V_3|) - 2$. Similarly, we have $|V_2| - 1 \leq 2n - (|V_1| + |V_3|) - 2$ and $|V_3| - 1 \leq 2n - (|V_1| + |V_2|) - 2$. Therefore, $|V_1| + |V_2| + |V_3| \leq 6n - 2(|V_1| + |V_2| + |V_3|) - 3$, that is $|V_1| + |V_2| + |V_3| \leq 2n - 1$, a contradiction. ■

4. Results involving rainbow P_5 or P_4^+

In this section, we give the Gallai-Ramsey numbers for complete bipartite graphs involving rainbow P_5 or P_4^+. In proving $\mathrm{gr}_4(P_5 : H)$, we need to use the results of $\mathrm{gr}_4(K_{1,3} : H)$ in Section 3. Next, we briefly describe the proof technique. According to the definition of Gallai-Ramsey number, if we know that $\mathrm{gr}_k(K_{1,3} : H) = N$, then for all integers $n \geq N$, if K_n does not contain the rainbow subgraph $K_{1,3}$, then K_n must contain the monochromatic subgraph H. According to Theorem 7 (iii), it is uniquely determined that when $k \geq 4$, the coloring structure of K_n does not contain a rainbow subgraph $K_{1,3}$, which is the structure described in Theorem 6 (ii). Therefore, if Theorem 6 (ii) holds, then K_n indeed has neither a rainbow subgraph $K_{1,3}$ nor a rainbow subgraph P_5, but it must have a monochromatic subgraph H, which contradicts the contradiction method we use in the following proofs. So we will not repeat this basic technique in the following proofs.

Theorem 27. *For an integer $n \geq 3$, we have*

$$\mathrm{gr}_4(P_5 : K_{2,n}) = \begin{cases} n+3, & 3 \leq n \leq 8; \\ n+a, & 2a+1 \leq n \leq 2(a+1) \text{ where } a \geq 4 \text{ is an integer.} \end{cases}$$

Proof. We distinguish the following two cases to proceed with our proof.

Case 1. $3 \leq n \leq 8$.

Let G_1 be a monochromatic copy of K_{n+1} with color 1, and G_2 be a copy of K_1. We construct a K_{n+2} by making use of G_1, G_2 by inserting all edges between these copies such that the edges from G_1 to G_2 are colored by 2, 3, and 4. One can easily check that there is neither a rainbow subgraph P_5 nor a monochromatic subgraph $K_{2,n}$ under such a 4-edge-colored K_{n+2}, and so $\mathrm{gr}_4(P_5 : K_{2,n}) \geq n+3$.

Consider any 4-edge-colored K_N where $N \geq n+3$ and suppose to the contrary that K_N contains neither a rainbow subgraph P_5 nor a monochromatic subgraph $K_{2,n}$. It follows from the fact that $k = 4$ and Theorem 20 that Theorem 6 (i), (ii), and (vi) do not hold.

Suppose that Theorem 6 (iii) holds. Noticing that $K_N - v$ is monochromatic for some vertex v, there is a monochromatic subgraph $K_{2,n}$, a contradiction. Suppose that Theorem 6 (iv) holds. Noticing that $\{a, b, c, v_1, v_2, \ldots, v_n\} \subseteq V(K_N)$, there is a monochromatic subgraph $K_{2,n}$ with bipartition $\{b, c\}$ and $\{v_1, v_2, \ldots, v_n\}$ of $V(K_N)$ with color 1, a contradiction. Suppose that Theorem 6 (v) holds. Noticing that $\{a, b, c, d, v_1, v_2, \ldots, v_{n-1}\} \subseteq V(K_N)$, there is a monochromatic subgraph $K_{2,n}$ with bipartition $\{v_1, v_2\}$ and $\{a, b, c, d, v_3, v_4, \ldots, v_{n-2}\}$ with color 1, a contradiction.

Case 2. $2a + 1 \leq n \leq 2(a+1)$ where $a \geq 4$ is an integer.

From Lemma 13 and Theorem 20, we have $\mathrm{gr}_4(P_5 : K_{2,n}) \geq n + a$. Consider any 4-edge-colored K_N where $N \geq n + a$ ($a \in \{4, 5, \ldots\}$) and suppose to the contrary that K_N contains neither a rainbow subgraph P_5 nor a monochromatic subgraph $K_{2,n}$. It follows from the fact that $k = 4$ and Theorem 20 that Theorem 6 (i), (ii), and (vi) do not hold.

Suppose that Theorem 6 (iii) holds. Noticing that $K_N - v$ is monochromatic for some vertex v, there is a monochromatic subgraph $K_{2,n}$, a contradiction. Suppose that Theorem 6 (iv) holds. Noticing that $\{a, b, c, v_1, v_2, \ldots, v_{n+a-3}\} \subseteq V(K_N)$, then there is a monochromatic subgraph $K_{2,n}$ with bipartition $\{b, c\}$ and $\{v_1, v_2, \ldots, v_n\}$ with color 1, a contradiction. Suppose that Theorem 6 (v) holds. Noticing that $\{a, b, c, d, v_1, v_2, \ldots, v_{n+a-4}\} \subseteq V(K_N)$, then there is a monochromatic subgraph $K_{2,n}$ with bipartition $\{a, b\}$ and $\{v_1, v_2, \ldots, v_n\}$ with color 1, a contradiction. ■

Theorem 28. *For an integer $n \geq 9$, we have*

$$\mathrm{gr}_4(P_5 : K_{3,n}) = \mathrm{gr}_4(P_5 : K_{4,n}) = 2n - 3.$$

Proof. It follows from Lemma 13, Theorems 21 and 22 that $\mathrm{gr}_4(P_5 : K_{3,n}) \geq 2n-3$ and $\mathrm{gr}_4(P_5 : K_{4,n}) \geq 2n-3$. Consider any 4-edge-colored K_N ($N \geq 2n-3$) and suppose to the contrary that K_N contains neither a rainbow subgraph P_5

nor a monochromatic subgraph $K_{3,n}$ or $K_{4,n}$. It follows from the fact that $k = 4$ and Theorem 21 that Theorem 6 (i), (ii), and (vi) do not hold.

Suppose that Theorem 6 (iii) holds. Noticing that $2n-3-1 > n+4$ $(n \geq 9)$, $K_N - v$ is monochromatic for some vertex v, there is a monochromatic subgraph $K_{4,n}$, a contradiction. Suppose that Theorem 6 (iv) holds. Noticing that $2n-3 > n+5$ $(n \geq 9)$, $\{a, b, c, v_1, v_2, \ldots, v_{n+2}\} \subseteq V(K_N)$, there is a monochromatic subgraph $K_{4,n}$ with bipartition $\{v_1, v_2, b, c\}$ and $\{v_3, v_4, \ldots, v_{n+2}\}$ with color 1, a contradiction. Suppose that Theorem 6 (v) holds. Noticing that $2n - 3 > n+5$ $(n \geq 9)$, $\{a, b, c, d, v_1, v_2, \ldots, v_{n+1}\} \subseteq V(K_N)$, there is a monochromatic subgraph $K_{4,n}$ with bipartition $\{a, b, c, d\}$ and $\{v_1, v_2, \ldots, v_n\}$ with color 1, a contradiction. ■

Lemma 29. *For integers $n \geq m \geq 2$, we have*

$$\mathrm{gr}_4(P_4^+ : K_{m,n}) \geq m + n + 2.$$

Proof. Let $K_{m+n+1} = G_2(m+n+1)$. It follows from Theorem 8 (ii) that there is neither a rainbow subgraph P_4^+ nor a monochromatic subgraph $K_{m,n}$ under such a 4-edge-colored K_{m+n+1}, and so $\mathrm{gr}_4(P_4^+ : K_{m,n}) \geq m + n + 2$. ■

Theorem 30. *For an integer $n \geq 3$, we have*

$$\mathrm{gr}_4(P_4^+ : K_{2,n}) = \begin{cases} n+4, & 3 \leq n \leq 8; \\ n+a, & 2a+1 \leq n \leq 2(a+1) \text{ where } a \geq 4 \text{ is an integer.} \end{cases}$$

Proof. We distinguish the following two cases to proceed with our proof.

Case 1. $3 \leq n \leq 8$.

It follows from Lemma 29 that $\mathrm{gr}_4(P_4^+ : K_{2,n}) \geq n+4$. Consider any 4-edge-colored K_N $(N \geq n+4)$ and suppose to the contrary that K_N contains neither a rainbow subgraph P_4^+ nor a monochromatic subgraph $K_{2,n}$. It follows from the fact that $k = 4$ and Theorem 20 that Theorem 8 (i) and (iii) do not hold.

Next, suppose that Theorem 8 (ii) holds. If $K_N = G_2(N)$, then $K_N - x - y$ is monochromatic with color 1, and hence there is a monochromatic subgraph $K_{2,n}$, a contradiction. Suppose that $K_N = G_3(N)$. Noticing that $\{a, b, c, v_1, v_2, \ldots, v_{n+1}\} \subseteq V(K_N)$, there is a monochromatic $K_{2,n}$ with bipartition $\{a, b\}$ and $\{v_1, v_2, \ldots, v_n\}$ with color 4, a contradiction.

Case 2. $2a + 1 \leq n \leq 2(a+1)$ where $a \geq 4$ is an integer.

It follows from Lemma 16 and Theorem 20 that $\mathrm{gr}_4(P_4^+ : K_{2,n}) \geq n + a$. Consider any 4-edge-colored K_N $(N \geq n + a)$ and suppose to the contrary that K_N contains neither a rainbow subgraph P_4^+ nor a monochromatic subgraph

$K_{2,n}$. It follows from the fact that $k = 4$ and Theorem 20 that Theorem 8 (i) and (iii) do not hold.

Next, suppose that Theorem 8 (ii) holds. Assume that $K_N = G_2(N)$. Since $n+a \geq n+4$ ($n \geq 9$), it follows that $K_N - x - y$ is monochromatic with color 1, and hence there is a monochromatic subgraph $K_{2,n}$, a contradiction. Suppose that $K_N = G_3(N)$. Noticing that $n + a \geq n + 4$ ($n \geq 9$), $\{a, b, c, v_1, v_2, \ldots, v_{n+1}\} \subseteq V(K_N)$, there is a monochromatic subgraph $K_{2,n}$ with bipartition $\{a, b\}$ and $\{v_1, v_2, \ldots, v_n\}$ with color 4, a contradiction. ■

Theorem 31. *For an integer $n \geq 10$, we have*

$$\mathrm{gr}_4(P_4^+ : K_{3,n}) = \mathrm{gr}_4(P_4^+ : K_{4,n}) = 2n - 3.$$

Proof. It follows from Lemma 16, Theorems 21 and 22 that $\mathrm{gr}_4(P_4^+ : K_{3,n}) \geq 2n-3$ and $\mathrm{gr}_4(P_4^+ : K_{4,n}) \geq 2n-3$. Consider any 4-edge-colored K_N ($N \geq 2n-3$) and suppose to the contrary that K_N contains neither a rainbow subgraph P_4^+ nor a monochromatic subgraph $K_{3,n}$ or $K_{4,n}$. It follows from the fact that $k = 4$ and Theorem 21 that Theorem 8 (i) and (iii) do not hold.

Next, suppose that Theorem 8 (ii) holds. Assume that $K_N = G_2(N)$. Since $2n - 3 > n + 6$ ($n \geq 10$), it follows that $K_N - x - y$ is monochromatic with color 1, and hence there is a monochromatic subgraph $K_{4,n}$, a contradiction. Suppose that $K_N = G_3(N)$. Noticing that $2n - 3 > n + 6$ ($n \geq 10$) and $\{a, b, c, v_1, v_2, \ldots, v_{n+3}\} \subseteq V(K_N)$. Then there is a monochromatic subgraph $K_{4,n}$ with bipartition $\{a, b, c, v_1\}$ and $\{v_2, v_3, \ldots, v_{n+1}\}$ with color 4, a contradiction. ■

Remark 32. For integers $k \geq 5$, $1 \leq m \leq 4$ and $n \geq 3$, we can get $\mathrm{gr}_k(P_5 : K_{m,n})$ directly from Lemma 14, and we can get $\mathrm{gr}_k(P_4^+ : K_{m,n})$ directly from Lemma 15. For a small integer $n \leq 9$, the method for proving the exact value of Gallai-Ramsey number for rainbow P_5 or P_4^+ and monochromatic $K_{1,n}$, $K_{3,n}$ or $K_{4,n}$ is very trivial. So this paper will not give these results.

5. CONCLUSION

Gallai-Ramsey number involving rainbow $K_{1,3}$ plays a very significant role in Gallai-Ramsey number involving rainbow P_5 or P_4^+. That is, if one can determine the exact value of $\mathrm{gr}_k(K_{1,3} : H)$ for an integer $k \geq 4$ and a graph H, then one can easily determine the exact value of $\mathrm{gr}_k(P_5 : H)$ and $\mathrm{gr}_k(P_4^+ : H)$. However, we have not completely solved all the exact values of Gallai-Ramsey number for rainbow trees and monochromatic complete bipartite graphs. We end this section with two open problems.

Problem 33. For integers $n \geq m \geq 2$, determine the exact value of $\mathrm{gr}_3(K_{1,3} : K_{m,n})$.

Problem 34. For integers $n \geq m \geq 5$ and $k \geq 4$, determine the exact value of $\mathrm{gr}_k(K_{1,3} : K_{m,n})$.

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