EDGE-TRANSITIVE CUBIC GRAPHS OF TWICE SQUARE-FREE ORDER

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Abstract. A graph is edge-transitive if its automorphism group acts transitively on the edge set. This paper presents a complete classification for connected edge-transitive cubic graphs of order $2n$, where n is even and square-free. In particular, it is shown that such a graph is either symmetric or isomorphic to one of the following graphs: a semisymmetric graph of order 420, a semisymmetric graph of order 29260 and five families of semisymmetric graphs constructed from the simple group $\mathrm{PSL}_2(p)$.

Keywords. Edge-transitive graph, symmetric graph, semisymmetric graph, coset graph, bi-coset graph.

1. Introduction

All graphs in this paper are finite, simple and undirected, and have no isolated vertex.

Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E, and denote by $\mathrm{Aut}\Gamma$ the automorphism group of Γ. Let G be a subgroup of $\mathrm{Aut}\Gamma$, written as $G \leqslant \mathrm{Aut}\Gamma$. Then Γ is said to be G-vertex-transitive or G-edge-transitive if G acts transitively on V or E, respectively. If Γ is G-edge-transitive but not G-vertex-transitive then Γ is a bipartite graph with a bipartition given by the G-orbits on V; in this case, Γ is called G-semisymmetric if further it is a regular graph. Recall that an arc in Γ is an ordered pair of adjacent vertices. Then Γ is said to be G-symmetric if G acts transitively on the set of arcs. For a vertex $v \in V$, set $\Gamma(v) = \{v' \in V \mid \{v, v'\} \in E\}$ and $G_v = \{g \in G \mid v^g = v\}$, called the neighborhood and stabilizer of v in Γ and G, respectively. Clearly, if Γ is either G-symmetric or G-semisymmetric then G_v acts transitively on $\Gamma(v)$ for all $v \in V$.

A graph Γ is called vertex-transitive, edge-transitive, symmetric and semisymmetric if it is $\mathrm{Aut}\Gamma$-vertex-transitive, $\mathrm{Aut}\Gamma$-edge-transitive, $\mathrm{Aut}\Gamma$-symmetric and $\mathrm{Aut}\Gamma$-semisymmetric, respectively. Clearly, symmetric graphs are both edge-transitive and vertex-transitive, and by [31, p.55, 7.31], the converse is also true for regular graphs of odd valency. In particular, edge-transitive cubic graphs (regular graphs of valency 3) are either symmetric or semisymmetric.

In this paper, we focus on connected edge-transitive cubic graphs. Interest in edge-transitive cubic graphs stems from the classical result on symmetric cubic graphs due to Tutte. In [29, 30], Tutte considered the automorphism groups of connected symmetric cubic graphs, and proved that the order of a vertex-stabilizer is a divisor of $2^4 \cdot 3$. Tutte's result was generalized by Goldschmidt in [16] where it is proved that the stabilizers of two adjacent vertices in a connected edge-transitive cubic graph are isomorphic to one of

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fifteen pairs of groups; in particular, the order of a vertex-stabilizer is a divisor of $2^7 \cdot 3$. Following these two classical results, edge-transitive cubic graphs have been extensively studied from different perspectives over the decades, see [5, 6, 7, 8, 9, 12, 18, 24, 26, 27, 28] for example. In recent papers [21] and [23], connedcted edge-transitive cubic graphs of square-free order were classified. This motivates us to classify connected edge-transitive cubic graphs of order $2n$, where n is even and square-free.

Let Γ be an arbitrary connected edge-transitive cubic graph of order $2n$ with n even and square-free. The group-theoretic structure of Γ is investigated in Section 2, where it is proved that, with four exceptions for Γ, an edge-transitive group of Γ has a unique insolvable minimal normal subgroup say T, which is isomorphic to J_1 or $\mathrm{PSL}_2(p)$. In Section 3, we collect two group-theoretic constructions for edge-transitive graphs, and present some improvements on the automorphisms or isomorphisms of coset graphs and bi-coset graphs. Then Γ is determined in Section 4 for the case where $T = \mathrm{J}_1$, followed by the classifications for $\mathrm{PSL}_2(p)$-symmetric Γ and $\mathrm{PSL}_2(p)$-semisymmetric Γ in Sections 5 and 6, respectively. Finally, the case where Γ is not $\mathrm{PSL}_2(p)$-edge-transitive is settled in Section 7, and then our main result stated as follows is proved.

Theorem 1.1. *Assume that $\Gamma = (V, E)$ is a connected edge-transitive cubic graph of order $2n$, where n is even and square-free. Let p be the largest prime divisor of n, and choose $\varepsilon, \eta \in \{1, -1\}$ for those odd p with $p+\varepsilon$ and $p+\eta$ divisible by 3 and 4, respectively. Let $\delta = 1$ if $p \equiv \pm 1 \pmod{10}$, or $\delta = 0$ otherwise.*

1. *If Γ is not bipartite then Γ is isomorphic to either the complete graph K_4 of order 4 or one of the graphs described as Table 1, where $v \in V$, $T = \mathrm{PSL}_2(p)$ and ω is the number of non-isomorphic graphs with isomorphic automorphism groups.*
2. *If Γ is bipartite then Γ is isomorphic to one of the graphs described as Table 2, where $\{u, w\} \in E$, $T = \mathrm{PSL}_2(p)$ and ν is the number of non-isomorphic graphs with isomorphic automorphism groups.*

2. On the automorphism groups

In this and the following sections, G is a finite group. Denote by $\mathrm{Aut}(G)$ the automorphism group of G. If α is a subset or an element of G, then we write $g^{-1}\alpha g$ to denote the conjugation of α under some $g \in G$. For subsets $X, Y \subseteq G$, we write $\mathbf{C}_X(Y) = \{x \in X \mid x^{-1}yx = y \text{ for all } y \in Y\}$ and $\mathbf{N}_X(Y) = \{x \in X \mid x^{-1}Yx = Y\}$, called the centralizer and normalizer of Y in X, respectively.

In the following, $\Gamma = (V, E)$ is assumed to be a connected G-edge-transitive cubic graph. Note that Γ is either G-symmetric or G-semisymmetric. Let $\{u, w\} \in E$. If Γ is G-semisymmetric then Γ is bipartite, and $G = \langle G_u, G_v \rangle$. Suppose that Γ is G-symmetric. Then Γ is $\langle G_u, G_v \rangle$-edge-transitive, and $|G : \langle G_u, G_v \rangle| \leqslant 2$, where the equality holds if and only if Γ is bipartite, refer to [32, Exercise 3.8]. Clearly, if $|G : \langle G_u, G_v \rangle| = 2$ then Γ is $\langle G_u, G_v \rangle$-semisymmetric. Thus, replacing G by $\langle G_u, G_v \rangle$ if necessary, we assume further that

(C1) Γ is either G-semisymmetric, or non-bipartite and G-symmetric, where $G \leqslant \mathrm{Aut}\Gamma$; and

(C2) $|V| = 2n$, where n is even and square-free.

	$G = \mathrm{Aut}\Gamma$	G_v	ω	Comments
1	A_6	S_3	1	F60, cf. [6]
2	$\mathrm{PSL}_2(8)$	S_3	1	F84, cf. [6]
3	J_1	S_3	10	Example 3.5
4	$\mathrm{PGL}_2(p)$	S_3	$\frac{p-\eta-6}{4}$	Theorem 5.7 $p\equiv\pm3\,(\mathrm{mod}\ 8)$
5	$\mathrm{PSL}_2(p)\times\mathbb{Z}_2$	S_3	$\frac{p+\eta-2\|\varepsilon+\eta\|}{4}-2\delta$	Theorem 5.7 $p\equiv\pm3\,(\mathrm{mod}\ 8)$
6	$\mathrm{PGL}_2(p)$	D_{12}	$1-\frac{\|\varepsilon+\eta\|}{2}$	Theorem 5.12 $p\equiv\pm7\,(\mathrm{mod}\ 16)$
7	$\mathrm{PSL}_2(p)\times\mathbb{Z}_2$	D_{12}	$\|\varepsilon+\eta\|$	Theorem 5.12 $p\equiv\pm7\,(\mathrm{mod}\ 16)$
8	$\mathrm{PSL}_2(p)$	S_3	$\frac{p+\eta-4\|\varepsilon+\eta\|}{8}-1-\delta$	Theorem 5.12 $p\equiv\pm7\,(\mathrm{mod}\ 16)$
9	$\mathrm{PSL}_2(p)$	D_{12}	1	Theorem 5.13 $p\equiv\pm47\,(\mathrm{mod}\ 96)$
10	$\mathrm{PSL}_2(p)$	S_4	1	Theorem 5.14 $p\equiv\pm31\,(\mathrm{mod}\ 64)$
11	$(\mathrm{PSL}_2(p)\times\mathbb{Z}_3){:}\mathbb{Z}_2$	S_3	$\frac{p-\eta-6}{4}$	$T{=}\mathrm{PSL}_2(p), T_v{=}1$ $p\equiv\pm3\,(\mathrm{mod}\ 8)$
12	$\mathrm{PSL}_2(p)\times\mathrm{S}_3$	S_3	$\frac{p+\eta-2\|\varepsilon+\eta\|}{4}-2\delta$	$T{=}\mathrm{PSL}_2(p), T_v{=}1$ $p\equiv\pm3\,(\mathrm{mod}\ 8)$

TABLE 1. Non-bipartite symmetric cubic graphs.

2.1. **Preliminaries.** Let $\{u, w\} \in E$. If Γ is G-symmetric then G_u and G_w are conjugate in G and, by [2, p.147, 18f], $G_u \cong \mathbb{Z}_3$, S_3, D_{12}, S_4 or $\mathbb{Z}_2\times\mathrm{S}_4$; in particular, $|G_u|$ is a divisor of 48. Suppose that Γ is G-semisymmetric. Then G has exactly two orbits on V, $G = \langle G_u, G_w\rangle$, and G_{uw} is a Sylow 2-subgroup of G_u (and G_w). The triple (G_u, G_{uw}, G_w) was determined by Goldschmidt in [16] where it is shown that (G_u, G_{uw}, G_w) is isomorphic to one of fifteen triples, see also [28, Table 3]. Then we have the following lemma.

Lemma 2.1. *Let $\{u, w\} \in E$. Then one of the following holds:*

(1) $G_u \cong G_w \cong \mathbb{Z}_3, \mathrm{S}_3, \mathrm{D}_{12}, \mathrm{S}_4$ *or* $\mathbb{Z}_2 \times \mathrm{S}_4$*;*
(2) Γ *is G-semisymmetric, $G_u \not\cong G_w$, and either $|G_u| = |G_w| = 2^i \cdot 3$ with $i \in \{5, 6, 7\}$ or (G_u, G_w) is isomorphic to one of* $(\mathrm{S}_3, \mathbb{Z}_6)$, $(\mathrm{D}_{12}, \mathrm{A}_4)$, $(\mathrm{D}_{24}, \mathrm{S}_4)$, $((\mathbb{Z}_2^2 \times \mathbb{Z}_3).\mathbb{Z}_2, \mathrm{S}_4)$, $(\mathbb{Z}_2 \times \mathrm{D}_{12}, \mathbb{Z}_2 \times \mathrm{A}_4)$ *and* $(\mathrm{D}_8 \times \mathrm{S}_3, \mathbb{Z}_2 \times \mathrm{S}_4)$.

In particular,

(i) *if $|G_u| > 3$ then G contains at least two involutions; if $|G_u| > 12$ then either $(G_u, G_w) \cong (\mathbb{Z}_2 \times \mathrm{D}_{12}, \mathbb{Z}_2 \times \mathrm{A}_4)$, or G contains nonabelian Sylow 2-subgroups;*
(ii) *if Γ is G-symmetric then $|G|$ is a divisor of $2^5 \cdot 3n$; if Γ is G-semisymmetric then $|G|$ is a divisor of $2^8 \cdot 3n$.*

Let N be a normal subgroup of G, written as $N \trianglelefteq G$. Suppose that N is intransitive on V. For $v \in V$, denote by $\bar{v}$ the N-orbit containing v. Put $\bar{V} = \{\bar{v} \mid v \in V\}$. The normal quotient graph Γ_N of Γ relative to G and N is defined on $\bar{V}$ with edge set

	$G=\mathrm{Aut}\Gamma$	G_u, G_w	ν	Symmetric?	Comments
1	$\mathrm{S}_7\times\mathbb{Z}_2$	$\mathrm{S}_4\times\mathbb{Z}_2$, $\mathrm{S}_3\times\mathrm{D}_8$	1	No	S420, cf. [8]
2	J_1	D_{12}, D_{12}	1	No	Example 3.11
3	$\mathrm{PSL}_2(p)\times\mathbb{Z}_2$	D_{12}, D_{12}	$\frac{\lvert\varepsilon+\eta\rvert}{2}$	No	Theorem 6.5 $p\equiv\pm3\pmod 8$
4	$\mathrm{PGL}_2(p)\times\mathbb{Z}_2$	D_{12}, D_{12}	1	Yes	Theorem 6.5 $p\equiv\pm3\pmod 8$
5	$\mathrm{PGL}_2(p)$	S_3, S_3	$\frac{p+\eta-4}{8}$	Yes	Theorem 6.5 $p\equiv\pm3\pmod 8$
6	$\mathrm{PSL}_2(p)\times\mathbb{Z}_2$	S_3, S_3	$\frac{p+\eta-4}{8}-\delta$	Yes	Theorem 6.5 $p\equiv\pm3\pmod 8$
7	$\mathrm{PSL}_2(p)\times\mathbb{Z}_2$	D_{24}, S_4	1	No	Theorem 6.7 $p\equiv\pm23\pmod{48}$
8	$\mathrm{PSL}_2(p)\times\mathbb{Z}_2$	D_{12}, D_{12}	1	Yes	Theorem 6.7 $p\equiv\pm23\pmod{48}$
9	$\mathrm{PSL}_2(p)$	D_{24}, S_4	1	No	Theorem 6.8 $p\equiv\pm47\pmod{96}$
10	$\mathrm{PSL}_2(p)\times\mathbb{Z}_2$	S_4, S_4	1	Yes	Theorem 6.8 $p\equiv\pm15\pmod{32}$
11	$\mathrm{PSL}_2(p)\times\mathrm{S}_3$	D_{12}, D_{12}	1	No	$T=\mathrm{PSL}_2(p), T_u\cong\mathrm{S}_3, T_w\cong\mathbb{Z}_2$ $p\equiv\pm11\pmod{24}$
12	$\mathrm{PSL}_2(p)\times\mathrm{S}_3$	D_{24}, S_4	1	No	$T=\mathrm{PSL}_2(p), T_u\cong\mathrm{D}_{12}, T_w\cong\mathbb{Z}_2^2$ $p\equiv\pm23\pmod{48}$

TABLE 2. Bipartite edge-transitive cubic graphs.

$\bar{E}:=\{\{\bar{u},\bar{w}\}\mid\{u,w\}\in E\}$. Denote by $G^{\bar{V}}$ (by $\bar{G}$ for short) the permutation group induced by G on $\bar{V}$. Recall that N is said to be semiregular (on V) if all its orbits have length $|N|$, i.e., $N_v=1$ for all $v\in V$. We have the following lemma, see [22, Lemma 2.6] for example.

Lemma 2.2. *Let $N\trianglelefteq G$. Assume that N is intransitive on each G-orbit on V. Then Γ_N is cubic and $\bar{G}$-edge-transitive, N is semiregular on V, and $\bar{G}\cong G/N$.*

Lemma 2.3. *Let $N\trianglelefteq G$. Assume that N is not semiregular on V. Then either Γ is N-edge-transitive, or Γ is bipartite and the following hold:*

(1) *N acts transitively on one part say U of Γ and has three orbits on the other part;*
(2) *$|G:N|$ is divisible by 3, $|N|$ is indivisible by 9 and, for $u\in U$, the stabilizer N_u is a 2-group and acts trivially on $\Gamma(u)$.*

Proof. Assume first that N is transitive on each G-orbit on V. Then $|N:N_u|=|N:N_w|=2n$ or n, in particular, $|N_u|=|N_w|$, where $u,w\in V$. Suppose that N_u acts trivially on $\Gamma(u)$. Then, letting $w\in\Gamma(u)$, we have $N_u=N_w$. Since $N_w\trianglelefteq G_w$ and G_w acts transitively on $\Gamma(w)$, we deduce that N_w acts trivially on $\Gamma(w)$. It follows from the connectedness of Γ that N_u fixes V point-wise, and so $N_u=1$, a contradiction. Thus N_u acts transitively on $\Gamma(u)$ for all $u\in V$, and hence Γ is N-edge-transitive.

Assume now that Γ is bipartite, and N is not transitive on one part of Γ, say W. Since N is not semiregular, by Lemma 2.2, N is transitive on $U := V \setminus W$. By [15, Lemma 5.5], N has three orbits on W and, for $u \in U$, the stabilizer N_u is contained in the kernel of G_u acting on $\Gamma(u)$. It follows that N_u is a 2-group, and $|G_u : N_u|$ is divisible by 3. Noting that $|G : G_u| = n = |N : N_u|$, we have that $|G : N| = |G_u : N_u|$, and $|N|$ is indivisible by 9. Then the lemma follows. □

2.2. **The solvable case.** For a prime divisor p, denote by $\mathbf{O}_p(G)$ the maximal normal p-subgroup of G.

Lemma 2.4. *Either* $\Gamma \cong \mathsf{K}_4$*, or* $|\mathbf{O}_p(G)| \in \{1, p\}$ *for every prime divisor* p *of* $|G|$*.*

Proof. Assume first that p is an odd prime. Since each G-orbit on V has even length n or $2n$, we know that $\mathbf{O}_p(G)$ is intransitive on each G-orbit on V. By Lemma 2.2, $\mathbf{O}_p(G)$ has order a divisor of $2n$, yielding $|\mathbf{O}_p(G)| \in \{1, p\}$.

Now consider the case where $p = 2$. Assume that $\mathbf{O}_2(G)$ is not transitive on each G-orbit. By Lemma 2.2, $\mathbf{O}_2(G)$ is semiregular on V, and so $|\mathbf{O}_2(G)| \in \{1, 2, 4\}$. If $|\mathbf{O}_2(G)| = 4$ then we get a cubic graph $\Gamma_{\mathbf{O}_2(G)}$ of odd order, which is impossible. Thus $|\mathbf{O}_2(G)| \in \{1, 2\}$. Assume that $\mathbf{O}_2(G)$ is transitive on one of G-orbits, say U. Then $|U|$ is a divisor of $|\mathbf{O}_2(G)|$, which forces that either $|U| = n = 2$ or $|V| = |U| = 4$. It follows that $\Gamma \cong \mathsf{K}_4$. This completes the proof. □

Theorem 2.5. *Assume that* G *is solvable. Then* $\Gamma \cong \mathsf{K}_4$*.*

Proof. Let F be the Fitting subgroup of G, i.e., the direct product of all $\mathbf{O}_p(G)$, where p runs over the prime divisors of $|G|$. Since G is solvable, every minimal normal subgroup of G has prime power order, and so $F \neq 1$.

Suppose that $\Gamma \ncong \mathsf{K}_4$. Then $2n = |V| > 4$ and, by Lemma 2.4, F is cyclic and has order a divisor of n. In particular, F is intransitive on V as $|V| = 2n$. Let B be an arbitrary F-orbit on V, and let K be the kernel of F acting on B. Since F is cyclic, K is characteristic in G, and so $K \trianglelefteq G$. If G is transitive on V then, since all K-orbits have equal length, K acts trivially on V, and so $K = 1$. Assume that G is intransitive on V. Then G has exactly two orbits on V, say U and W. Without loss of generality, let $B \subseteq U$. Then K acts trivially on U. If $K \neq 1$ then it is easily shown that Γ is isomorphic to the complete bipartite graph $\mathsf{K}_{3,3}$, and so $2n = 6$, which is not the case. Therefore, F is faithful and hence regular on each of its orbits; in particular, F is semiregular on V.

Assume that F has two orbits on V. Then Γ is bipartite and $|F| = n$. Let L be the $2'$-Hall subgroup of F. Then L is a normal subgroup of G. Clearly, L is intransitive on both the F-orbits. By Lemma 2.2, the quotient graph Γ_L has valency 3. However, Γ_L is a bipartite graph of order 4, a contradiction.

Assume that F has at least three orbits on V. In this case, it is easy to see that F is intransitive on each G-orbit on V. Then, by Lemma 2.2, the quotient graph Γ_F is cubic, and G induces an edge-transitive subgroup of $\mathrm{Aut}\Gamma_F$, which is isomorphic to G/F. Since G is solvable, $\mathbf{C}_G(F) \leq F$, refer to [1, p.158, (31.10)]. Thus $\mathbf{C}_G(F) = F$. Noting that G induces a subgroup $\mathrm{Aut}(F)$ by conjugation, we have $G/F = \mathbf{N}_G(F)/\mathbf{C}_G(F) \lesssim \mathrm{Aut}(F)$. Since F is cyclic, $\mathrm{Aut}(F)$ is abelian, and so does G/F. It follows that $\mathrm{Aut}\Gamma_F$ has an abelian edge-transitive subgroup. Then the only possibility is that $\Gamma_F \cong \mathsf{K}_{3,3}$ and $G/F \cong \mathbb{Z}_3^2$. In particular, $n = 3|F|$, and Γ is bipartite. Let L be the $2'$-Hall subgroup of F. Then L is normal in G and intransitive on each of F-orbits. By Lemma 2.2, G induces an edge-transitive subgroup of $\mathrm{Aut}\Gamma_L$, which is isomorphic to G/L. Noting that F/L is a normal subgroup of G/L of order 2, we have $G/L \cong \mathbb{Z}_2 \times \mathbb{Z}_3^2$. It follows that $\mathrm{Aut}\Gamma_L$ has an abelian edge-transitive subgroup, and thus $\Gamma_L \cong \mathsf{K}_{3,3}$, which is impossible as Γ_L has order divisible by 4. Therefore, $\Gamma \cong \mathsf{K}_4$, and the result follows. □

2.3. **The insolvable case.** In this subsection, the group G is assumed to be insolvable. Denote by $\mathsf{rad}(G)$ the maximal solvable normal subgroup of G. Then $\mathsf{rad}(G)$ is a characteristic subgroup G. If $\mathsf{rad}(G)$ is transitive on one of G-orbits on V, then $G = \mathsf{rad}(G)G_v$ for some $v \in V$, which implies that G is solvable, a contradiction. Then Lemma 2.2 is available for the triple $(\Gamma, G, \mathsf{rad}(G))$. For $v \in V$, denote by $\bar{v}$ the $\mathsf{rad}(G)$-orbit containing v. Put $\bar{V} = \{\bar{v} \mid v \in V\}$, and $\bar{G} = G^{\bar{V}}$. We have the following lemma.

Lemma 2.6. *Assume that G is insolvable. Then $\Gamma_{\mathsf{rad}(G)}$ is a connected $\bar{G}$-edge-transitive cubic graph, $|\mathsf{rad}(G)|$ is a divisor of n, $|\bar{V}| = \frac{2n}{|\mathsf{rad}(G)|}$ and $\bar{G} \cong G/\mathsf{rad}(G)$.*

Lemma 2.7. *Assume that G is insolvable. Then $\bar{G}$ has a unique minimal normal subgroup say $\bar{N}$, $\Gamma_{\mathsf{rad}(G)}$ is $\bar{N}$-edge-transitive, and $\bar{N}$ is isomorphic to one of the following simple groups:* A_6, A_7, J_1, $\mathrm{PSL}_2(8)$ *and* $\mathrm{PSL}_2(p)$, *where $p \geqslant 5$ is a prime.*

Proof. Let $\bar{N}$ be a minimal normal subgroup of $\bar{G}$. Then $\bar{N}$ is insolvable, and $|\bar{N}|$ is a divisor of $2^8 \cdot 3n$. Note that $\bar{N}$ is a direct product of isomorphic nonabelian simple groups. If $\bar{N}$ is not simple then $|\bar{N}|$ has a divisor r^2 for some prime $r > 3$, and so n is divisible by r^2, which contradicts the assumption that n is square-free. Thus $\bar{N}$ is simple. If $|\mathsf{rad}(G)|$ is even then, noting that $\Gamma_{\mathsf{rad}(G)}$ has square-free order $|\bar{V}|$, our lemma follows from [21, Lemma 6.3] and [23, Lemma 4.3]; in this case, $\bar{N} \cong \mathrm{A}_6$, A_7 or $\mathrm{PSL}_2(p)$. Thus, we assume next that $|\mathsf{rad}(G)|$ is an odd divisor of n.

If $\bar{N}$ is intransitive on each $\bar{G}$-orbit on $\bar{V}$ then, by Lemma 2.2, the quotient graph of $\Gamma_{\mathsf{rad}(G)}$ with respect to $\bar{N}$ is cubic and of order $|\bar{V}|/|\bar{N}|$; however, $|\bar{N}|$ is divisible by 4, and so $|\bar{V}|/|\bar{N}|$ is odd, a contradiction. Thus $\bar{N}$ is transitive on at least one of $\bar{G}$-orbits, say $\bar{U}$. Then $\bar{G} = \bar{N}\bar{G}_{\bar{u}}$ for some $\bar{u} \in \bar{U}$. Let $C = \mathbf{C}_{\bar{G}}(\bar{N})$. We have $\bar{N} \cap C = 1$, and so $C \cong \bar{N}C/\bar{N} \leqslant \bar{G}/\bar{N} \cong \bar{G}_{\bar{u}}/\bar{N}_{\bar{u}}$. It follows that C is solvable, and so $C = 1$ as $\mathsf{rad}(\bar{G}) = 1$ and $C \trianglelefteq \bar{G}$. This says that $\bar{N}$ is the unique minimal normal subgroup of $\bar{G}$.

Note that $|\bar{N}|$ is not divisible by 2^{10}, 3^3 or r^2, where r is an arbitrary prime with $r \geqslant 5$. Inspecting the orders of finite simple groups (refer to [19, Tables 5.1.A-C]), we deduce that $\bar{N}$ is isomorphic to one of the following groups: A_6, A_7, A_8, M_{11}, M_{22}, M_{23}, J_1, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_2(2^f)$ and $\mathrm{PSL}_2(p)$, where $3 \leqslant f \leqslant 8$, and $p \geqslant 5$ is a prime.

Suppose that $\bar{N}$ is isomorphic to one of A_6, A_7, $\mathrm{PSL}_2(8)$, A_8, M_{11}, M_{22}, M_{23}, $\mathrm{PSL}_3(4)$ and $\mathrm{PSL}_2(2^6)$. Then $|\bar{N}|$ is divisible by 9. It follows from Lemma 2.3 that $\Gamma_{\mathsf{rad}(G)}$ is $\bar{N}$-edge-transitive. If $\bar{N} \cong \mathrm{PSL}_2(2^6)$ then $|\bar{N}_{\bar{v}}|$ is divisible by $2^4 \cdot 3$, by Lemma 2.1 (i), $\bar{N}$ has nonabelian Sylow 2-subgroups, which is impossible. Assume that $\bar{N} \cong \mathrm{M}_{22}$ or M_{23}. Then $|\bar{N}_{\bar{u}}|$ is divisible by $2^5 \cdot 3$. By Lemma 2.1, $\Gamma_{\mathsf{rad}(G)}$ is $\bar{N}$-semisymmetric, and then $|\bar{N}_{\bar{u}}| = 2^6 \cdot 3$. Since $\Gamma_{\mathsf{rad}(G)}$ is connected, $\bar{N} = \langle L, R \rangle$, where R and L are the stabilizers of two adjacent vertices. For such a pair (L, R), noting that $|L| = |R| = 2^6 \cdot 3$ and $|L \cap R| = 64$, computation with GAP [14] shows that either $|\langle L, R \rangle| = 1344$, or $\bar{N} \cong \mathrm{M}_{23}$ and $|\langle L, R \rangle| \in \{576, 1920, 40320\}$, and so $\bar{N} \neq \langle L, R \rangle$, a contradiction. Assume that $\bar{N} \cong \mathrm{PSL}_3(4)$, A_8 or M_{11}. Then $|\bar{V}| = 2\frac{n}{|\mathsf{rad}(G)|} = 420$, 420 or 660, respectively.

By [6, 8], up to graph isomorphisms, there exist one connected edge-transitive cubic graph of order 420, and two connected edge-transitive cubic graphs of order 660, which have automorphism groups of order 10080, 3960 and 3960 respectively. Then $|\bar{N}| > |\mathrm{Aut}\Gamma_{\mathsf{rad}(G)}|$, a contradiction. Thus, in this case, $\Gamma_{\mathsf{rad}(G)}$ is $\bar{N}$-edge-transitive, and $\bar{N}$ is one of A_6, A_7 and $\mathrm{PSL}_2(8)$.

Finally, suppose that $\bar{N} \cong \mathrm{J}_1$, $\mathrm{PSL}_2(2^4)$, $\mathrm{PSL}_2(2^5)$, $\mathrm{PSL}_2(2^7)$, $\mathrm{PSL}_2(2^8)$ or $\mathrm{PSL}_2(p)$. Recalling that $\mathbf{C}_{\bar{G}}(\bar{N}) = 1$, we know that $\bar{G}$ is almost simple, and $\bar{G} = \bar{N}.O$, where O is a subgroup of the outer automorphism group of $\bar{N}$. Checking [19, Tables 5.1.A and 5.1.C], we conclude that $|O|$ is a divisor of 1, 4, 5, 7, 8 or 2, respectively. Then $|\bar{G} : \bar{N}| = |O|$ is indivisible by 3. Noting that $|\bar{G}_{\bar{v}} : \bar{N}_{\bar{v}}| = |\bar{N}G_{\bar{v}} : \bar{N}|$, it follows that $|\bar{N}_{\bar{v}}|$ is divisible by 3 for all $\bar{v} \in \bar{V}$. By Lemma 2.3, $\Gamma_{\mathsf{rad}(G)}$ is $\bar{N}$-edge-transitive. If $\bar{N} \cong \mathrm{PSL}_2(2^4)$ then $|\bar{V}| = 340$; however, by [6, 8], there exists no connected edge-transitive cubic graph of order 340. Suppose that $\bar{N} \cong \mathrm{PSL}_2(2^f)$, where $f \in \{5, 7, 8\}$. Then $f - 2 \geqslant 3$, and $|\bar{N}_{\bar{v}}|$ is divisible by $2^{f-2} \cdot 3$. Noting that $\mathrm{PSL}_2(2^f)$ has abelian Sylow 2-subgroups, by Lemma 2.1 (i), we conclude that $f = 5$, $\bar{N}_{\bar{v}} \cong \mathbb{Z}_2 \times \mathrm{D}_{12}$ or $\mathbb{Z}_2 \times \mathrm{A}_4$. This contradicts that $\mathrm{PSL}_2(2^5)$ has no subgroup isomorphic to $\mathbb{Z}_2 \times \mathrm{D}_{12}$ or $\mathbb{Z}_2 \times \mathrm{A}_4$, see Lemma 5.1. Therefore, $\Gamma_{\mathsf{rad}(G)}$ is $\bar{N}$-edge-transitive, and $\bar{N} \cong \mathrm{J}_1$ or $\mathrm{PSL}_2(p)$. This completes the proof. □

Denote by $G^{(\infty)}$ the intersection of all terms appearing in the derived series of G.

Lemma 2.8. *Assume that G is insolvable. Let $T = G^{(\infty)}$. Then $T \cong \mathrm{A}_6$, A_7, J_1, $\mathrm{PSL}_2(8)$ or $\mathrm{PSL}_2(p)$, $\mathsf{rad}(G) = \mathbf{C}_G(T)$ and Γ is $\mathsf{rad}(G)T$-edge-transitive.*

Proof. By Lemma 2.7, $\bar{G}$ has a unique minimal normal subgroup $\bar{N} \cong \mathrm{A}_6$, A_7, J_1, $\mathrm{PSL}_2(8)$ or $\mathrm{PSL}_2(p)$, and $\Gamma_{\mathsf{rad}(G)}$ is $\bar{N}$-edge-transitive. By the edge-transitivity of $\bar{N}$, we conclude that $\bar{N}$ is transitive on each of $\bar{G}$-orbits on $\bar{V}$. Then $\bar{G} = \bar{N}\bar{G}_{\bar{v}}$ for $\bar{v} \in \bar{V}$. Since $\bar{G}_{\bar{v}}$ is solvable, we have $\bar{N} = \bar{G}^{(\infty)}$. Noting that $\mathsf{rad}(G)T/\mathsf{rad}(G) = (G/\mathsf{rad}(G))^{(\infty)} \cong \bar{G}^{(\infty)} = \bar{N}$, it follows that $\mathsf{rad}(G)T$ is the primage of $\bar{N}$ in G. Then, considering $\Gamma_{\mathsf{rad}(G)}$ as a normal quotient of Γ with respect $\mathsf{rad}(G)T$ and $\mathsf{rad}(G)$, it is easily shown that Γ is $\mathsf{rad}(G)T$-edge-transitive.

Note that $T/(\mathsf{rad}(G) \cap T) \cong \mathsf{rad}(G)T/\mathsf{rad}(G) \cong \bar{N}$. Suppose that $\mathsf{rad}(G) \cap T = 1$. Then $T \cong \bar{N} \cong \mathrm{A}_6$, A_7, J_1, $\mathrm{PSL}_2(8)$ or $\mathrm{PSL}_2(p)$. In addition, $\mathsf{rad}(G) \leqslant \mathbf{C}_G(T)$. Since $(\mathbf{C}_G(T))^{(\infty)} \leqslant G^{(\infty)} = T$ and $\mathbf{C}_G(T) \cap T = 1$, we have $(\mathbf{C}_G(T))^{(\infty)} = 1$, and so $\mathbf{C}_G(T)$ is a solvable normal subgroup of G. It follows that $\mathsf{rad}(G) = \mathbf{C}_G(T)$. Thus, to complete the proof, it suffices to show that $\mathsf{rad}(G) \cap T = 1$.

Clearly, $|\mathsf{rad}(G) \cap T|$ is square-free, and so $\mathrm{Aut}(\mathsf{rad}(G) \cap T)$ is solvable. Note that T induces a subgroup of $\mathrm{Aut}(\mathsf{rad}(G) \cap T)$ by conjugation with kernel equal to $\mathbf{C}_T(\mathsf{rad}(G) \cap T)$. Since T is simple, $\mathbf{C}_T(\mathsf{rad}(G) \cap T) = 1$ or T. If $\mathbf{C}_T(\mathsf{rad}(G) \cap T) = 1$ then $\mathrm{Aut}(\mathsf{rad}(G) \cap T)$ has a subgroup isomorphic to T, and so $\mathrm{Aut}(\mathsf{rad}(G) \cap T)$ is insolvable, a contradiction. We have $T = \mathbf{C}_T(\mathsf{rad}(G) \cap T)$, and thus T is a covering group of the simple group $\bar{N}$ with center $\mathsf{rad}(G) \cap T$. Then $\mathsf{rad}(G) \cap T$ is a homomorphic image of the Schur multiplier of $\bar{N}$, refer to [1, p.168, (33.8)]. If $\bar{N} \cong \mathrm{PSL}_2(8)$ or J_1 then $\bar{N}$ has Schur multiplier 1 (see [19, p. 173, Theorem 5.14]), and so $\mathsf{rad}(G) \cap T = 1$.

Next we suppose that $\mathsf{rad}(G) \cap T \neq 1$, and produce a contradiction. By the above argument, we have that $\bar{N} \cong \mathrm{A}_6$, A_7 or $\mathrm{PSL}_2(p)$, and $\bar{N}$ has Schur multiplier $\mathbb{Z}_6$, $\mathbb{Z}_6$ or $\mathbb{Z}_2$ respectively, refer to [19, p.173, Theorem 5.14]. For $\bar{N} \cong \mathrm{A}_6$ or A_7, recalling that $|G|$ is indivisible by 3^3, we have $\mathsf{rad}(G) \cap T \cong \mathbb{Z}_2$; in this case, computation with GAP shows that T contains a unique involution. If $\bar{N} \cong \mathrm{PSL}_2(p)$ then $\mathsf{rad}(G) \cap T \cong \mathbb{Z}_2$ and $T \cong \mathrm{SL}_2(p)$; in this case, T also contains a unique involution.

Let $N = \mathsf{rad}(G)T$, the primage of $\bar{N}$ in G. Recall that Γ is N-edge-transitive. Since $|\mathsf{rad}(G)|$ is square-free, $\mathsf{rad}(G)$ has a unique Hall $2'$-subgroup say L. Then $L \trianglelefteq N$, and L is not transitive on each of N-orbits on V. Then, by Lemma 2.2, Γ_L is a cubic graph, and N induces an edge-transitive subgroup say X of $\mathrm{Aut}\Gamma_L$ with kernel equal to L. By the choice of L, we have $\mathsf{rad}(G) = L \times (\mathsf{rad}(G) \cap T)$, and so $X \cong N/L = TL/L \cong T$. In particular, $|X|$ is divisible by 8, and so X_α has order divisible by 6, where α is an L-orbit. By Lemma 2.1 (i), X contains at least two involutions, and hence so does T, a contradiction. Therefore, $\mathsf{rad}(G) \cap T = 1$. This completes the proof. □

Assume that G is insolvable. Let $M = \mathsf{rad}(G)$ and $T = G^{(\infty)}$. For $v \in V$, denote by $\bar{v}$ the M-orbit containing v. Put $\bar{V} = \{\bar{v} \mid v \in V\}$, and $\bar{T} = T^{\bar{V}}$. Then $MT = M \times T$ and $\bar{T} \cong MT/M \cong T$. Considering the set-wise stabilizers $T_{\bar{v}}$ and $(MT)_{\bar{v}}$ of $\bar{v}$ in T and MT respectively, we have $M(MT)_v = (MT)_{\bar{v}} = MT_{\bar{v}}$, and so

$$T_{\bar{v}} \cong (MT)_v \cong (MT)_{\bar{v}}/M \cong \bar{T}_{\bar{v}}. \tag{2.1}$$

Choose a G-orbit on V, say W, such that T is transitive on W. For $w \in W$, it is easily shown that $T_{\bar{w}}$ is transitive on $\bar{w}$. Noting that M is regular on $\bar{w}$ and centralizes $T_{\bar{w}}$, it follows from [11, p.109, Theorem 4.2A] that

$$T_w \trianglelefteq T_{\bar{w}},\ M \cong T_{\bar{w}}/T_w. \tag{2.2}$$

In particular, since $|M|$ is square-free and $|T_{\bar{w}}| = 2^s \cdot 3$ for some integer s, we have

$$|M| \in \{1, 2, 3, 6\}. \tag{2.3}$$

Lemma 2.9. *Assume that G is insolvable. Let $M = \mathsf{rad}(G)$ and $T = G^{(\infty)}$. Then Γ is MT-edge-transitive, and either Γ is T-edge-transitive, or $|M| \in \{3, 6\}$ and one of the following holds:*

(1) *Γ is bipartite, $T \in \{\mathrm{J}_1, \mathrm{PSL}_2(p)\}$, and T is transitive on one part of Γ and has three orbits on the other part;*
(2) *$T = \mathrm{PSL}_2(p)$ is regular on V, and $p \equiv \pm 3 \pmod 8$.*

Proof. By Lemma 2.8, Γ is MT-edge-transitive. Note that $|MT : T| = |M|$. If T is not semiregular on V then, applying Lemmas 2.3 and 2.8 to the triple (Γ, MT, T), either Γ is T-edge-transitive, or $|M| \in \{3, 6\}$ and (1) occurs.

Assume that T is semiregular on V. Then T has an odd number of orbits on V. Since there exists no cubic graph of odd order, by Lemma 2.2, we conclude that T is transitive on V, and so T is regular on V. In particular, $|T|$ is not divisible by 8 or 9, and so $T = \mathrm{PSL}_2(p)$ with $p \equiv \pm 3 \pmod 8$, desired as in (2). □

Theorem 2.10. *Let $A = \mathrm{Aut}\Gamma$, and $T = G^{(\infty)}$. Assume that G is insolvable. Then*

(1) *either $T \in \{\mathrm{J}_1, \mathrm{PSL}_2(p)\}$ or one of the following holds:*
   (i) *$\Gamma \cong \mathsf{F}60$ and $\mathrm{Aut}\Gamma = \mathrm{A}_6$;*
   (ii) *$\Gamma \cong \mathsf{S}420$ and $\mathrm{Aut}\Gamma = \mathbb{Z}_2 \times \mathrm{S}_7$;*
   (iii) *$\Gamma \cong \mathsf{F}84$ and $\mathrm{Aut}\Gamma = \mathrm{PSL}_2(8)$;*
(2) *$A^{(\infty)} = T$, and either $|\mathsf{rad}(G)| = 2$ or $\mathsf{rad}(G) \trianglelefteq A$.*

Proof. By Lemma 2.8, $T \cong \mathrm{A}_6$, A_7, $\mathrm{PSL}_2(8)$, J_1 or $\mathrm{PSL}_2(p)$, where $p \geqslant 5$ is a prime. Suppose that $T \cong \mathrm{A}_6$, A_7 or $\mathrm{PSL}_2(8)$. Then $|T|$ has a divisor 9, and so Γ is T-edge-transitive by Lemma 2.3. We have $|V| = 60$, 420 or 84, respectively. Employing [6, 8], we conclude that Γ is desired as in (i), (ii) or (iii), and part (1) follows.

Let $X = \langle A_u, A_w\rangle$ for an edge $\{u, w\} \in E$. Then $|A : X| \leqslant 2$, where the equality holds if and only if Γ is bipartite, refer to [32, Exercise 3.8]. In particular, $A^{(\infty)} = X^{(\infty)}$. Clearly, $G \leqslant X$, and Γ is either non-bipartite or X-semisymmetric. Then, by Lemma 2.8, $A^{(\infty)} = X^{(\infty)} \cong \mathrm{A}_6$, A_7, $\mathrm{PSL}_2(8)$, J_1 or $\mathrm{PSL}_2(p)$. By Lemma 2.3, we may choose a G-orbit U such that T acts transitively on it. Noting that $T = G^{(\infty)} \leqslant A^{(\infty)}$, we know that U is also a $A^{(\infty)}$-orbit. In particular, $|T : T_u| = |U| = |A^{(\infty)} : (A^{(\infty)})_u|$, where $u \in U$. Then $|T|$ and $|A^{(\infty)}|$ have the same prime divisors no less than 5. It follows that $A^{(\infty)} = T$, desired as in (2).

Finally, by (2.3), $|\mathsf{rad}(X)|$ is a divisor of 6. Noting that $\mathsf{rad}(G) = \mathbf{C}_G(T) \leqslant \mathbf{C}_X(T) = \mathsf{rad}(X)$, if $|\mathsf{rad}(G)| \neq 2$ then $|\mathsf{rad}(G)| = 1$, 3 or 6, and so $\mathsf{rad}(G)$ is a characteristic subgroup of $\mathsf{rad}(X)$, yielding $\mathsf{rad}(G) \trianglelefteq A$. This completes the proof. □

3. Coset graphs and bi-coset graphs

Let G be a finite group. If G is normal in some group A then each $a \in A$ induces an automorphism $\mathsf{conj}(a)$ of G by conjugation:

$$x^{\mathsf{conj}(a)} := a^{-1}xa, \forall x \in G.$$

For $X_1, \ldots, X_m \subseteq G$, we write

$$\begin{aligned}
\mathbf{N}_G(X_1, \ldots, X_m) &= \cap_{i=1}^m \mathbf{N}_G(X_i),\\
\mathbf{N}_G(\{X_1, \ldots, X_m\}) &= \{g \in G \mid \{g^{-1}X_1g, \ldots, g^{-1}X_mg\} = \{X_1, \ldots, X_m\}\},\\
\mathrm{Aut}(G, X_1, \ldots, X_m) &= \{\sigma \in \mathrm{Aut}(G) \mid X_i^\sigma = X_i, 1 \leqslant i \leqslant m\},\\
\mathrm{Aut}(G, \{X_1, \ldots, X_m\}) &= \{\sigma \in \mathrm{Aut}(G) \mid \{X_1^\sigma, \ldots, X_m^\sigma\} = \{X_1, \ldots, X_m\}\}.
\end{aligned}$$

3.1. **Coset actions.** Assume that H is a core-free subgroup of G, that is, H contains no nontrivial normal subgroup of G. Then G acts faithfully and transitively on $[G : H] := \{Hx \mid x \in G\}$ by right multiplication:

$$(Hx)^g := Hxg, \forall x, g \in G. \tag{3.1}$$

The resulting transitive subgroup of $\mathrm{Sym}([G : H])$ is still denoted by G in the following.

Note that the group $\mathrm{Aut}(G, H)$ has a natural action on $[G : H]$ by

$$(Hx)^\sigma := Hx^\sigma,\ x \in G, \sigma \in \mathrm{Aut}(G, H).$$

For $\sigma \in \mathrm{Aut}(G, H)$, we denote by σ_H the permutation induced by σ on $[G : H]$. Clearly,

$$\mathsf{conj}(h)_H = h, \forall h \in H. \tag{3.2}$$

The next lemma says that $\sigma \mapsto \sigma_H$ is an embedding from $\mathrm{Aut}(G, H)$ into $\mathrm{Sym}([G : H])$.

Lemma 3.1. $\mathrm{Aut}(G, H)$ *acts faithfully on* $[G : H]$.

Proof. Clearly, if $H = 1$ then the action of $\mathrm{Aut}(G, H)$ is faithful. Thus let $H \neq 1$. Pick $\sigma \in \mathrm{Aut}(G, H)$ such that $Hx^\sigma = Hx$, i.e., $x^\sigma x^{-1} \in H$, for all $x \in G$. For x, $y \in G$,

$$Hyx = H(yx)^\sigma = Hy^\sigma x^\sigma = Hyx^\sigma \Rightarrow yx^\sigma x^{-1}y^{-1} \in H.$$

Then, for each $x \in G$, the subgroup H contains a normal subgroup $\langle yx^\sigma x^{-1}y^{-1} \mid y \in G\rangle$ of G. Since H is core-free, we have $x^\sigma x^{-1} = 1$, i.e., $x^\sigma = x$ for all $x \in G$. Thus $\sigma = 1$, and the lemma follows. □

If $g \in \mathbf{N}_G(H)$, then g induces a permutation $\hat{g}$ on $[G : H]$ by

$$(Hx)^{\hat{g}} := Hg^{-1}x, \forall x \in G. \tag{3.3}$$

In fact, $\hat{g}g = \mathsf{conj}(g)_H = g\hat{g}$, where g acts on $[G : H]$ by the way described as in (3.1).

Lemma 3.2. $\mathbf{N}_G(H)/H \cong \mathbf{C}_{\mathrm{Sym}([G:H])}(G) = \{\hat{g} \mid g \in \mathbf{N}_G(H)\}$, *and* $\mathbf{N}_{\mathrm{Sym}([G:H])}(G) = G\{\sigma_H \mid \sigma \in \mathrm{Aut}(G, H)\}$.

Proof. The first part of this lemma follows directly from [11, p.108, Lemma 4.2A].

Let $N = \mathbf{N}_{\mathrm{Sym}([G:H])}(G)$, and K be the point-stabilizer of H in N. Then $G \leqslant N$ and, since G is transitive on $[G : H]$, we have $N = GK$. Clearly, $\mathrm{Aut}(G, H) \cong \{\sigma_H \mid \sigma \in \mathrm{Aut}(G, H)\} \leqslant K$. For $t \in K$, considering the point-stabilizers of H^t and H in G, we have $t^{-1}Ht = H$, and so $\mathsf{conj}(t) \in \mathrm{Aut}(G, H)$. Thus we have a group homomorphism: $K \to \mathrm{Aut}(G, H)$, $t \mapsto \mathsf{conj}(t)$, and the kernel equals to $\mathbf{C}_K(G)$. Noting that $\mathbf{C}_K(G)$ is semiregular on $[G : H]$, we have $\mathbf{C}_K(G) = 1$. Thus K is isomorphic to a subgroup of $\mathrm{Aut}(G, H)$, and so $|K| \leqslant |\mathrm{Aut}(G, H)|$. We have $K = \{\sigma_H \mid \sigma \in \mathrm{Aut}(G, H)\}$, and the lemma follows. □

3.2. **Coset graphs.** Let $G \neq 1$ be a finite group, and let H be a core-free subgroup of G. Suppose that H has a subgroup K with index $k > 1$, and

(I) there exists $o \in \mathbf{N}_G(K) \setminus H$ such that $o^2 \in K$ and $H \cap o^{-1}Ho = K$.

The coset graph $\mathsf{Cos}(G, H, K, o)$ is defined on $[G : H]$ such that Hx and Hy are adjacent if and only if $yx^{-1} \in HoH$. Then $\mathsf{Cos}(G, H, K, o)$ is a well-defined G-symmetric graph of valency k. It is well-known that every connected symmetric graph of valency k is isomorphic to a coset graph defined as above. The following facts are easily shown, see also [20] for example.

(II) $\mathsf{Cos}(G, H, K, o)$ is connected if and only if $G = \langle H, o\rangle$.

(III) If $\sigma \in \mathrm{Aut}(G)$ then $Hx \mapsto H^\sigma x^\sigma$ defines an isomorphism from $\mathsf{Cos}(G, H, K, o)$ to $\mathsf{Cos}(G, H^\sigma, K^\sigma, o^\sigma)$. In particular, if $\sigma \in \mathrm{Aut}(G, H)$ then σ_H is an automorphism of $\mathsf{Cos}(G, H, K, o)$ if and only if $Ho^\sigma H = HoH$. (Note, for $h \in H$, we have $\mathsf{Cos}(G, H, K, o) = \mathsf{Cos}(G, H, h^{-1}Kh, h^{-1}oh)$.)

In view of (III), up to isomorphism of graphs, H, K and o may be chosen up to the conjugacy under $\mathrm{Aut}(G)$, $\mathrm{Aut}(G, H)$ and $\mathrm{Aut}(G, H, K)$, respectively.

Lemma 3.3. *Let* $\Gamma = \mathsf{Cos}(G, H, K, o)$ *and* $\Sigma = \mathsf{Cos}(G, H, K, o')$. *Suppose that both* $\mathrm{Aut}\Gamma$ *and* $\mathrm{Aut}\Sigma$ *have a unique subgroup isomorphic to* G. *Then* $\Gamma \cong \Sigma$ *if and only if* $Ho^\sigma H = Ho'H$ *for some* $\sigma \in \mathrm{Aut}(G, H, K)$.

Proof. The sufficiency of $\Gamma \cong \Sigma$ is immediate from the above (III). Now let λ be an isomorphism from $\mathsf{Cos}(G, H, K, o)$ to $\mathsf{Cos}(G, H, K, o')$. Then $\mathrm{Aut}\Sigma = \lambda^{-1}\mathrm{Aut}\Gamma\lambda$. It follows that $G = \lambda^{-1}G\lambda$. Since G is transitive on the arc sets of Γ and Σ, without loss of generality, we choose λ with $(H, Ho)^\lambda = (H, Ho')$. Considering the stabilizers of H, (H, Ho) and (H, Ho') in G, we have $H = \lambda^{-1}H\lambda$ and $K = \lambda^{-1}K\lambda$. Then $\sigma := \mathsf{conj}(\lambda) \in \mathrm{Aut}(G, H, K)$. For $Hx \in [G : H]$, since λ fixes the vertex H, we have

$$(Hx)^\lambda = H^{x\lambda} = H^{\lambda^{-1}x\lambda} = H(\lambda^{-1}x\lambda) = Hx^\sigma.$$

Considering the neighborhoods of H in Γ and Σ, we have

$$\{Ho'h \mid h \in H\} = \{Hoh \mid h \in H\}^\lambda = \{H\lambda^{-1}oh\lambda \mid h \in H\} = \{Ho^\sigma h^\sigma \lambda \mid h \in H\}.$$

This implies that $Ho'H = Ho^\sigma H$, and the lemma follows. □

Using Lemma 3.2, the following lemma is easily shown.

Lemma 3.4. *Let $\Gamma = \mathsf{Cos}(G, H, K, o)$, and view G as a subgroup of $\mathrm{Aut}\Gamma$. Then $\mathbf{C}_{\mathrm{Aut}\Gamma}(G) = \{\hat{g} \mid g \in \mathbf{N}_G(H, HoH)\}$, and $\mathbf{N}_{\mathrm{Aut}\Gamma}(G) = G\{\sigma_H \mid \sigma \in \mathrm{Aut}(G, H, HoH)\}$.*

Example 3.5. Let $T = \mathrm{J}_1$, the first Janko group. Computation with GAP [14] shows that, up to conjugacy, J_1 has two subgroup isomorphic to S_3, and only one of them say H has a subgroup K which has order 2 and satisfies the condition that $\mathbf{N}_T(K) \setminus K$ contains elements o with $o^2 \in K$ and $\langle H, o\rangle = T$. Fix such a pair (H, K). Then $\mathbf{N}_T(K) = \mathbb{Z}_2 \times \mathrm{A}_5$, and thus every desired o should be an involution. Further computation shows that there exist exactly 20 desired involutions, which are conjugate in pairs under $\mathbf{N}_T(H, K)$ and produce 10 distinct double cosets HoH. Thus we get ten connected T-symmetric cubic graphs of order $4 \cdot 5 \cdot 7 \cdot 11 \cdot 19$. It is shown in Section 4 that these graphs are not isomorphic to each other. □

3.3. **Bi-coset graphs.** Let G be a finite group, and $L, R < G$ with $L \neq R$, $|L| = |R|$ and $L \cap R$ core-free in G. The bi-coset graph $\mathsf{BC}(G, L, R)$ is defined with bipartition $([G : L], [G : R])$ such that Lx and Ry are adjacent if and only if $yx^{-1} \in RL$, i.e., $xy^{-1} \in LR$. Then $\mathsf{BC}(G, L, R)$ is a well-defined regular graph of valency $|L : (L \cap R)|$, and $\mathsf{BC}(G, L, R) = \mathsf{BC}(G, R, L)$. View G as a subgroup of $\mathrm{Aut}\mathsf{BC}(G, L, R)$, where G acts on $[G : L]$ and $[G : R]$ by right multiplications:

$$(Lx)^g := Lxg,\ (Ry)^g := Ryg,\ \forall g, x, y \in G. \tag{3.4}$$

Then $\mathsf{BC}(G, L, R)$ is G-semisymmetric. It is easily shown that $\mathsf{BC}(G, L, R)$ is connected if and only if $G = \langle L, R\rangle$. The reader is referred to [13, 25] for more information about bi-coset graphs.

Each $\sigma \in \mathrm{Aut}(G)$ defines an isomorphism from $\mathsf{BC}(G, L, R)$ to $\mathsf{BC}(G, L^\sigma, R^\sigma)$ by

$$Lx \mapsto L^\sigma x^\sigma,\ Ry \mapsto R^\sigma y^\sigma,\ \forall x,\, y \in G. \tag{3.5}$$

Thus, up to isomorphism of graphs, the subgroups L and R may be chosen under $\mathrm{Aut}(G)$-conjugacy and $\mathrm{Aut}(G, L)$-conjugacy, respectively.

Lemma 3.6. *Assume that $G = \langle L_1, R_1\rangle = \langle L_2, R_2\rangle$, and $\Gamma_i = \mathsf{BC}(G, L_i, R_i)$ for $i = 1, 2$.*

1. *If $\{L_1^\sigma, R_1^\sigma\} = \{L_2, R_2\}$ for some $\sigma \in \mathrm{Aut}(G)$ then $\Gamma_1 \cong \Gamma_2$.*
2. *Suppose that both $\mathrm{Aut}\Gamma_1$ and $\mathrm{Aut}\Gamma_2$ have a unique subgroup isomorphic to G. If $\Gamma_1 \cong \Gamma_2$ then $\{L_1^\sigma, R_1^\sigma\} = \{L_2, R_2\}$ for some $\sigma \in \mathrm{Aut}(G)$, and σ is chosen from $\mathrm{Aut}(G, L_1)$ for the case where $L_1 = L_2$ and either Γ_1 is symmetric or L_1 and R_1 are not conjugate under $\mathrm{Aut}(G)$.*

Proof. Part (1) of the lemma is pretty obvious. Suppose that both $\mathrm{Aut}\Gamma_1$ and $\mathrm{Aut}\Gamma_2$ have a unique subgroup isomorphic to G, and let λ be an isomorphism from Γ_1 to Γ_2. Then $\mathrm{Aut}\Gamma_2 = \lambda^{-1}\mathrm{Aut}\Gamma_1\lambda$, and $G = \lambda^{-1}G\lambda$. Since G acts transitively on the edge sets, we choose λ such that $\{L_1, R_1\}^\lambda = \{L_2, R_2\}$. Let σ be the automorphism of G induced by λ. Considering the vertex-stabilizers of L_1, L_2, R_1 and R_2 in G, we deduce that

$$\{L_2, R_2\} = \{\lambda^{-1}L_1\lambda, \lambda^{-1}R_1\lambda\} = \{L_1^\sigma, R_1^\sigma\}.$$

Assume further that $L_1 = L_2$, and either Γ_1 is symmetric or L_1 and R_1 are not conjugate under $\mathrm{Aut}(G)$. It is easily shown that λ may be chosen such that $(L_1, R_1)^\lambda = (L_1, R_2)$. This implies that $L_1^\sigma = L_1$ and $R_2 = R_1^\sigma$, and so part (2) of the lemma follows. □

Note that $\mathrm{Aut}(G, \{L, R\})$ induces a subgroup of $\mathrm{Aut}\mathsf{BC}(G, L, R)$, see (3.5). Denote $\sigma_{\{L,R\}}$ the graph automorphism induced by $\sigma \in \mathrm{Aut}(G, \{L, R\})$. Clearly,

$$\mathsf{conj}(h)_{\{L,R\}} = h, \ \forall h \in L \cap R.$$

Lemma 3.7. $\mathrm{Aut}(G, \{L, R\})$ *acts faithfully on* $[G : L] \cup [G : R]$.

Proof. Let K be the kernel of $\mathrm{Aut}(G, \{L, R\})$ acting on $[G : L] \cup [G : R]$. Then $K \leqslant \mathrm{Aut}(G, L, R)$. Let $\sigma \in K$ and $x \in G$. It is easily shown that both L and R contains a normal subgroup $\langle yx^\sigma x^{-1}y^{-1} \mid y \in G\rangle$ of G, see the proof of Lemma 3.1. Since $L \cap R$ is core-free in G, we have $x^\sigma x^{-1} = 1$. Thus $x^\sigma = x$ for all $x \in G$ and $\sigma \in K$. Then $K = 1$, and the lemma follows. □

Lemma 3.8. *Let* $\Gamma = \mathsf{BC}(G, L, R)$ *and* $N = \mathbf{N}_{\mathrm{Aut}\Gamma}(G)$. *Then* $N = G\{\sigma_{\{L,R\}} \mid \sigma \in \mathrm{Aut}(G, \{L, R\})\}$.

Proof. Let H be the edge-stabilizer of $\{L, R\}$ in N. We have $H \geqslant \{\sigma_{\{L,R\}} \mid \sigma \in \mathrm{Aut}(G, \{L, R\})\} \cong \mathrm{Aut}(G, \{L, R\})$ and, since Γ is G-edge-transitive, $N = GH$. Considering the conjugation of H on G, we have a homomorphism $\rho : H \to \mathrm{Aut}(G)$ with kernel equal to $\mathbf{C}_H(G)$. Note that Γ has valency $|L : (L \cap R)| > 1$. It follows that N acts faithfully on the edge set of Γ. Then $\mathbf{C}_H(G)$ is faithful and semiregular on the edge set of Γ. Thus $\mathbf{C}_H(G) = 1$, and ρ is injective. In particular, $|H| = |\rho(H)|$.

Let $t \in H$. Then either $L^t = L$ and $R^t = R$, or $L^t = R$ and $R^t = L$. Now consider the vertex-stabilizers of L, R, L^t and R^t in G. If $L^t = L$ and $R^t = R$, then $L^{\rho(t)} = t^{-1}Lt = L$ and $R^{\rho(t)} = t^{-1}Rt = R$; if $L^t = R$ and $R^t = L$ then $L^{\rho(t)} = t^{-1}Lt = R$ and $R^{\rho(t)} = t^{-1}Rt = L$. For both cases, $\rho(t) \in \mathrm{Aut}(G, \{L, R\})$. Thus $|H| = |\rho(H)| \leqslant |\mathrm{Aut}(G, \{L, R\})| = |\{\sigma_{\{L,R\}} \mid \sigma \in \mathrm{Aut}(G, \{L, R\})\}|$. Recalling that $\{\sigma_{\{L,R\}} \mid \sigma \in \mathrm{Aut}(G, \{L, R\})\} \leqslant H$, it follows that $\{\sigma_{\{L,R\}} \mid \sigma \in \mathrm{Aut}(G, \{L, R\})\} = H$. Then the lemma follows. □

For $g_1 \in \mathbf{N}_G(L)$ and $g_2 \in \mathbf{N}_G(R)$, define

$$\begin{aligned} \tilde{g}_1 : \quad & [G : L] \cup [G : R] \to Lx \mapsto Lg_1^{-1}x, Ry \mapsto Ry; \\ \hat{g}_2 : \quad & [G : L] \cup [G : R] \to Lx \mapsto Lx, Ry \mapsto Rg_2^{-1}y. \end{aligned}$$

Then

$$\mathbf{C}_{\mathrm{Sym}([G:L])\times\mathrm{Sym}([G:R])}(G) = \{\tilde{g}_1\hat{g}_2 \mid g_1 \in \mathbf{N}_G(L), g_2 \in \mathbf{N}_G(R)\}.$$

Further, we have the following lemma.

Lemma 3.9. *Let* $\Gamma = \mathsf{BC}(G, L, R)$. *If* $g_1 \in \mathbf{N}_G(L)$ *and* $g_2 \in \mathbf{N}_G(R)$, *then* $\tilde{g}_1\hat{g}_2 \in \mathbf{C}_{\mathrm{Aut}\Gamma}(G)$ *if and only if* $Rg_2^{-1}g_1L = RL$, *and* $\tilde{g}_1\hat{g}_2 = 1$ *if and only if* $g_1 \in L$ *and* $g_2 \in R$.

Lemma 3.10. *Let $\Gamma = (V, E)$ be a connected G-semisymmetric graph of valency $k > 1$. Then $\Gamma \cong \mathsf{BC}(G, L, R)$ for some $L, R < G$ with $|L| = |R|$, $k = |L : (L \cap R)|$, $G = \langle L, R\rangle$ and $L \cap R$ core-free in G.*

Proof. Clearly, for $v \in V$, the stabilizer G_v acts transitively $\Gamma(v)$, and so $k = |G_v : (G_v \cap G_{v'})|$ for $v' \in \Gamma(v)$. Let U and W be the G-orbits on V, and fix an edge $\{u, w\} \in E$ with $u \in U$ and $w \in W$. Since Γ is regular, we have $|G : G_u| = |U| = |W| = |G : G_w|$, and so $|G_u| = |G_w|$. Since Γ is connected, $G = \langle G_u, G_w\rangle$. Since Γ has valency $k > 1$, it is easily shown that G acts faithfully on E. If $G_u \cap G_w$ contains a normal subgroup N of G then N fixes E point-wise, and so $N = 1$. Thus $G_u \cap G_w$ is core-free in G. Put $L = G_u$ and $R = G_w$. Noting that $U = \{u^x \mid x \in G\}$ and $W = \{w^y \mid y \in G\}$, define

$$\rho : U \cup W \to [G : L] \cup [G : R],\ u^x \mapsto Lx,\ w^y \mapsto Ry.$$

Then ρ is a bijection and, for $u^x \in U$ and $w^y \in W$,

$$\{u^x, w^y\} \in E \Leftrightarrow w^{yx^{-1}} \in \Gamma(u) \Leftrightarrow yx^{-1} \in G_wG_u = RL.$$

Thus ρ is an isomorphism from Γ to $\mathsf{BC}(G, L, R)$, and the lemma follows. □

Example 3.11. Let $T = \mathrm{J}_1$. Computation with GAP [14] shows that

(i) T has a unique conjugacy class of subgroups isomorphic to D_{12}, and each subgroup D_{12} is self-normalized in T; and

(ii) fixing a subgroup $L \cong \mathrm{D}_{12}$, there exist exactly 6 subgroups $R \cong \mathrm{D}_{12}$ with $|L\cap R| = 4$ and $\langle L, R\rangle = G$, which form two classes under the conjugation of L.

Thus, up to isomorphism of graphs, we get two connected T-semisymmetric cubic graphs, say $\Gamma_1 = \mathsf{BC}(T, L, R_1)$ and $\Gamma_2 = \mathsf{BC}(T, L, R_2)$ with the stabilizers of two adjacent vertices isomorphic to D_{12}. We next show that $\Gamma_1 \cong \Gamma_2$.

Since $\mathbf{N}_T(L) = L$, there is a unique $o \in G$ with $R_1 = o^{-1}Lo$. Set $R = oLo^{-1}$. Then $\langle L, R\rangle = T$ and $|L \cap R| = 4$. Suppose that $R = x^{-1}R_1x$ for some $x \in L$. We have $oLo^{-1} = x^{-1}o^{-1}Lox$, yielding $o^{-1} = ox$, and so $o^2 = x^{-1} \in L$. Then there exists a connected T-symmetric cubic graph $\mathsf{Cos}(T, L, L \cap L^o, o)$, which is impossible by [21, Lemma 6.3]. Therefore, R and R_1 are not conjugate under L, and so we may choose $R_2 = oLo^{-1}$. Noting that $\{L, R_2\}^{\mathsf{conj}(o)} = \{L, R_1\}$, we have $\Gamma_1 \cong \Gamma_2$ by Lemma 3.6. □

4. The graphs arising from J_1

In this section, we assume that $\Gamma = (V, E)$ is a connected edge-transitive cubic graph of order $2n$ with n even and square-free. Assume further that $\mathrm{J}_1 \leqslant \mathrm{Aut}\Gamma$.

Lemma 4.1. *Suppose that Γ is J_1-edge-transitive. Then $\mathrm{Aut}\Gamma = \mathrm{J}_1$, and either*

(1) *Γ is isomorphic to one of ten non-isomorphic graphs in Example 3.5; or*

(2) *Γ is semisymmetric and isomorphic to the graph constructed in Example 3.11.*

Proof. Let $T = \mathrm{J}_1$. We discuss in two cases according whether Γ is bipartite or not.

Case 1. Assume that Γ is not bipartite. Then Γ is T-symmetric, and $2n = |V| = |T : T_u|$ for $u \in V$. We have $|T_u| = 6$, and so $T_u \cong \mathrm{S}_3$ by Lemma 2.1. Then Γ is isomorphic one of the ten coset graphs $\mathsf{Cos}(T, H, K, o)$ given as in Example 3.5. Let $A = \mathrm{Aut}\mathsf{Cos}(T, H, K, o)$. Then $T = A^{(\infty)}$ by Theorem 2.10. In particular, $\mathbf{N}_A(T) = \mathrm{Aut}\mathsf{Cos}(T, H, K, o)$. Note that every automorphism of T is induced by the conjugation of some element in T. Computation with GAP shows that $\mathrm{Aut}(T,H)\cong \mathrm{D}_{12}$, and if $\sigma\in\mathrm{Aut}(T,H)$ such that $Ho^{\sigma}H=HoH$ then $\sigma=\mathsf{conj}(h)$ for some $h\in H$. We deduce from Lemma 3.4 that $\mathrm{Aut}\mathsf{Cos}(T,H,K,o)=T$. Thus every graph in Example 3.5 has automorphism group T. By Lemma 3.3, these coset graphs are not isomorphic to each other, and part (1) if the lemma follows.

Case 2. Assume that Γ is bipartite. Then T is intransitive on V; otherwise, T has a subgroup of index 2, and so T is not simple, a contradiction. Thus Γ is T-semisymmetric, and $n=|T:T_u|$ for $u\in V$. We have $|T_u|=12$. By Lemma 2.1, we assume that $T_u\cong \mathrm{D}_{12}$ and $T_w\cong\mathrm{D}_{12}$ or A_4, where $w\in\Gamma(u)$. If $T_u\not\cong T_w$ then computation with GAP shows that $|\langle T_u,T_w\rangle|=660\neq|T|$, which contradicts the fact that Γ is connected. We have $T_u\cong T_w\cong\mathrm{D}_{12}$. By Lemma 3.10, Γ is isomorphic to the bi-coset graph $\mathsf{BC}(T,L,R_1)$ given in Example 3.11. By Theorem 2.10, we have $T\trianglelefteq\mathrm{Aut}\mathsf{BC}(T,L,R_1)$. Computation with GAP shows that $\mathrm{Aut}(T,\{L,R_1\})=\{\mathsf{conj}(h)\mid h\in L\cap R_1\}$. It follows from Lemma 3.8 that $\mathrm{Aut}\mathsf{BC}(T,L,R_1)=T$. Then Γ is semisymmetric, and part (2) of the lemma follows. □

Theorem 4.2. *Let $A=\mathrm{Aut}\Gamma$. Assume that $A^{(\infty)}=\mathrm{J}_1$. Then Γ is J_1-edge-transitive, and Γ is described as in Lemma 4.1.*

Proof. By Lemma 4.1, it suffices to show that Γ is J_1-edge-transitive. We next suppose that Γ is not J_1-edge-transitive, and produce a contradiction. By Lemma 2.9, Γ is bipartite, and $T:=\mathrm{J}_1$ is transitive on one part of Γ say W and has three orbits on the other part U. Let $\{u,w\}\in E$ with $u\in U$ and $w\in W$. Then $n=|T:T_w|$ and $n=3|T:T_u|$. It follows that $|T_w|=4$ and $|T_u|=12$.

Let $G=\langle A_u,A_w\rangle$ and $M=\mathsf{rad}(G)$. By Lemma 2.9, $|M|=3$ or 6. Clearly, the quotient graph Γ_M is bipartite. Then, by Lemma 2.7, Γ_M is $\bar{T}$-semisymmetric. In addition, $|\bar{T}:\bar{T}_{\bar{v}}|=\frac{n}{|M|}$ is square-free, where $v\in V$. By Lemma 2.1 and inspecting the subgroups of J_1, we conclude that $\bar{T}_{\bar{u}}$ and $\bar{T}_{\bar{w}}$ are isomorphic to D_{12} or A_4. In particular, $\frac{n}{|M|}$ is even, and so $|M|$ is odd. We have $|M|=3$. Recall that $\bar{T}_{\bar{w}}\cong T_{\bar{w}}$ and $M\cong T_{\bar{w}}/T_w$, see (2.1) and (2.2). This implies that $\bar{T}_{\bar{w}}\cong\mathrm{A}_4$, and so $\bar{T}_{\bar{u}}\cong\mathrm{D}_{12}$ by Lemma 2.1. However, since $|\bar{T}:\bar{T}_{\bar{v}}|$ is even and square-free, (2) of Lemma 4.1 is available for the pair $(\bar{T},\Gamma_M)$, which leads to $\bar{T}_{\bar{w}}\cong\bar{T}_{\bar{u}}\cong\mathrm{D}_{12}$, a contradiction. This completes the proof. □

5. $\mathrm{PSL}_2(p)$-symmetric graphs

In this section, $\Gamma=(V,E)$ is a connected T-symmetric cubic graph of order $2n$, where $T=\mathrm{PSL}_2(p)$ for some prime $p\geqslant 5$, and n is even and square-free. Choose $\varepsilon,\eta\in\{1,-1\}$ with $p+\varepsilon$ and $p+\eta$ divisible by 3 and 4, respectively. Our discussion is based on the subgroup structure of $\mathrm{PSL}_2(p)$ and $\mathrm{PGL}_2(p)$. The reader is referred to [17, II.8.27] and [3, Theorem 3] for the subgroups of $\mathrm{PSL}_2(p)$, and to [4, Theorem 2] for the subgroups of $\mathrm{PGL}_2(p)$. For convenience, we list the subgroups of $\mathrm{PSL}_2(p)$ and $\mathrm{PGL}_2(p)$ in the following two lemmas.

Lemma 5.1. *Let $p\geqslant 5$ be a prime. Then the subgroups of $\mathrm{PSL}_2(p)$ are listed as follows.*

(1) *One conjugacy class of $\frac{p(p-\eta)}{2}$ cyclic subgroups $\mathbb{Z}_2$.*
(2) *One conjugacy class of $\frac{p(p\mp1)}{2}$ cyclic subgroups $\mathbb{Z}_d$, where $d\,\big|\,\frac{p\pm1}{2}$ and $d>2$.*

(3) $\frac{p(p^2-1)}{24}$ *elementary abelian subgroups* $\mathbb{Z}_2^2$.
(4) $\frac{p(p^2-1)}{4d}$ *dihedral subgroups* D_{2d}, *where* $d \,\big|\, \frac{p\pm1}{2}$ *and* $d > 2$.
(5) *One conjugacy class of* $p+1$ *subgroups* $\mathbb{Z}_p{:}\mathbb{Z}_d$, *where* $d \,\big|\, \frac{p-1}{2}$ *and* $d \geqslant 1$.
(6) $\frac{p(p^2-1)}{24}$ *subgroups* A_4.
(7) *Two conjugacy classes of subgroups* S_4, *each consists of* $\frac{p(p^2-1)}{48}$ *subgroups, where* $p \equiv \pm1 \pmod 8$.
(8) *Two conjugacy classes of subgroups* A_5, *each consists of* $\frac{p(p^2-1)}{120}$ *subgroups, where* $p \equiv \pm1 \pmod{10}$.

Moreover, isomorphic subgroups of $\mathrm{PSL}_2(p)$ *are conjugate in* $\mathrm{PGL}_2(p)$.

Lemma 5.2. *Let* $p \geqslant 5$ *be a prime. Then the subgroups of* $\mathrm{PGL}_2(p)$ *are listed as follows.*

(1) *The subgroup* $\mathrm{PSL}_2(p)$.
(2) *Two conjugacy classes of cyclic subgroup* $\mathbb{Z}_2$, *one class consists of* $\frac{p(p-\eta)}{2}$ *subgroups which lie in* $\mathrm{PSL}_2(p)$, *and the other one consists of* $\frac{p(p+\eta)}{2}$ *subgroups.*
(3) *One conjugacy class of* $\frac{p(p\mp1)}{2}$ *cyclic subgroups* $\mathbb{Z}_d$, *where* $d \,\big|\, p\pm1$ *and* $d > 2$.
(4) *Two conjugacy classes of subgroups* $\mathbb{Z}_2^2$, *one class consists of* $\frac{p(p^2-1)}{24}$ *subgroups which lie in* $\mathrm{PSL}_2(p)$, *and the other one consists of* $\frac{p(p^2-1)}{8}$ *subgroups.*
(5) *Two conjugacy classes of subgroups* D_{2d}, *one class consists of* $\frac{p(p^2-1)}{4d}$ *subgroups which lie in* $\mathrm{PSL}_2(p)$, *and the other one consists of* $\frac{p(p^2-1)}{4d}$ *subgroups, where* $d \,\big|\, \frac{p\pm1}{2}$ *and* $d > 2$.
(6) *One conjugacy class of* $\frac{p(p^2-1)}{2d}$ *subgroups* D_{2d}, *where* $d > 2$ *and* $\frac{p\pm1}{d}$ *is an odd integer.*
(7) *One conjugacy class of* $p+1$ *subgroups* $\mathbb{Z}_p{:}\mathbb{Z}_d$, *where* $d \,\big|\, (p-1)$ *and* $d \geqslant 1$.
(8) *One conjugacy class of* $\frac{p(p^2-1)}{24}$ *subgroups* A_4.
(9) *One conjugacy class of* $\frac{p(p^2-1)}{24}$ *subgroups* S_4.
(10) *One conjugacy classes of* $\frac{p(p^2-1)}{60}$ *subgroups* A_5, *where* $p \equiv \pm1 \pmod{10}$.

By Lemma 2.1 and inspecting the subgroups of $\mathrm{PSL}_2(p)$, we have $T_v \cong \mathbb{Z}_3$, S_3, D_{12} or S_4, where $v \in V$. Then

$$p \equiv 2^{i+2} \pm 1 \pmod{2^{i+3}} \text{ and } |T_v| = 2^i \cdot 3 \text{ for } 0 \leqslant i \leqslant 3. \tag{5.1}$$

We deduce from Lemmas 5.1 and 5.2 that T contains at most two conjugacy classes of subgroups isomorphic to T_v, and these subgroups are all conjugate in $\mathrm{PGL}_2(p)$. Thus up to isomorphism of graphs, we fix two subgroups K, H of T, and write

$$\Gamma \cong \mathsf{Cos}(T, H, K, o),$$

where $K < H \cong T_v$, $|H : K| = 3$ and $o \in \mathbf{N}_T(K)$ with $o^2 \in K$ and $\langle o, H\rangle = T$.

By Theorem 2.10, $T \trianglelefteq \mathrm{Aut}\Gamma$. Noting that $\mathrm{Aut}(T) = \{\mathsf{conj}(g) \mid g \in \mathrm{PGL}_2(p)\}$, we have

$$\mathrm{Aut}\mathsf{Cos}(T, H, K, o) = T\{\mathsf{conj}(g)_H \mid g \in \mathbf{N}_{\mathrm{PGL}_2(p)}(H, HoH)\}, \tag{5.2}$$

by Lemma 3.4. Recall that $\mathsf{conj}(g)_H = g\hat{g}$ for $g \in \mathbf{N}_T(H)$.

5.1. $|H| = 3$**.** Assume that $H \cong \mathbb{Z}_3$. Then $p \equiv \pm 3 \pmod 8$ by (5.1), $K = 1$, and o is an involution. Let S and O be the sets of involutions $x \in T$ with $\langle x, H\rangle \neq T$ and $\langle x, H\rangle = T$, respectively. Then $|S| + |O| = \frac{p(p-\eta)}{2}$, see Lemma 5.1 (1).

Lemma 5.3.

$$|S| = \begin{cases} \frac{3p+3\varepsilon+|\varepsilon+\eta|}{2} & \textit{if } p \not\equiv \pm 1 \pmod{10}, \varepsilon+\eta \neq -2, \\ \frac{7p-5}{2} & \textit{if } p \not\equiv \pm 1 \pmod{10}, \varepsilon = \eta = -1, \\ \frac{7p+7\varepsilon+|\varepsilon+\eta|}{2} & \textit{if } p \equiv \pm 1 \pmod{10}, \varepsilon+\eta \neq -2, \\ \frac{11p-9}{2} & \textit{if } p \equiv \pm 1 \pmod{10}, \varepsilon = \eta = -1. \end{cases}$$

Proof. For an arbitrary $x \in S$, inspecting the subgroups of $\mathrm{PSL}_2(p)$, we deduce that $\langle x, H\rangle \cong \mathrm{S}_3$, $\mathbb{Z}_6$ (if $\varepsilon = \eta$), $\mathbb{Z}_p{:}\mathbb{Z}_6$ (if $\varepsilon = \eta = -1$), A_4, or A_5 (if $p \equiv \pm 1 \pmod{10}$). Let $\Delta_1 = \{X < \mathrm{PSL}_2(p) \mid H < X \cong \mathrm{S}_3\}$, $\Delta_2 = \{X < \mathrm{PSL}_2(p) \mid H < X \cong \mathbb{Z}_6\}$ when $\varepsilon = \eta$, $\Delta_3 = \{X < \mathrm{PSL}_2(p) \mid H < X \cong \mathbb{Z}_p{:}\mathbb{Z}_6\}$ when $\varepsilon = \eta = -1$, $\Delta_4 = \{X < \mathrm{PSL}_2(p) \mid H < X \cong \mathrm{A}_4\}$, and $\Delta_5 = \{X < \mathrm{PSL}_2(p) \mid H < X \cong \mathrm{A}_5\}$ when $p \equiv \pm 1 \pmod{10}$. Then $x \in S$ if and only if x is an involution contained in one member of Δ_i for some i.

By Lemma 5.1, $\mathrm{PSL}_2(p)$ contains exactly $\frac{p(p-\varepsilon)}{2}$ subgroups $\mathbb{Z}_3$, $\frac{p(p^2-1)}{12}$ subgroups S_3, $\frac{p(p-\varepsilon)}{2}$ subgroups $\mathbb{Z}_6$, $p-\varepsilon$ subgroups $\mathbb{Z}_p{:}\mathbb{Z}_6$, $\frac{p(p^2-1)}{24}$ subgroups A_4, and $\frac{p(p^2-1)}{60}$ subgroups A_5. Note that S_3, $\mathbb{Z}_6$, $\mathbb{Z}_p{:}\mathbb{Z}_6$, A_4 and A_5 contain exactly 1, 1, p, 4 and 10 subgroups $\mathbb{Z}_3$, respectively. Enumerating the pairs (Y, X) with $\mathbb{Z}_3 \cong Y < X \cong \mathrm{S}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_p{:}\mathbb{Z}_6$, A_4 or A_5, we have

$$\frac{p(p-\varepsilon)}{2}|\Delta_i| = \begin{cases} \frac{p(p^2-1)}{12}, & i = 1; \\ \frac{p(p-\varepsilon)}{2}, & i = 2, \varepsilon = \eta; \\ p(p-\varepsilon), & i = 3, \varepsilon = \eta = -1; \\ 4\frac{p(p^2-1)}{24}, & i = 4; \\ 10\frac{p(p^2-1)}{60}, & i = 5. \end{cases}$$

It follows that $|\Delta_1| = \frac{p+\varepsilon}{6}$, $|\Delta_2| = 1$ if $\varepsilon = \eta$, $|\Delta_3| = 2$ if $\varepsilon = \eta = -1$, $|\Delta_4| = \frac{p+\varepsilon}{3}$, and $|\Delta_5| = \frac{p+\varepsilon}{3}$ if $p \equiv \pm 1 \pmod{10}$.

Let S_i be the set of involutions contained in the members of Δ_i, where $1 \leqslant i \leqslant 5$. Then $x \in S$ if and only if $x \in S_i$ for some i. Note that none of S_3, A_4 and A_5 contains elements of order 6, and A_4 has no subgroup isomorphic to S_3. It is easily shown that the following hold: $|S_1| = \frac{p+\varepsilon}{2}$; $|S_2| = 1$ and $(S_1 \cup S_4 \cup S_5) \cap S_2 = \emptyset$ when $\varepsilon = \eta$; $(S_1 \cup S_4 \cup S_5) \cap S_3 = \emptyset$ when $\varepsilon = \eta = -1$; $|S_4| = p + \varepsilon$ and $S_1 \cap S_4 = \emptyset$. Moreover, for $\varepsilon = \eta = -1$, putting $\Delta_2 = \{X\}$ and $\Delta_3 = \{X_1, X_2\}$, it is easily shown that $X_1 \cap X_2 = X$, this implies that $S_2 \subset S_3$ and $|S_3| = 2p - 1$.

Assume first that $p \not\equiv \pm 1 \pmod{10}$. If $\varepsilon = \eta = 1$ then $S = S_1 \cup S_2 \cup S_4$, and so $|S| = \frac{p+1}{2} + 1 + p + 1 = \frac{3p+3\varepsilon+|\varepsilon+\eta|}{2}$. If $\varepsilon \neq \eta$, i.e., $\varepsilon + \eta = 0$ then $S = S_1 \cup S_4$, and so $|S| = \frac{p+\varepsilon}{2} + p + \varepsilon = \frac{3p+3\varepsilon+|\varepsilon+\eta|}{2}$. If $\varepsilon = \eta = -1$ then $S = S_1 \cup S_3 \cup S_4$, and so $|S| = \frac{p-1}{2} + 2p - 1 + p - 1 = \frac{7p-5}{2}$.

Assume next that $p \equiv \pm 1 \pmod{10}$. In this case, each subgroup of $\mathrm{PSL}_2(p)$ which is isomorphic to S_3 or A_4 is contained in a subgroup isomorphic to A_5. It follows that each member of $\Delta_1 \cup \Delta_4$ is a subgroup of some member of Δ_5. Then one of the following holds: $S = S_5$ if $\varepsilon \neq \eta$; $S = S_2 \cup S_5$ if $\varepsilon = \eta = 1$; $S = S_3 \cup S_5$ if $\varepsilon = \eta = -1$. For a given subgroup of order 3 in A_5, it is easily checked that A_5 contains exactly one subgroup which is isomorphic to S_3 and contains the subgroup of order 3, and two subgroups which are isomorphic to A_4 and contain the subgroup of order 3. From this observation, we deduce that each member of Δ_5 contributes $15-3-2\cdot 3=6$ involutions to $S_5\setminus(S_1\cup S_4)$. Thus $|S_5\setminus(S_1\cup S_4)|=6\frac{p+\varepsilon}{3}=2(p+\varepsilon)$. If $\varepsilon\neq\eta$ then $\varepsilon+\eta=0$, and $|S|=|S_5|=|S_5\setminus(S_1\cup S_4)|+|S_1|+|S_4|=2(p+\varepsilon)+\frac{p+\varepsilon}{2}+p+\varepsilon=\frac{7p+7\varepsilon}{2}=\frac{7p+7\varepsilon+|\varepsilon+\eta|}{2}$. If $\varepsilon=\eta=1$ then $|S|=|S_2|+|S_5|=1+\frac{7p+7\varepsilon}{2}=\frac{7p+7\varepsilon+|\varepsilon+\eta|}{2}$. If $\varepsilon=\eta=-1$ then $|S|=|S_3|+|S_5|=2p-1+\frac{7p+7\varepsilon}{2}=\frac{11p-9}{2}$. This completes the proof. □

It is easy to see that $|S|<\frac{p(p-\eta)}{2}$. We have $|O|=\frac{p(p-\eta)}{2}-|S|>0$. Clearly, O is invariant under the conjugation of $\mathbf{N}_{\mathrm{PGL}_2(p)}(H)$. Noting that $\mathbf{N}_{\mathrm{PGL}_2(p)}(H)\cong \mathrm{D}_{2(p+\varepsilon)}$, we write

$$\mathbf{N}_{\mathrm{PGL}_2(p)}(H)=\langle a,b\rangle,$$

where a has order $p+\varepsilon$ and b is an involution not contained in T. Then

$$H\leqslant\langle a^2\rangle<\langle a\rangle,\ \mathbf{N}_T(H)=\langle a^2,ab\rangle.$$

Lemma 5.4. (1) *If $o\in O$ then $\mathbf{C}_{\mathrm{PGL}_2(p)}(o)\cap\langle a\rangle=1$.*
(2) *If $Ho_1H=Ho_2H$ for $o_1,o_2\in O$, then o_1 and o_2 are conjugate under $\langle a\rangle$.*

Proof. Assume that $o\in O$ and $y\in\mathbf{C}_{\mathrm{PGL}_2(p)}(o)\cap\langle a\rangle$. Then $\mathrm{PSL}_2(p)=\langle o,H\rangle\leqslant\mathbf{C}_{\mathrm{PGL}_2(p)}(y)$, forcing that $y=1$. Thus (1) of the lemma follows.

Assume that $Ho_1H=Ho_2H$ for some $o_1,o_2\in O$. Then $o_2=xo_1y$ for some $x,y\in H$. If $xy=1$ then $x=y^{-1}$, and (2) follows. Suppose that $yx\neq 1$, and so $H=\langle yx\rangle$. Since o_2 is an involution, we have $xo_1yxo_1y=o_2^2=1$, yielding $o_1yxo_1=x^{-1}y^{-1}=(yx)^{-1}$. Then $T=\langle o_1,H\rangle=\langle o_1,yx\rangle\cong\mathrm{S}_3$, a contradiction. This completes the proof. □

By (1) of Lemma 5.4, if $o\in O$ then either $\mathbf{N}_{\mathrm{PGL}_2(p)}(H)\cap\mathbf{C}_{\mathrm{PGL}_2(p)}(o)=1$ or $o\in\mathbf{C}_{\mathrm{PGL}_2(p)}(a^ib)$ for some integer i. For the latter case, $o\in\mathbf{C}_T(a^ib)$ as $o\in T$. Define

$$O_1=\{o\in O\mid \exists\, i \text{ s.t. } o\in\mathbf{C}_T(a^{2i+1}b)\},$$
$$O_2=\{o\in O\mid \exists\, i \text{ s.t. } o\in\mathbf{C}_T(a^{2i}b)\}.$$

Clearly, $O_1\cap O_2=\emptyset$.

Lemma 5.5.

$$|O_1|=\begin{cases}\frac{(p+\varepsilon)(p+\eta-2|\varepsilon+\eta|)}{4} & \textit{if } p\not\equiv\pm1\ (\mathrm{mod}\ 10),\\ \frac{(p+\varepsilon)(p+\eta-2|\varepsilon+\eta|-8)}{4} & \textit{if } p\equiv\pm1\ (\mathrm{mod}\ 10).\end{cases}$$

Proof. Let $x\in\mathbf{C}_T(a^{2i+1}b)\setminus\{a^{2i+1}b\}$ be an involution. Then $x\in O_1$ if and only if $\langle x,H\rangle=T$, or equivalently, $\langle x,H,a^{2i+1}b\rangle=T$. Note that $\langle H,a^{2i+1}b\rangle\cong\mathrm{S}_3$. Suppose that $\langle x,H,a^{2i+1}b\rangle\neq T$. Inspecting the subgroups of T, we deduce that either $\langle x,H,a^{2i+1}b\rangle\leqslant\mathbf{N}_T(H)$, or $p\equiv\pm1\ (\mathrm{mod}\ 10)$ and $\langle x,H,a^{2i+1}b\rangle\cong\mathrm{A}_5$. The former case implies that x lies in the center of $\mathbf{N}_T(H)$, and then $\varepsilon=\eta$, $x=a^{\frac{p+\varepsilon}{2}}$ or $a^{\frac{p+\varepsilon}{2}}a^{2i+1}b$. Assume that the latter case occurs. Enumerating the subgroups A_5 which contain a given subgroup S_3, we deduce that $\langle H,a^{2i+1}b\rangle$ is contained exactly in two subgroups A_5. It follows that there exist exactly four choices of x with $\langle x,H,a^{2i+1}b\rangle\cong\mathrm{A}_5$. Thus

$$|\mathbf{C}_T(a^{2i+1}b)\cap O_1|=\begin{cases}\frac{p+\eta-2|\varepsilon+\eta|}{2} & \text{if } p\not\equiv\pm1\ (\mathrm{mod}\ 10),\\ \frac{p+\eta-2|\varepsilon+\eta|-8}{2} & \text{if } p\equiv\pm1\ (\mathrm{mod}\ 10).\end{cases}$$

Assume that $o \in \mathbf{C}_T(a^{2i+1}b) \cap \mathbf{C}_T(a^{2j+1}b) \cap O_1$. Then $o \in \mathbf{C}_T(a^{2(i-j)})$. If $a^{2(i-j)} \neq 1$ then $o \in \mathbf{C}_T(a^{2(i-j)}) = \mathbf{N}_T(H)$, which is impossible as $\langle o, H\rangle = T$. Thus $a^{2(i-j)} = 1$, and so $a^{2i+1}b = a^{2j+1}b$. This says that every $o \in O_1$ centralizes exactly one of $\frac{p+\varepsilon}{2}$ involutions $a^{2i+1}b$. Then $|O_1|$ is desired as in the lemma. □

Lemma 5.6. $|O_2| = \frac{(p+\varepsilon)(p-\eta-6)}{4}$.

Proof. Let $x \in \mathbf{C}_T(a^{2i}b)$ be an involution. Then $x \in O_2$ if and only if $\langle x, H\rangle = T$, or equivalently, $\langle x, H, a^{2i}b\rangle = \mathrm{PGL}_2(p)$. Note that $\langle H, a^{2i}b\rangle \cong \mathrm{S}_3$. Suppose that $\langle x, H, a^{2i}b\rangle \neq \mathrm{PGL}_2(p)$. Inspecting the subgroups of $\mathrm{PGL}_2(p)$, either $\langle x, H, a^{2i}b\rangle \leqslant \mathbf{N}_{\mathrm{PGL}_2(p)}(H)$, or $\langle x, H, a^{2i}b\rangle \cong \mathrm{S}_4$. The former case implies that either $\varepsilon = \eta$ and $x = a^{\frac{p+\varepsilon}{2}}$, or $\varepsilon \neq \eta$ and $x = a^{\frac{p+\varepsilon}{2}}a^{2i}b$. For $\langle x, H, a^{2i}b\rangle \cong \mathrm{S}_4$, enumerating the subgroups S_4 which contain a given subgroup S_3, we deduce that $\langle H, a^{2i}b\rangle$ is contained exactly in two subgroups S_4. Noting that $\langle x, H, a^{2i}b\rangle \cap T \cong \mathrm{A}_4$, it follows that there exist exactly two choices of x with $\langle x, H, a^{2i}b\rangle \cong \mathrm{S}_4$. Since $\mathbf{C}_T(a^{2i}b) \cong \mathrm{D}_{p-\eta}$, we have $|\mathbf{C}_T(a^{2i}b) \cap O_2| = \frac{p-\eta-6}{2}$. Similarly as in the proof of Lemma 5.5, it is easily shown that every $o \in O_2$ centralizes exactly one of $\frac{p+\varepsilon}{2}$ involutions $a^{2i}b$. Then $|O_2|$ is desired as in the lemma. □

It is easy to check that $|O_1| + |O_2| = \frac{p(p-\eta)}{2} - |S| = |O|$, and so $O = O_1 \cup O_2$. Clearly, O_1 and O_2 are invariant under the conjugation of $\langle a\rangle$, and so each of them is the union of some $\langle a\rangle$-conjugacy classes. Selecting a representative o from each $\langle a\rangle$-conjugacy class in O such that $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) \cap \mathbf{C}_{\mathrm{PGL}_2(p)}(o) = \langle ab\rangle$ or $\langle b\rangle$, we have a set O_0 of ω_0 involutions, where

$$\omega_0 = \begin{cases} \frac{p-|\varepsilon+\eta|-3}{2} & \text{if } p \not\equiv \pm 1 \pmod{10}, \\ \frac{p-|\varepsilon+\eta|-7}{2} & \text{if } p \equiv \pm 1 \pmod{10}. \end{cases}$$

Then O_0 consists of $\omega_0 - \frac{p-\eta-6}{4}$ involutions from O_1, and $\frac{p-\eta-6}{4}$ involutions from O_2.

Theorem 5.7. *Assume that $H \cong \mathbb{Z}_3$. Then Γ is isomorphic to one of ω_0 non-isomorphic symmetric cubic graphs, $\frac{p-\eta-6}{4}$ of them have automorphism group $T\langle \mathsf{conj}(b)_H\rangle \cong \mathrm{PGL}_2(p)$, and the others have automorphism group $\langle \hat{ab}\rangle \times T$.*

Proof. By the foregoing argument, $\Gamma \cong \mathsf{Cos}(T, H, 1, o)$ for some $o \in O_0$.

Let $o \in O_0$. Then $\mathrm{Aut}\mathsf{Cos}(T, H, 1, o) \geqslant \langle \hat{ab}\rangle \times T$ or $T\langle \mathsf{conj}(b)_H\rangle$ depending on $o \in O_1$ or $o \in O_2$, respectively. Pick an arbitrary element $z \in \mathbf{N}_{\mathrm{PGL}_2(p)}(H)\setminus H$ with $Hz^{-1}ozH = HoH$. We have $z^{-1}oz = xoy$ for some $x, y \in H$, and so $xoyxoy = 1$, yielding $oyxo = (yx)^{-1}$. If $yx \neq 1$ then $T = \langle o, H\rangle = \langle o, yx\rangle \cong \mathrm{S}_3$, a contradiction. Then $yx = 1$, i.e, $y = x^{-1}$. Thus $z^{-1}oz = xoy = xox^{-1}$, and so $(zx)^{-1}ozx = o$. By the choice of O_0, we have $\langle zx\rangle = \mathbf{N}_{\mathrm{PGL}_2(p)}(H) \cap \mathbf{C}_{\mathrm{PGL}_2(p)}(o) = \langle ab\rangle$ or $\langle b\rangle$. It follows that $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, HoH) = H\langle ab\rangle$ or $H\langle b\rangle$. Thus, by (5.2), $\mathrm{Aut}\mathsf{Cos}(T, H, 1, o) = \langle \hat{ab}\rangle \times T$ or $T\langle \mathsf{conj}(b)_H\rangle$.

By Lemma 5.4 and the choice of O_0, distinct elements in O_0 produce distinct coset graphs $\mathsf{Cos}(T, H, 1, o)$. Then, by Lemma 3.3, we have ω_0 non-isomorphic symmetric cubic graphs $\mathsf{Cos}(T, H, 1, o)$. This completes the proof. □

5.2. $|H| = 6$. Assume that $H \cong \mathrm{S}_3$. Then $p \equiv \pm 7 \pmod{16}$ by (5.1), $K \cong \mathbb{Z}_2$, and $o \in \mathbf{N}_T(K) = \mathbf{C}_T(K) \cong \mathrm{D}_{p+\eta}$. Since $o^2 \in K$, either o is an involution or o has order 4. Let

$$O = \{o \in \mathbf{C}_T(K) \mid o^2 \in K, \langle o, H\rangle = T\}.$$

Lemma 5.8. *O contains two inverse elements of order 4 and $|O|-2$ involutions, and*

$$|O| = \begin{cases} \frac{p+\eta-2|\varepsilon+\eta|}{2} - 2 & \text{if } p \not\equiv \pm 1 \pmod{10}, \\ \frac{p+\eta-2|\varepsilon+\eta|}{2} - 6 & \text{if } p \equiv \pm 1 \pmod{10}. \end{cases}$$

Proof. Let $S = \{x \in \mathbf{C}_T(K) \setminus K \mid x^2 \in K, \langle o, H\rangle \neq T\}$. Then $|S| + |O| = \frac{p+\eta+4}{2}$, and $S \cup O$ consists of two inverse elements of order 4 and $\frac{p+\eta}{2}$ involutions in $\mathbf{C}_T(K) \setminus K$.

Let $x \in S$. Then $\langle x, H\rangle \cong \mathrm{D}_m$, S_4, or A_5 (if $p \equiv \pm 1 \pmod{10}$), where $m > 6$ is a divisor of $p+\varepsilon$ and divisible by 6. By the choice of x and inspecting the elements of D_m, S_4 and A_5, we deduce that x is an involution. By Lemma 5.1, all subgroups S_3 of T are conjugate in $\mathrm{PGL}_2(p)$. Enumerating the maximal subgroups of T which contain H, we deduce that H is contained exactly in one subgroup $\mathrm{D}_{p+\varepsilon}$, two subgroups S_4, and two subgroups A_5 if $p \equiv \pm 1 \pmod{10}$. Let L be a maximal subgroup of T with $\langle x, H\rangle \leqslant L$. If $L \cong \mathrm{D}_{p+\varepsilon}$ then $|S \cap L| = |\varepsilon + \eta|$. If $L \cong \mathrm{S}_4$ or A_5 then $|S \cap L| = 2$. We deduce that $|S| = |\varepsilon+\eta| + 8$ if $p \equiv \pm 1 \pmod{10}$, or $|S| = |\varepsilon + \eta| + 4$ otherwise. Then $|O|$ is given as in this lemma. Clearly, S consists of involutions. Then the lemma follows. □

Note that $Ko \subseteq O$ for $o \in O$. It follows that O is the union of $\frac{|O|}{2}$ cosets of K.

Lemma 5.9. *Let o, $o' \in O$. Then $Ho'H = HoH$ if and only of $Ko = Ko'$.*

Proof. Clearly, if $Ko' = Ko$ then $Ho'H = HoH$. Conversely, suppose that $Ho'H = HoH$ for distinct $o, o' \in O$. If o and o' are of order 4 then $K = \langle o^2\rangle$ and $o' \in \{o, o^{-1}\}$, we have $Ko = Ko'$. Thus, without loss of generality, we assume that o is an involution. Write $o = xo'y$ for some $x, y \in H$. Then $xo'yxo'y = o^2 = 1$, yielding $o'yxo' = (yx)^{-1}$.

If yx has order 3, then $o' \in \mathbf{N}_T(\langle yx\rangle) = \mathbf{N}_T(H)$, which contradicts that $\langle o', H\rangle = T$. Assume that $yx = 1$. Then $1 \neq y \notin K$, and $o = y^{-1}o'y \in \mathbf{C}_T(K) \cap \mathbf{C}_T(y^{-1}Ky)$. This implies that o centralizes $\langle K, y^{-1}Ky\rangle = H$. We have $\langle o, H\rangle \neq T$, a contradiction. Thus $yx \neq 1$. It follows that yx is an involution, and so $o' \in \mathbf{C}_T(yx)$. In addition, $yx \in K$ since, otherwise, o' centralizes $\langle K, yx\rangle = H$, which will give a contradiction.

Now we have $K = \langle yx\rangle$. Then $o = xo'y = y^{-1}(yx)o'y \in \mathbf{C}_T(K) \cap \mathbf{C}_T(y^{-1}Ky)$, and so o centralizes $\langle K, y^{-1}Ky\rangle$. If $y \notin K$ then $\langle K, y^{-1}Ky\rangle = H$, and so o centralizes $T = \langle o, H\rangle$, a contradiction. Then $y \in K$, and $x \in K$. Thus $o = xo'y = yxo' \in Ko'$, yielding $Ko = Ko'$. This completes the proof. □

Note that $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) \cong \mathrm{D}_{12}$, which has center of order 2. Let c be the involution in the center of $\mathbf{N}_{\mathrm{PGL}_2(p)}(H)$. Clearly, $o \in \mathbf{C}_{\mathrm{PGL}_2(p)}(K)$. Then $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, K) = \langle c\rangle \times K$, and $c \in T$ if and only if $\varepsilon = \eta$. Consider the conjugation of $\langle c\rangle$ on $\Omega := \{Ko \mid o \in O\}$.

Lemma 5.10. *The action of $\langle c\rangle$ on Ω produces $\frac{2+|\varepsilon+\eta|}{2}$ orbits of size 1, and $\frac{|O|-|\varepsilon+\eta|-2}{4}$ orbits of size 2.*

Proof. Pick an element $o_0 \in O$ of order 4. Then $co_0c = o_0^{-1}$, c fixes Ko_0, and $\langle o_0, c\rangle \cong \mathrm{D}_8$. It is easily shown that $\langle o_0, c\rangle \cap O = \{o_0, o_0^{-1}, o_0c, o_0^{-1}c\}$ or $\{o_0, o_0^{-1}\}$ depending on whether $\varepsilon = \eta$ or not. Note that $Ko_0 = Ko_0^{-1}$ and $Ko_0c = Ko_0^{-1}c$. It follows $\langle o_0, c\rangle$ contributes $\frac{2+|\varepsilon+\eta|}{2}$ fixed-points of $\langle c\rangle$ on Ω.

Now assume that Ko is fixed by $\langle c\rangle$, where $o \in O$. Then $Kcoc = Ko = Ko^{-1}$, yielding $coco \in K$, and so co has order 2 or 4. Recall that c, $o \in \mathbf{C}_{\mathrm{PGL}_2(p)}(K) \setminus K$ and $\mathbf{C}_{\mathrm{PGL}_2(p)}(K) \cong \mathrm{D}_{2(p+\eta)}$. If co has order 4 then $co \in \{o_0, o_0^{-1}\}$, and so $o \in \langle c, o_0\rangle$. Assume that co is an involution. Then either co or $o = cco$ is contained in the cyclic subgroup of $\mathbf{C}_{\mathrm{PGL}_2(p)}(K)$ of index 2. This implies that either co or o lies in $\langle o_0\rangle$, and hence $o \in \langle c, o_0\rangle$. Therefore, $\langle c\rangle$ has exactly $\frac{2+|\varepsilon+\eta|}{2}$ fixed-points on Ω. Since $\langle c\rangle \cong \mathbb{Z}_2$, every $\langle c\rangle$-orbit on Ω has length 1 or 2. Then the lemma follows. □

Choosing a coset Ko from each $\langle c\rangle$-orbit on Ω and a representative from Ko, we have a set O_1 of size

$$\omega_1 = \begin{cases} \frac{p+\eta}{8} & \text{if } p \not\equiv \pm 1 \pmod{10}, \\ \frac{p+\eta}{8} - 1 & \text{if } p \equiv \pm 1 \pmod{10}. \end{cases}$$

By the foregoing argument, the following statements hold:

(i) $\Gamma \cong \mathsf{Cos}(T, H, K, o)$ for some $o \in O_1$, and $HoH \neq Ho'H$ for distinct o, $o' \in O_1$;
(ii) O_1 contains a unique element of order 4, say o_0, and $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, Ho_0H) \geqslant \langle c\rangle \times K = \mathbf{N}_{\mathrm{PGL}_2(p)}(H, K)$;
(iii) if $o \in O_1$ is an involution then $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, K, HoH) = K$, except that $\varepsilon = \eta$, $Ko = Ko_0c$, and $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, Ho_0H) \geqslant \langle c\rangle \times K = \mathbf{N}_{\mathrm{PGL}_2(p)}(H, K)$.

Lemma 5.11. *Let $o \in O_1$. Then $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, HoH) = K$, except that*

(1) *$o = o_0$, in this case, $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, HoH) = K \times \langle c\rangle$; and*
(2) *$\eta = \varepsilon$ and $Ko = Ko_0c$, in this case, $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, HoH) = K \times \langle c\rangle$.*

Proof. Let g be an arbitrary element in $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, HoH) \setminus H$. Noting that $Hg^{-1}ogH = HoH$, by Lemma 5.9, $Ko = Kg^{-1}og$. Then $\langle Kg^{-1}og\rangle = \langle Ko\rangle = \langle o\rangle \times K$. This implies that $g^{-1}og \in \mathbf{C}_T(K)$, and so $o \in \mathbf{C}_T(gKg^{-1})$. Then o centralizes $\langle K, gKg^{-1}\rangle$. Since $\langle o, H\rangle = T$ and $\langle K, gKg^{-1}\rangle \leqslant H$, we have $K = gKg^{-1}$, i.e., $g \in \mathbf{N}_{\mathrm{PGL}_2(p)}(K)$. Thus $g \in \mathbf{N}_{\mathrm{PGL}_2(p)}(H, K, HoH)$. Then the lemma follows from (ii) and (iii) listed as above. □

Theorem 5.12. *Assume that $H \cong \mathrm{S}_3$. Then Γ is isomorphic to one of ω_1 non-isomorphic symmetric cubic graphs, and $\mathrm{Aut}\Gamma = \mathrm{PSL}_2(p)$ except that*

(1) *$\Gamma \cong \mathsf{Cos}(T, H, K, o_0)$, and $\mathrm{Aut}\Gamma = \mathbb{Z}_2 \times \mathrm{PSL}_2(p)$ or $\mathrm{PGL}_2(p)$ depending on whether $\eta = \varepsilon$ or not; and*
(2) *$\eta = \varepsilon$, $\Gamma \cong \mathsf{Cos}(T, H, K, o_0c)$, and $\mathrm{Aut}\Gamma = \mathbb{Z}_2 \times \mathrm{PSL}_2(p)$.*

Proof. Recall that $\Gamma \cong \mathsf{Cos}(T, H, K, o)$ for some $o \in O_1$. By (5.2) and Lemma 5.11, we deduce that $\mathrm{Aut}\Gamma$ is described as in this lemma. Then it suffices to show that if $\mathsf{Cos}(T, H, K, o) \cong \mathsf{Cos}(T, H, K, o')$ for o, $o' \in O_1$ then $o = o'$.

Suppose that $\mathsf{Cos}(T, H, K, o) \cong \mathsf{Cos}(T, H, K, o')$ for some o, $o' \in O_1$. By Lemma 5.11, we deduce from (5.2) that $A := \mathrm{Aut}\mathsf{Cos}(T, H, K, o) = \mathrm{Aut}\mathsf{Cos}(T, H, K, o')$. It follows from Lemma 3.3 that $Hg^{-1}ogH = Ho'H$ for some $g \in \mathbf{N}_{\mathrm{PGL}_2(p)}(H)$. By Lemma 5.9, $Kg^{-1}og = Ko'$, which forces that $g^{-1}og$ centralizes K. Then o centralizes $\langle K, gKg^{-1}\rangle$. Noting that $\langle K, gKg^{-1}\rangle \leqslant H$ and $\langle o, H\rangle = T$, we have $K = gKg^{-1}$, and so $g \in \mathbf{N}_{\mathrm{PGL}_2(p)}(H, K)$. By the choice of O_1, we have $o = o'$, and the result follows. □

5.3. $|H| = 12$**.** Assume that $H \cong \mathrm{D}_{12}$. Then $p \equiv \pm 15 \pmod{32}$ by (5.1), and $\varepsilon = \eta$. This implies that $p \equiv \pm 47 \pmod{96}$. Since $K \cong \mathbb{Z}_2^2$, by [17, II.8.16], $\mathbf{N}_T(K) \cong \mathrm{S}_4$, and thus o is either an involution or of order 4. Clearly, o lies in some Sylow 2-subgroup of $\mathbf{N}_T(K)$.

Theorem 5.13. *Assume that $H \cong \mathrm{D}_{12}$. Then Γ is isomorphic to a unique symmetric cubic graph, which has automorphism group $\mathrm{PSL}_2(p)$.*

Proof. By the choice of η, we know that $p+\eta$ is divisible by 4, and so $p-\eta$ is indivisible by 4. Noting that $(p+\eta)(p-\eta) = p^2-1 \equiv 0 \pmod{32}$, we have $p \equiv -\eta \pmod{16}$. Thus $p+\varepsilon = p+\eta$ is divisible by 16. We have $\mathbf{N}_T(H) \cong \mathrm{D}_{24}$ and $\mathbf{N}_T(H,K) \cong \mathrm{D}_8$. Let $P := \mathbf{N}_T(H,K)$, P_0 and P_1 be the three Sylow 2-subgroups of $\mathbf{N}_T(K)$. It is easily shown that there exists an involution $x \in P \setminus K$ such that $xP_0x = P_1$. Pick an involution $o_0 \in P_0 \setminus K$. Suppose that $\langle o_0, H\rangle \neq T$. Inspecting the subgroups of $\mathrm{PSL}_2(p)$, we deduce that $\langle o_0, H\rangle \leqslant \mathbf{N}_T(H)$. Then $o_0 \in \mathbf{N}_T(H,K) = P$, and so $P_0 = \langle o_0, K\rangle \leqslant P$, a contradiction. Thus $\langle o_0, H\rangle = T$. Recalling that $o \in P \cup P_0 \cup P_1$, since $\langle o, H\rangle = T$, we have $o \in P_0 \cup P_1$. Then $HoH = Ho_0H$ or Hxo_0xH. Since $x \in \mathbf{N}_T(H)$, we have $\mathsf{Cos}(T,H,K,o_0) \cong \mathsf{Cos}(T,H,K,xo_0x)$, and so $\Gamma \cong \Sigma := \mathsf{Cos}(T,H,K,o_0)$.

Choose a maximal subgroup L of $\mathrm{PGL}_2(p)$ with $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) \leqslant L$. Then $L \cong \mathrm{D}_{2(p+\varepsilon)}$, and $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) = \mathbf{N}_L(H) \cong \mathrm{D}_{24}$. Recalling that $\mathbf{N}_T(H) \cong \mathrm{D}_{24}$, we have $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) = \mathbf{N}_T(H)$. Then $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) = HP = H\langle x\rangle$. By (5.2), we deduce that $\mathrm{Aut}\Sigma = T\langle \mathsf{conj}(x)\rangle$ or T depending on whether $Hxo_0xH = Ho_0H$ or not.

Suppose that $\mathrm{Aut}\Sigma = T\langle \mathsf{conj}(x)\rangle$. Then $\mathrm{Aut}\Sigma = T \times \langle \hat{x}\rangle$, where $\hat{x}$ is defined as in (3.3). Let $M = \langle \hat{x}\rangle$, and consider the quotient graph Σ_M. Let $\bar{T}$ be the subgroup of $\mathrm{Aut}\Sigma_M$ induced by T. Then Σ_M is a $\bar{T}$-symmetric cubic graph of square-free order n. Let $\bar{v}$ be the M-orbit on $[T:H]$ containing $v := H$. We have $n = |\bar{T} : \bar{T}_{\bar{v}}|$. Since $\bar{T} \cong \mathrm{PSL}_2(p)$ has order divisible by 16, it follows that $|\bar{T}_{\bar{v}}|$ is divisible by 8. By Lemma 2.1, $\bar{T}_{\bar{v}} \cong \mathrm{S}_4$, and so $T_{\bar{v}} \cong \mathrm{S}_4$ by (2.1). By (2.2), T_v has index 2 in $T_{\bar{v}}$, forcing $T_v \cong \mathrm{A}_4$, which is impossible as Σ is T-symmetric. Therefore, $\mathrm{Aut}\Sigma = T$, and our result follows. □

5.4. $|H| = 24$**.** Assume that $H \cong \mathrm{S}_4$. Then $p \equiv \pm 31 \pmod{64}$ by (5.1). In this case, H is maximal in T, $K \cong \mathrm{D}_8$ and $\mathbf{N}_G(K) \cong \mathrm{D}_{16}$. Fix an involution $o_0 \in \mathbf{N}_G(K) \setminus K$. We have $\langle H, o_0\rangle = T$, and $H\mathbf{N}_G(K)H = H \cup Ho_0H$. Then $\Gamma \cong \mathsf{Cos}(T,H,K,o_0)$. Checking the subgroups of $\mathrm{PGL}_2(p)$, we deduce that $\mathbf{N}_{\mathrm{PGL}_2(p)}(H) = H$, and so $\mathbf{N}_{\mathrm{PGL}_2(p)}(H, Ho_0H) = \mathbf{N}_T(H, Ho_0H) = H$. Then we have the following result.

Theorem 5.14. *Assume that $H \cong \mathrm{S}_4$. Then Γ is isomorphic to a unique symmetric cubic graph, which has automorphism group $\mathrm{PSL}_2(p)$.*

6. $\mathrm{PSL}_2(p)$-semisymmetric graphs

In this section, $\Gamma = (V,E)$ is a connected T-semisymmetric cubic graph of order $2n$, where $T = \mathrm{PSL}_2(p)$ for some prime $p \geqslant 5$, and n is even and square-free. Choose $\varepsilon, \eta \in \{1,-1\}$ with $p+\varepsilon$ and $p+\eta$ divisible by 3 and 4, respectively.

Let $\{u,w\} \in E$. By Lemma 2.1 and inspecting the subgroups of $\mathrm{PSL}_2(p)$, we may assume that $(T_u, T_w) \cong (\mathrm{S}_3,\mathrm{S}_3)$, $(\mathrm{D}_{12},\mathrm{D}_{12})$, $(\mathrm{S}_4,\mathrm{S}_4)$, $(\mathrm{S}_3,\mathbb{Z}_6)$, $(\mathrm{D}_{12},\mathrm{A}_4)$ or $(\mathrm{S}_4,\mathrm{D}_{24})$. By Lemma 3.10, $\Gamma \cong \mathsf{BC}(T,L,R)$, where $L \cong T_u$ and $R \cong T_w$. Note that $|T:L| = n$ is even and square-free. We have

$$p \equiv 2^{i+1} \pm 1 \pmod{2^{i+2}} \text{ and } |L| = 2^i \cdot 3 \text{ for } 1 \leqslant i \leqslant 3. \tag{6.1}$$

In addition, $\eta = \varepsilon$ if L or R has a subgroup isomorphic to $\mathbb{Z}_6$.

It follows from Lemma 5.2 that T contains at most two conjugacy classes of subgroup isomorphic to L, and these subgroups are conjugate in $\mathrm{PGL}_2(p)$. Then, up to isomorphism of graphs, we may fix a subgroup L. Note that $L \cap R$ is a Sylow 2-subgroup of

L, and $\mathsf{BC}(T, L, R) \cong \mathsf{BC}(T, L, h^{-1}Rh)$ for $h \in L$. Thus, fixing a Sylow 2-subgroup P of L, one of our main tasks is to determine those subgroups R with $|R| = |L|$, $L \cap R = P$ and $\langle L, R\rangle = T$. Put

$$\mathcal{R} = \{R < T \mid |R| = |L|, L \cap R = P\}.$$

Lemma 6.1. *Let $L \cong R < T$. Then $R \in \mathcal{R}$ if and only if $R = z^{-1}Lz$ for some $z \in \mathbf{N}_{\mathrm{PGL}_2(p)}(P) \setminus \mathbf{N}_{\mathrm{PGL}_2(p)}(L, P)$.*

Proof. The sufficiency is trivial. Now assume that $L \cong R \in \mathcal{R}$. By Lemma 5.2, L and R are conjugate in $\mathrm{PGL}_2(p)$. Then $R = x^{-1}Lx$ for some $x \in \mathrm{PGL}_2(p)$. We have P, $xPx^{-1} \leqslant L$, and so $xPx^{-1} = y^{-1}Py$ for some $y \in L$. Then $yx \in \mathbf{N}_{\mathrm{PGL}_2(p)}(P)$, and so $x = y^{-1}z$ for some $z \in \mathbf{N}_{\mathrm{PGL}_2(p)}(P)$. Thus $R = x^{-1}Lx = z^{-1}Lz$. Since $L \cap R = P \neq L$, we know that L is not normalized by z, and so $z \in \mathbf{N}_{\mathrm{PGL}_2(p)}(P) \setminus \mathbf{N}_{\mathrm{PGL}_2(p)}(L, P)$. Then the lemma follows. □

6.1. $|L| = 6$**.** Assume that $L \cong \mathrm{S}_3$. Then $p \equiv \pm 3 \pmod 8$ by (6.1), $\mathbf{N}_{\mathrm{PGL}_2(p)}(L) \cong \mathrm{D}_{12}$, $P \cong \mathbb{Z}_2$ and $\mathbf{N}_{\mathrm{PGL}_2(p)}(P) = \mathbf{C}_{\mathrm{PGL}_2(p)}(P) \cong \mathrm{D}_{2(p+\eta)}$. Clearly, the center of $\mathbf{N}_{\mathrm{PGL}_2(p)}(L)$ has order 2 and is contained in $\mathbf{C}_{\mathrm{PGL}_2(p)}(P)$. Write

$$\mathbf{C}_{\mathrm{PGL}_2(p)}(P) = \langle a, c\rangle,$$

where a has order $p + \eta$ and c generates the center of $\mathbf{N}_{\mathrm{PGL}_2(p)}(L)$. Then

$$P = \langle a^{\frac{p+\eta}{2}}\rangle,\ \mathbf{N}_{\mathrm{PGL}_2(p)}(L, P) = \langle c, a^{\frac{p+\eta}{2}}\rangle \cong \mathbb{Z}_2^2.$$

In addition, $c \in T$ if and only if $\varepsilon = \eta$.

Lemma 6.2. *If $\varepsilon \neq \eta$ then $\mathcal{R} = \{a^{-i}La^i \mid 1 \leqslant i < \frac{p+\eta}{2}\}$, if $\varepsilon = \eta$ then $\mathcal{R} = \{\langle a^{\frac{p+\eta}{6}}\rangle\} \cup \{a^{-i}La^i \mid 1 \leqslant i < \frac{p+\eta}{2}\}$.*

Proof. Recalling that $P = \langle a^{\frac{p+\eta}{2}}\rangle$, we have $P < a^{-i}La^i$ for an arbitrary integer i. If $i \equiv j \pmod{\frac{p+\eta}{2}}$ then it is easily shown that $a^{-i}La^i = a^{-j}La^j$. Conversely, suppose that $a^{-i}La^i = a^{-j}La^j$ for some integers i and j. Then $a^{i-j} \in \mathbf{N}_{\mathrm{PGL}_2(p)}(L) \cap \mathbf{N}_{\mathrm{PGL}_2(p)}(P) = \mathbf{N}_{\mathrm{PGL}_2(p)}(L, P) = \langle c, P\rangle$. This implies that $a^{i-j} \in P$, and so $i \equiv j \pmod{\frac{p+\eta}{2}}$. By Lemma 6.1, all members S_3 of $\mathcal{R}$ are contained in $\{a^{-i}La^i \mid 1 \leqslant i < \frac{p+\eta}{2}\}$.

Assume that $R \in \mathcal{R}$ and $R \not\cong \mathrm{S}_3$. Then $R \cong \mathbb{Z}_6$, and so $R < \mathbf{C}_{\mathrm{PGL}_2(p)}(P) = \langle a, c\rangle \cong \mathrm{D}_{2(p+\eta)}$. In particular, $p + \eta$ is divisible by 3, and so $\varepsilon = \eta$. Note that $\mathrm{D}_{2(p+\eta)}$ has a unique subgroup $\mathbb{Z}_6$, which is generated by $a^{\frac{p+\eta}{6}}$. Then the lemma follows. □

Lemma 6.3. *Let $R_i = a^{-i}La^i$ for $1 \leqslant i < \frac{p+\eta}{2}$, and $R_0 = \langle a^{\frac{p+\eta}{6}}\rangle$ if further $\varepsilon = \eta$. Then*

(1) $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_0\}) = \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R_0) = \langle a^{\frac{p+\eta}{2}}, c\rangle < T$, *in this case,* $\varepsilon = \eta$*;*
(2) $\mathbf{N}_{\mathrm{PGL}_2(p)}(L, R_i) = P$ *and* $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_i\}) = \langle a^{\frac{p+\eta}{2}}, a^i c\rangle$, *where* $i \neq \frac{p+\eta}{4}$ *and* $1 \leqslant i < \frac{p+\eta}{2}$.
(3) $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_{\frac{p+\eta}{4}}\}) = \langle a^{\frac{p+\eta}{4}}, c\rangle$, *and* $\mathbf{N}_{\mathrm{PGL}_2(p)}(L, R_{\frac{p+\eta}{4}}) = \langle a^{\frac{p+\eta}{2}}, c\rangle$.

Proof. Clearly, $|\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R\}) : \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R)| \leqslant 2$, and if the equality holds then $R \cong \mathrm{S}_3$. In particular, since $L \not\cong R_0$, we have $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_0\}) = \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R_0)$. Recall that $\mathbf{N}_{\mathrm{PGL}_2(p)}(L) = L \times \langle c\rangle$. If $\varepsilon = \eta$ then $c \in T$ and, noting that $\mathbf{N}_{\mathrm{PGL}_2(p)}(R_0) = \mathbf{C}_{\mathrm{PGL}_2(p)}(P)$, we have $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_0\}) = \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R_0) = \langle a^{\frac{p+\eta}{2}}, c\rangle$, desired as in (1).

Now let $R = R_i$, where $1 \leqslant i < \frac{p+\eta}{2}$. Note that $P \leqslant \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R) \leqslant \mathbf{N}_{\mathrm{PGL}_2(p)}(L, P) = \langle a^{\frac{p+\eta}{2}}, c\rangle \cong \mathbb{Z}_2^2$. If $R = R_{\frac{p+\eta}{4}}$ then $cRc = ca^{-\frac{p+\eta}{4}}La^{\frac{p+\eta}{4}}c = a^{\frac{p+\eta}{4}}La^{-\frac{p+\eta}{4}} = a^{-\frac{p+\eta}{4}}La^{\frac{p+\eta}{4}} = R$, and so $\mathbf{N}_{\mathrm{PGL}_2(p)}(L, R) = \langle a^{\frac{p+\eta}{2}}, c\rangle$. Suppose that $\mathbf{N}_{\mathrm{PGL}_2(p)}(L, R) = \langle a^{\frac{p+\eta}{2}}, c\rangle$. Then $a^{-i}La^i = R = cRc = ca^{-i}La^ic = a^iLa^{-i}$, and so $a^{-2i}La^{2i} = L$. This implies that $2i \equiv 0 \pmod{\frac{p+\eta}{2}}$, yielding $i = \frac{p+\eta}{4}$. Thus $\mathbf{N}_{\mathrm{PGL}_2(p)}(L, R) = \langle a^{\frac{p+\eta}{2}}, c\rangle$ if and only if $R = R_{\frac{p+\eta}{4}}$. Noting that $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R\}) = \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R)\langle a^ic\rangle$, we obtain (2) or (3). Then the lemma follows. □

Lemma 6.4. *Let $R \in \mathcal{R}$. Then either $\langle L, R\rangle = T$, or $p \equiv \pm 1 \pmod{10}$ and $\langle L, R\rangle \cong \mathrm{A}_5$. For the latter case, $R = a^{-i}La^i$ or $a^{-(\frac{p+\eta}{2}-i)}La^{\frac{p+\eta}{2}-i}$ for a unique i with $1 < i < \frac{p+\eta}{2}$, $i \neq \frac{p+\eta}{4}$ and $a^ic \in T$; in particular, i is odd or even depending on whether $\varepsilon = \eta$ or not.*

Proof. Assume that $\langle L, R\rangle \neq T$. Inspecting the subgroups of $\mathrm{PSL}_2(p)$, we deduce that either $\langle L, R\rangle$ is isomorphic to a subgroup of $\mathrm{D}_{p+\varepsilon}$, or $p \equiv \pm 1 \pmod{10}$ and $\langle L, R\rangle \cong \mathrm{A}_5$. For the former case, noting that $\mathrm{D}_{p+\varepsilon}$ has a unique subgroup of order 3, we have $|L \cap R| \geqslant 3$, a contradiction. Then the latter case occurs; in particular, L and R are conjugate in T. It is easily shown that for each subgroup of A_5 that isomorphic to S_3, there exists a unique subgroup isomorphic to S_3 such that their intersection is a subgroup of order 2. Then R is uniquely determined by L in $\langle L, R\rangle$. Enumerating the subgroups A_5 of T which contain L, it follows that L is contained exactly in two subgroups A_5. Then R has exactly two choices.

Fix an $R \in \mathcal{R}$ with $\langle L, R\rangle \cong \mathrm{A}_5$. Then $cRc \in \mathcal{R}$ and $\langle L, cRc\rangle \cong \mathrm{A}_5$. Write $R = a^{-i}La^i$, where $1 \leqslant i < \frac{p+\eta}{2}$. Then $cRc = a^{-(\frac{p+\eta}{2}-i)}La^{\frac{p+\eta}{2}-i}$. By (2) and (3) of Lemma 6.3, the involution a^ic normalizes $\langle L, R\rangle$. Noting that $\mathrm{PGL}_2(p)$ has no proper subgroup isomorphic to S_5 or $\mathbb{Z}_2 \times \mathrm{A}_5$, it follows that $a^ic \in \langle L, R\rangle < T$. Suppose that $i = \frac{p+\eta}{4}$. Noting that $a^{\frac{p+\eta}{4}} \notin T$, we have $c \notin T$. By (3) of Lemma 6.3, c normalizes $\langle L, R\rangle$. Then $\langle L, R, c\rangle \cong \mathrm{S}_5$ or $\mathbb{Z}_2 \times \mathrm{A}_5$, which is impossible. Thus $i \neq \frac{p+\eta}{4}$, and the lemma follows. □

Define

$$\nu_1 = \begin{cases} \frac{p+\eta+2|\varepsilon+\eta|}{4} & \text{if } p \not\equiv \pm 1 \pmod{10}, \\ \frac{p+\eta+2|\varepsilon+\eta|}{4} - 1 & \text{if } p \equiv \pm 1 \pmod{10}. \end{cases}$$

Theorem 6.5. *Assume that $L \cong \mathrm{S}_3$. Then Γ is isomorphic to one of ν_1 non-isomorphic connected edge-transitive cubic bipartite graphs described as follows:*

(1) *$\frac{|\varepsilon+\eta|}{2}$ semisymmetric graphs with automorphism group isomorphic to $\mathbb{Z}_2 \times T$;*
(2) *a unique symmetric graph with automorphism graph isomorphic to $\mathbb{Z}_2 \times \mathrm{PGL}_2(p)$;*
(3) *$\nu_1 - 1 - \frac{|\varepsilon+\eta|}{2}$ non-isomorphic symmetric graphs, $\frac{p+\eta-4}{8}$ of these graphs have automorphism group isomorphic to $\mathrm{PGL}_2(p)$, and the others have automorphism group isomorphic to $\mathbb{Z}_2 \times T$.*

Proof. Let $R_0, R_1, \ldots, R_{\frac{p+\eta}{2}-1}$ be defined as in Lemma 6.3. Put $I = \{0, 1, 2 \ldots, \frac{p+\eta}{2}-1\}$, and choose an $i_0 \in I$ with $\langle L, R_{i_0}\rangle \cong \mathrm{A}_5$. For each $i \in I$, by Lemma 6.4, $\langle L, R_i\rangle = T$ if and only if $i \in I_0 := I \setminus \{i_0, \frac{p+\eta}{2} - i_0\}$. Then $|I_0| = 2\nu_1 - 1 - \frac{|\varepsilon+\eta|}{2}$, and we get $|I_0|$ distinct connected T-semisymmetric cubic graphs $\Gamma_i := \mathsf{BC}(T, L, R_i)$, where i runs over I_0. Moreover, $\Gamma \cong \Gamma_i$ for some $i \in I_0$.

By Theorem 2.10, since Γ_i is T-semisymmetric, T is the unique insolvable minimal normal subgroup of $\mathrm{Aut}\Gamma_i$. In particular, by Lemma 3.8, $\mathrm{Aut}\Gamma_i = T\{\mathsf{conj}(g)_{\{L,R\}} \mid g \in$

$\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L,R_i\})\}$. Let $c_i = a^i c$. It follows from Lemmas 3.9 and 6.3 that

$$\mathrm{Aut}\Gamma_i = \begin{cases} T\times\langle\hat{c}\tilde{c}\rangle \cong T\times\mathbb{Z}_2 & \text{if } \varepsilon=\eta,\ i=0;\\ T\langle\mathsf{conj}(c)_{\{L,R_i\}}\rangle\times\langle\hat{c}_i\tilde{c}_i\rangle \cong \mathrm{PGL}_2(p)\times\mathbb{Z}_2 & \text{if } \varepsilon\neq\eta,\ i=\frac{p+\eta}{4};\\ T\langle\mathsf{conj}(c_i)_{\{L,R_i\}}\rangle\times\langle\hat{c}\tilde{c}\rangle \cong \mathrm{PGL}_2(p)\times\mathbb{Z}_2 & \text{if } \varepsilon=\eta,\ i=\frac{p+\eta}{4};\\ T\times\langle\hat{c}_i\tilde{c}_i\rangle \cong T\times\mathbb{Z}_2 & \text{if } i\neq\frac{p+\eta}{4}, i+\frac{\varepsilon+\eta}{2} \text{ is odd};\\ T\langle\mathsf{conj}(c_i)_{\{L,R_i\}}\rangle \cong \mathrm{PGL}_2(p) & \text{if } i\neq\frac{p+\eta}{4}, i+\frac{\varepsilon+\eta}{2} \text{ is even}.\end{cases}$$

Clearly, $\Gamma_0 \not\cong \Gamma_{\frac{p+\eta}{4}}$, and if $i\in I_1 := I_0\setminus\{0,\frac{p+\eta}{4}\}$ then $\Gamma_i\not\cong\Gamma_0$ or $\Gamma_{\frac{p+\eta}{4}}$. Thus, it remains to consider the isomorphisms among $2\nu_1-2-|\varepsilon+\eta|$ graphs Γ_i, where $i\in I_1$.

Let $I_2=\{i\in I_1 \mid \mathrm{Aut}\Gamma_i\cong\mathrm{PGL}_2(p)\}$ and $I_3=I_1\setminus I_2$. Then $\Gamma_i\not\cong\Gamma_j$ for all $i\in I_2$ and $j\in I_3$. It is easily shown that $|I_2|=\frac{p+\eta}{4}-1$. Let $i,j\in I_2$ or I_3 with $i\neq j$. Recall that $\mathbf{N}_{\mathrm{PGL}_2(p)}(L,P)=\langle c, a^{\frac{p+\eta}{2}}\rangle$. It follows from Lemma 3.6 that $\Gamma_i\cong\Gamma_j$ if and only if $cR_ic=R_j$, i.e., $ca^{-i}La^ic=a^{-j}La^j$. Noting that $ca^{-i}La^ic=a^iLa^{-i}$, it is easily shown that $ca^{-i}La^ic=a^{-j}La^j$ if and only if $j\equiv p+\eta-i \pmod{\frac{p+\eta}{2}}$, see the proof of Lemma 6.2. Since $1\leqslant i,j<\frac{p+\eta}{2}$, if $j\equiv p+\eta-i \pmod{\frac{p+\eta}{2}}$ then $i+j=\frac{p+\eta}{2}$. Thus $\Gamma_i\cong\Gamma_j$ if and only if $i+j=\frac{p+\eta}{2}$. On the other hand, it is easy to check that $I_2=\{\frac{p+\eta}{2}-i \mid i\in I_2\}$ and $I_3=\{\frac{p+\eta}{2}-i \mid i\in I_3\}$. Then we have $\frac{|I_2|}{2}$ or $\frac{|I_3|}{2}$ non-isomorphic graphs Γ_i when i runs over I_2 or I_3, respectively. This completes the proof. □

6.2. $|L|=12$. Assume that $L\cong \mathrm{D}_{12}$. Then $p\equiv\pm7 \pmod{16}$ and $\varepsilon=\eta$, see (6.1). In addition, $R\cong\mathrm{D}_{12}$ or A_4, and $P\cong\mathbb{Z}_2^2$. It is easily shown that $\mathbf{N}_{\mathrm{PGL}_2(p)}(P)=\mathbf{N}_T(P)\cong \mathrm{S}_4$, $\mathbf{N}_{\mathrm{PGL}_2(p)}(L)=\mathbf{N}_T(L)\cong\mathrm{D}_{24}$, and $\mathbf{N}_{\mathrm{PGL}_2(p)}(L,R)\leqslant\mathbf{N}_T(L,P)\cong\mathrm{D}_8$. Write $\mathbf{N}_T(P)=P{:}\langle a,b\rangle$, where a has order 3 and b is an involution such that $\mathbf{N}_T(L,P)=P:\langle b\rangle$.

Lemma 6.6. *Assume that $L\cong\mathrm{D}_{12}$. Then $\mathcal{R}=\{P{:}\langle a\rangle, a^{-1}La, aLa^{-1}\}$.*

Proof. Let $R\in\mathcal{R}$. If $R\cong\mathrm{A}_4$ then $R\leqslant\mathbf{N}_{\mathrm{PGL}_2(p)}(P)=P{:}\langle a,b\rangle$, yielding $R=P{:}\langle a\rangle$. Suppose that $R\cong\mathrm{D}_{12}$. Then $R=x^{-1}Lx$ for some $x\in\mathrm{PGL}_2(p)$. We have P, $xPx^{-1}\leqslant L$, and so $xPx^{-1}=y^{-1}Py$ for some $y\in L$. Then $yx\in\mathbf{N}_{\mathrm{PGL}_2(p)}(P)=P{:}\langle a,b\rangle$. It follows that $R=x^{-1}Lx=z^{-1}Lz$ for some $z\in\langle a,b\rangle$. Noting that $bLb=L$, we have $R=P{:}\langle a\rangle$, $a^{-1}La$ or aLa^{-1}. Clearly, $P{:}\langle a\rangle\neq a^{-1}La$ or aLa^{-1}. If $a^{-1}La=aLa^{-1}$ then $a\in\mathbf{N}_T(L)$, yielding $\mathrm{A}_4\cong P{:}\langle a\rangle\leqslant\mathbf{N}_T(L)\cong\mathrm{D}_{24}$, a contradiction. Then the lemma follows. □

Theorem 6.7. *Assume that $L\cong\mathrm{D}_{12}$. Then Γ is isomorphic to one of two edge-transitive cubic graphs with automorphism group isomorphic to $T\times\mathbb{Z}_2$, one of them is semisymmetric and the other one is symmetric.*

Proof. Inspecting the subgroups of T, we deduce that $\langle L,R\rangle=T$ for all $R\in\mathcal{R}$. Up to isomorphism of graphs, write $\Gamma=\mathsf{BC}(T,L,R)$ for some $R\in\mathcal{R}$. By Theorem 2.10 and Lemma 3.8, we have $\mathrm{Aut}\Gamma=T\{\mathsf{conj}(g)_{\{L,R\}}\mid g\in\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L,R\})\}$.

Assume that $R=P{:}\langle a\rangle$. Then $L\not\cong R$, and so $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L,R\})=\mathbf{N}_{\mathrm{PGL}_2(p)}(L,R)$. We have $P{:}\langle b\rangle\leqslant\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L,R\})=\mathbf{N}_{\mathrm{PGL}_2(p)}(L,R)\leqslant\mathbf{N}_T(L,P)=P{:}\langle b\rangle$, yielding $\mathbf{N}_{\mathrm{PGL}_2(p)}(L,R)=P{:}\langle b\rangle<T$. Then $\mathrm{Aut}\Gamma=T\times\langle\hat{b}\tilde{b}\rangle$, and Γ is semisymmetric.

Assume that $R\neq P{:}\langle a\rangle$. Noting that $ba^{-1}Lab=aLa^{-1}$, we have $\mathsf{BC}(T,L,a^{-1}La)\cong\mathsf{BC}(T,L,aLa^{-1})$. Thus, we may choose $R=a^{-1}La$. Note that $P\leqslant\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L,R\})\leqslant\mathbf{N}_{\mathrm{PGL}_2(p)}(P)=\mathbf{N}_T(P)=P{:}\langle a,b\rangle$. Calculation shows that $\mathbf{N}_{\mathrm{PGL}_2(p)}(L,R)=P$ and $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L,R\})=P\times\langle ba\rangle$. We get $\mathrm{Aut}\Gamma=T\langle\mathsf{conj}(ba)_{\{L,R\}}\rangle=T\times\langle ba\mathsf{conj}(ba)_{\{L,R\}}\rangle$.

Noting that $\mathsf{conj}(ba)_{\{L,R\}}$ interchanges two parts of Γ, it follows that Γ is symmetric. Then the result follows. □

6.3. $|L| = 24$. Assume that $L \cong \mathrm{S}_4$. Then $p \equiv \pm 15 \pmod{32}$ by (6.1). In addition, $\mathbf{N}_{\mathrm{PGL}_2(p)}(L) = L$, $P \cong \mathrm{D}_8$, and $\mathbf{N}_{\mathrm{PGL}_2(p)}(P) = \mathbf{N}_T(P) \cong \mathrm{D}_{16}$. For each $R \in \mathcal{R}$ we have $R \cong \mathrm{S}_4$ or D_{24}, and it is easily shown that $T = \langle L, R\rangle$. Note, if $R \cong \mathrm{D}_{24}$ then $\varepsilon = \eta$. Write $\mathbf{N}_T(P) = P{:}\langle b\rangle$, where b is an involution in T.

Let $R \in \mathcal{R}$. Since L is self-normalized in $\mathrm{PGL}_2(p)$, we have $R_1 := bLb \neq L$. If $R \cong \mathrm{S}_4$ then $R = R_1$ by Lemma 6.1. Assume that $R \cong \mathrm{D}_{24}$. Then $\varepsilon = \eta$, and $\mathbf{N}_{\mathrm{PGL}_2(p)}(R) = \mathbf{N}_T(R) \cong \mathrm{D}_{48}$. We deduce from Lemma 5.2 that T has two classes of subgroups D_8 and two classes subgroups D_{24}. Note that all subgroups D_8 in D_{24} are conjugate. It follows that, for the given pair (L, P), there exists a unique subgroup $R_0 < T$ with $R_0 \cong \mathrm{D}_{24}$ and $R_0 \cap L = P$. Thus $\mathcal{R} = \{R_0, R_1\}$.

Note that $\mathbf{N}_{\mathrm{PGL}_2(p)}(L) = L$ and $|\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_i\}) : \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R_i)| \leqslant 2$. We have $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_0\}) = \mathbf{N}_{\mathrm{PGL}_2(p)}(L, R_0) = P$, and $\mathbf{N}_{\mathrm{PGL}_2(p)}(\{L, R_1\}) = P{:}\langle b\rangle$. Then, by Theorem 2.10 and Lemma 3.8, we have the following result.

Theorem 6.8. *Assume that $L \cong \mathrm{S}_4$. Then Γ is isomorphic to one of two edge-transitive cubic graphs, one of them is semisymmetric with automorphism group $\mathrm{PSL}_2(p)$, and the other one is symmetric with automorphism group $\mathrm{PSL}_2(p) \times \mathbb{Z}_2$.*

7. Proof of Theorem 1.1

Let $\Gamma = (V, E)$ be a connected edge-transitive cubic graph of order $2n$ with n even and square-free, and let $A = \mathrm{Aut}\Gamma$. If A is solvable then $\Gamma \cong \mathsf{K}_4$ by Theorem 2.5. Assume that A is insolvable, and let $T = A^{(\infty)}$. By Theorem 2.10, either T is one of J_1 and $\mathrm{PSL}_2(p)$, or Γ is described as in Lines 1, 2 of Table 1 and Line 1 of Table 2. If $T = \mathrm{J}_1$ then Line 3 of Table 1 and Line 2 of Table 2 follow from Theorem 4.2. If $T = \mathrm{PSL}_2(p)$ and Γ is T-edge-transitive then we get Lines 4-10 of Table 1 by Theorems 5.7, 5.12-5.14, and Lines 3-10 of Table 2 by Theorems 6.5, 6.7 and 6.8.

In the following, we assume that $T = \mathrm{PSL}_2(p)$, and Γ is not T-edge-transitive. Fix an edge $\{u, w\} \in E$, and let $A^* = \langle A_u, A_w\rangle$. By Lemma 2.9, $|\mathsf{rad}(A^*)| \in \{3, 6\}$, Γ is $\mathsf{rad}(A^*)T$-edge-transitive, and one of the following holds:

(i) T is transitive on one part say W of Γ and has three orbits on the other part U;
(ii) T is regular on V, and $p \equiv \pm 3 \pmod 8$.

Let $M = \langle z\rangle$ be the unique Sylow 3-subgroup of $\mathsf{rad}(A^*)$, and put $G = MT$. For each $g \in \mathrm{PGL}_2(p)$, extend $\mathsf{conj}(g)$ to an automorphism of G by setting $y^{\mathsf{conj}(g)} = y$ for $y \in M$. Let $\mathrm{Aut}(M) = \langle \tau\rangle$, and extend τ to an automorphism of G by setting $x^\tau = x$ for $x \in T$. Then

$$\mathrm{Aut}(G) = \langle \tau\rangle \times \{\mathsf{conj}(g) \mid g \in \mathrm{PGL}_2(p)\}.$$

Clearly, G acts transitively on each A^*-orbit. This implies that Γ is G-edge-transitive. Let $\bar{T}$ be the subgroup of $\mathrm{Aut}\Gamma_M$ induced by T. For $v \in V$, let $\bar{v}$ be the M-orbit containing v. Then $T_{\bar{v}} \cong G_v \cong \bar{T}_{\bar{v}}$, see (2.1). We next discuss in two cases.

Case 1. Assume that (i) occurs, $u \in U$ and $w \in W$. Then $n = 3|T : T_u| = |T : T_w|$, and so $|T_u| = 3|T_w|$. Recall that $\bar{T}_{\bar{w}} \cong T_{\bar{w}}$, $T_w \trianglelefteq T_{\bar{w}}$ and $M \cong T_{\bar{w}}/T_w$, see (2.2). Since

$M \cong \mathbb{Z}_3$, it follows from Lemma 2.1 that either $G_w \cong \bar{T}_{\bar{w}} \cong \mathbb{Z}_6$ and $G_u \cong \bar{T}_{\bar{u}} \cong \mathrm{S}_3$, or $G_w \cong \bar{T}_{\bar{w}} \cong \mathrm{A}_4$ and $G_u \cong \bar{T}_{\bar{u}} \cong \mathrm{D}_{12}$, and so $T_w \cong \mathbb{Z}_2$ or $\mathbb{Z}_2^2$, respectively. In particular, $G_w \cap T = G_u \cap G_w = T_w$. Since $|T_u| = 3|T_w|$, we have $|T_u| = |T_{\bar{u}}|$. Then $T_u = T_{\bar{u}} \cong G_u$, yielding $G_u = T_{\bar{u}} < T$. It is easy to see that those subgroups of T isomorphic to $\bar{T}_{\bar{u}}$ are all conjugate under $\mathrm{Aut}(G)$. Up to isomorphism of graphs, we fix a subgroup $L < T$ and Sylow 2-subgroup P of L, and write $\Gamma \cong \mathsf{BC}(G, L, R)$, where $L \cong \bar{T}_{\bar{u}}$, $R \cong \bar{T}_{\bar{w}}$, $R \cap T = P$, and $\langle L, R\rangle = G$.

Noting that P is the unique Sylow 2-subgroup of R, we write $R = P{:}\langle yx\rangle$, where $y \in M$ and $x \in T$ with $\langle yx\rangle \cong \mathbb{Z}_3$. Since $\langle L, R\rangle = G$, we deduce that $M = \langle y\rangle$, and so $R = P{:}\langle zx\rangle$ or $P{:}\langle z^{-1}x\rangle$. Clearly, $\tau \in \mathrm{Aut}(G, L, P)$, and $(P{:}\langle zx\rangle)^\tau = P{:}\langle z^{-1}x\rangle$. Thus, up to isomorphism of graphs, we further choose $R = P{:}\langle zx\rangle$, and then Γ is determined completely by $R_0 := P{:}\langle x\rangle$.

Again by $\langle L, R\rangle = G$, we have that $\langle L, x\rangle = T$ and x has order 3. Then $\Gamma_0 := \mathsf{BC}(T, L, R_0)$ is a connected T-semisymmetric cubic graph, and $R_0 \cong R \cong G_w$. Conversely, if Γ_0 is connected then it is easily shown that $\mathsf{BC}(G, L, R)$ is also connected.

Let $A = \mathrm{Aut}\mathsf{BC}(G, L, R)$. Then T, $G \trianglelefteq A$ by Theorem 2.10. Noting that the normal subgroup T is transitive on one part of $\mathsf{BC}(G, L, R)$ but not transitive on the other one, it follows that $\mathsf{BC}(G, L, R)$ is semisymmetric. Further, by Lemma 3.8, we deduce that $A = G\{\sigma_{\{L,R\}} \mid \sigma \in \mathrm{Aut}(G, L, R)\}$. Clearly, $\mathrm{Aut}(G, L, R) \leqslant \langle\tau\rangle \times \mathrm{Aut}(T, L, R_0)$.

Suppose that $L \cong \mathrm{S}_3$ and $R \cong \mathbb{Z}_6$. By Lemma 6.2, $\varepsilon = \eta$, and R_0 is uniquely determined by L. By Lemma 6.3, we have $\mathrm{Aut}(G, L, R_0) = \{\mathsf{conj}(g) \mid g \in P \times \langle c\rangle\}$, where c generates the center of $\mathbf{N}_T(L)$ and $\langle R_0, c\rangle \cong \mathrm{D}_{12}$. Calculation shows that $\mathrm{Aut}(G, L, R) = \{\mathsf{conj}(g), \tau\mathsf{conj}(cg) \mid g \in P\}$. Noting that $\tau\mathsf{conj}(c)$ inverses z and centralizes T, we have $A = G\{\sigma_{\{L,R\}} \mid \sigma \in \mathrm{Aut}(G, L, R)\} \cong \mathrm{S}_3 \times T$, and then Γ is described as in Line 11 of Table 2.

Suppose that $L \cong \mathrm{D}_{12}$ and $R \cong \mathrm{A}_4$. Using Lemma 6.6 and Theorem 6.7, by a similar argument as above, we deduce that R_0 is uniquely determined by L, and $A \cong \mathrm{S}_3 \times T$. Then Γ is described as in Line 12 of Table 2.

Case 2. Assume that (ii) occurs. Then $G_v \cong \mathbb{Z}_3$, and $\Gamma \cong \mathsf{Cos}(G, H, 1, o)$, where o is an involution, $H \cong \mathbb{Z}_3$ and $\langle H, o\rangle = G$. Clearly, $o \in T$. Write $H = \langle yx\rangle$, where $y \in M$ and $x \in T$. Since $\langle yx, o\rangle = \langle H, o\rangle = G$, we deduce that $M = \langle y\rangle$, and $\langle x, o\rangle = T$. In particular, $\mathsf{Cos}(T, \langle x\rangle, 1, o)$ is a connect T-symmetric cubic graph. Conversely, for a connect T-symmetric cubic graph $\mathsf{Cos}(T, \langle x\rangle, 1, o')$, since $G = M \times T = \langle y\rangle \times T$, it is easily shown that $\langle yx, o'\rangle$ has a homomorphic image $\langle x, o'\rangle = T$. Then $|G : \langle yx, o'\rangle|$ is a divisor of $|G : T| = |M| = 3$, and hence either $G = \langle yx, o'\rangle$ or $|G : \langle yx, o'\rangle| = 3$. The latter case implies that $\langle yx, o'\rangle \cong T$ is simple, since $\langle yx, o'\rangle \nleqslant T$ and T is normal in G, we have $\langle yx, o'\rangle \cap T = 1$, and hence $3|T| = |G| \geqslant |T\langle yx, o'\rangle| = |T|^2$, yielding $|T| \leqslant 3$, a contradiction. Thus $G = \langle yx, o'\rangle$, and so $\mathsf{Cos}(G, H, 1, o')$ is connected.

Recalling that $\langle y\rangle = M = \langle z\rangle$, we have $y = z$ or z^{-1}. By the definition of τ, we have $y^\tau = y^{-1}$, $(yx)^\tau = y^{-1}x$, and $o^\tau = o$. Then $\mathsf{Cos}(G, H, 1, o) \cong \mathsf{Cos}(G, H^\tau, 1, o)$, see (III) in Subsection 3.2. Thus, up to isomorphism of graphs, we may choose $H = \langle zx\rangle$. Moreover, all elements of T with order 3 are all conjugate, this allows we fix an element $x \in T$ of order 3. Noting that $\mathsf{Cos}(T, \langle x\rangle, 1, o)$ is a connect T-symmetric cubic graph, the argument in Subsection 5.1 is available for $\mathsf{Cos}(T, \langle x\rangle, 1, o)$. In particular, we assume that $\mathsf{Cos}(T, \langle x\rangle, 1, o)$ is one of ω_0 non-isomorphic symmetric cubic graphs, $\frac{p-\eta-6}{4}$ of them have automorphism group $T\langle \mathsf{conj}(b)_{\langle x\rangle}\rangle \cong \mathrm{PGL}_2(p)$, and the others have automorphism group $\langle \hat{ab}\rangle \times T$, where ω_0, $o \in O_0$, a and b are defined as in Subsection 5.1.

Let $A = \mathrm{Aut}\mathsf{Cos}(G, H, 1, o)$. By Theorem 2.10, we have $T, G \trianglelefteq A$. It follows from Lemma 3.4 that $A = G\{\sigma_H \mid \sigma \in \mathrm{Aut}(G, H, HoH)\}$. Recall that $\mathrm{Aut}(G) = \langle\tau\rangle \times \{\mathsf{conj}(g) \mid g \in \mathrm{PGL}_2(p)\}$. It is easily shown that $\mathrm{Aut}(G, H, HoH) \leqslant \langle\tau\rangle\times\mathrm{Aut}(T, \langle x\rangle, \langle x\rangle o\langle x\rangle) = \langle\tau\rangle\times\{\mathsf{conj}(g) \mid g \in \mathbf{N}_{\mathrm{PGL}_2(p)}(\langle x\rangle, \langle x\rangle o\langle x\rangle)\}$. By calculation, see the proof of Theorem 5.7, we have $\mathbf{N}_{\mathrm{PGL}_2(p)}(\langle x\rangle, \langle x\rangle o\langle x\rangle) = \langle x\rangle\langle b\rangle$ or $\langle x\rangle\langle ab\rangle$ when $\mathrm{Aut}\mathsf{Cos}(T, \langle x\rangle, 1, o) \cong \mathrm{PGL}_2(p)$ or $\mathbb{Z}_2 \times \mathrm{PSL}_2(p)$, respectively. It follows that $\mathrm{Aut}(G, H, HoH) = \{\tau\mathsf{conj}(g) \mid g \in \langle x\rangle\langle b\rangle\}$ or $\{\tau\mathsf{conj}(g) \mid g \in \langle x\rangle\langle ab\rangle\}$, respectively. Since $ab \in T$ and $g\hat{g} = \mathsf{conj}(g)_H$ for $g \in \mathbf{N}_G(H)$, we have $A = G\{\sigma_H \mid \sigma \in \mathrm{Aut}(G, H, HoH)\} = G\langle\tau\mathsf{conj}(b)_H\rangle$ or $G\langle\tau\hat{ab}\rangle$, which is isomorphic to $(\mathrm{PSL}_2(p) \times \mathbb{Z}_3){:}\mathbb{Z}_2$ or $\mathrm{PSL}_2(p) \times \mathrm{S}_3$, respectively.

Finally, suppose that $\mathsf{Cos}(G, H, 1, o_1) \cong \mathsf{Cos}(G, H, 1, o_2)$ for $o_1, o_2 \in O_0$. Then, by Lemma 3.3, there is $\sigma \in \mathrm{Aut}(G, H)$ such that $Ho_1^{\sigma}H = Ho_2H$. This implies that $\langle x\rangle o_1^{\mathsf{conj}(g)}\langle x\rangle = \langle x\rangle o_2\langle x\rangle$ for some $g \in \mathrm{PGL}_2(p)$. Then $\mathsf{Cos}(T, \langle x\rangle, 1, o_1) \cong \mathsf{Cos}(T, \langle x\rangle, 1, o_2)$. By Theorem 5.7, we have $o_1 = o_2$. Thus distinct involutions o in O_0 produce non-isomorphic symmetric graphs $\mathsf{Cos}(G, H, 1, o)$. Therefore, Γ is described as in Lines 11 or 12 of Table 1. This completes the proof of Theorem 1.1.

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