HECKE-TYPE DOUBLE SUMS AND THE BAILEY TRANSFORM

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Abstract. Hecke-type double sums play a crucial role in proving many identities related to mock theta functions given by Ramanujan. In the literature, the Bailey pair machinery is an efficient tool to derive Hecke-type double sums for mock theta functions. In this paper, by using some Bailey pairs and conjugate Bailey pairs, and then applying the Bailey transform, we establish some trivariate identities which imply the Hecke-type double sums for some classical mock theta functions of orders 3, 6, and 10. Meanwhile, we generalize a bivariate Hecke-type identity due to Garvan.

1. Introduction

Here and throughout the paper, we adopt the standard q-series notation in [25]. Let q be a complex number such that $|q|<1$. For positive integer n, we define

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

Sometimes we use the compressed notation:

$$(a_1,a_2,\ldots,a_m;q)_n := (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, \quad n\in\mathbb{N}\cup\{\infty\},\ m\geq 1.$$

Let

$${}_r\phi_s\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix};q,x\right) := \sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,b_2,\ldots,b_s;q)_n}\left((-1)^nq^{\binom{n}{2}}\right)^{1+s-r}x^n.$$

Define

$$j(x;q) := (x,q/x,q;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^nq^{\binom{n}{2}}x^n,$$

where the second equality is the Jacobi triple product identity [25, Equation (1.6.1)]. For any positive integer m, let

$$J_{a,m} := j(q^a;q^m), \quad \overline{J}_{a,m} := j(-q^a;q^m), \quad J_m := J_{m,3m} = \prod_{i=1}^{\infty}(1-q^{mi}).$$

In his last letter to Hardy, Ramanujan gave a list of 17 functions which he called "mock theta functions" and separated them into four classes: one class of third order, two of fifth

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order, and one of seventh order. With the discovery of Ramanujan's lost notebook, more results related to mock theta functions came to light, such as sixth and tenth order mock theta functions and some related identities. In the development of mock theta functions, it is worth mentioning Hecke-type double sums which are defined as

$$\sum_{(m,n)\in D}(-1)^{H(m,n)}q^{Q(m,n)+L(m,n)}, \tag{1.1}$$

where $H(m,n)$ and $L(m,n)$ are linear forms, $Q(m,n)$ is an indefinite quadratic form, and D is some subset of $\mathbb{Z}\times\mathbb{Z}$ such that $Q(m,n)\geq 0$ for all $(m,n)\in D$. This type of representations plays a very important role in proving Ramanujan's mock theta function identities, and so has been widely studied. In [30], Hickerson and Mortenson gave the following definition of Hecke-type double sums.

Definition 1.1. Let $x,y\in\mathbb{C}^*:=\mathbb{C}\backslash\{0\}$ and define $\mathrm{sg}(r):=1$ for $r\geq 0$ and $\mathrm{sg}(r):=-1$ for $r<0$. Then

$$f_{a,b,c}(x,y,q):=\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)}\mathrm{sg}(r)(-1)^{r+s}x^ry^sq^{a\binom{r}{2}+brs+c\binom{s}{2}}. \tag{1.2}$$

For the third order mock theta functions, in addition to those in Ramanujan's last letter to Hardy, Watson [45,46] studied the other three functions, which appeared in Ramanujan's lost notebook [42], and derived some interesting related identities. In 2013, Mortenson [36] derived the Hecke-type double sums for some third order mock theta functions, such as

$$\begin{aligned}\phi^{(3)}(q)&=\frac{1}{\overline{J}_{1,4}}\left(f_{1,7,1}(-q,-q^2,q)+qf_{1,7,1}(-q^3,-q^4,q)\right)\\&=\frac{1}{\overline{J}_{1,4}}\left(f_{4,4,1}(q^3,-q^2,q)+qf_{4,4,1}(q^5,-q^4,q)\right),\\\psi^{(3)}(q)&=\frac{1}{J_1}f_{3,5,3}(q^2,q^3,q)-1,\end{aligned} \tag{1.3}$$

where $\phi^{(3)}(q)$ and $\psi^{(3)}(q)$ are defined as

$$\phi^{(3)}(q):=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n}\qquad\text{and}\qquad\psi^{(3)}(q):=\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q;q^2)_n}. \tag{1.4}$$

In 2020, Chen and Wang [12] provided a new method to establish Hecke-type double sums for all the classical mock theta functions except for those of seventh and tenth order. For example, they derived the following Hecke-type double sums for $\psi^{(3)}(q)$:

$$\psi^{(3)}(q)=-\frac{1}{J_1}f_{3,5,3}(1,q,q),$$

which is equivalent to (1.3) by using [30, Proposition 6.3] with $(\ell,k)=(1,1)$ and [30, Proposition 6.2].

Recently, Chen and Garvan [11] also found a new form of $\psi^{(3)}(q)$, namely,

$$\begin{aligned}\psi^{(3)}(q) &= \frac{J_2}{J_1^2}\sum_{n=1}^{\infty}\sum_{m=-n+1}^{n}(1-q^{2n})(-1)^{m-1}q^{n(3n-1)-2m^2+m}\\ &= \frac{J_2}{J_1^2}\left(f_{3,3,1}(-q^4,q^3,q^2)-f_{3,3,1}(-q^2,q,q^2)\right).\end{aligned}\tag{1.5}$$

Notice that with the aid of the proof of [36, Eq. (2.5)] and [30, Theorem 1.4], one can find the equivalence between (1.3) and (1.5).

In 1986, Andrews [3] employed the Bailey pair technology to successfully establish the Hecke-type double sums for the fifth and seventh order mock theta functions. Then, combining the Hecke-type double sums obtained by Andrews [3] and the constant term method, Hickerson [28] proved the mock theta conjectures, which are specifically identities that express the fifth order mock theta functions in terms of $g_3(z;q)$, where

$$g_3(z,q) := \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(z,q/z;q)_{n+1}}.\tag{1.6}$$

In [29], Hickerson proved analogous identities for the seventh order mock theta functions, which express the seventh order functions in terms of $g_3(z;q)$. In 2019, Garvan [24] discovered some new Hecke-type double sums for the fifth and seventh order mock theta functions by utilizing the Bailey pair method.

In [7], Andrews and Hickerson studied the sixth order mock theta functions which were found in Ramanujan's lost notebook. The following functions are four of the sixth order mock theta functions due to Ramanujan:

$$\phi^{(6)}(q) := \sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^nq^{n^2}}{(-q;q)_{2n}},\quad \psi^{(6)}(q) := \sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^nq^{(n+1)^2}}{(-q;q)_{2n+1}},\tag{1.7}$$

$$\rho^{(6)}(q) := \sum_{n=0}^{\infty}\frac{(-q;q)_nq^{\binom{n+1}{2}}}{(q;q^2)_{n+1}},\qquad \sigma^{(6)}(q) := \sum_{n=0}^{\infty}\frac{(-q;q)_nq^{\binom{n+2}{2}}}{(q;q^2)_{n+1}}.\tag{1.8}$$

By constructing a new Bailey pair, Andrews and Hickerson [7] proved some identities related to the sixth order mock theta functions given by Ramanujan. In the proofs, Hecke-type double sums still play an irreplaceable role. For example, they [7] established the following Hecke-type identities, which were later given new proofs by Chen and Wang [12]. The identities are as follows:

$$\begin{aligned}\phi^{(6)}(q) &= \frac{J_1}{J_2^2}\left(1+2\sum_{n=1}^{\infty}q^{2n^2-n}+\sum_{n=1}^{\infty}\sum_{j=-n}^{n}(1-q^{2n})(-1)^{n+j+1}q^{3n^2-n-j^2}\right)\\ &= \frac{J_1}{J_2^2}f_{1,2,1}(q,-q,q),\end{aligned}\tag{1.9}$$

$$\psi^{(6)}(q)=\frac{J_1}{J_2^2}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^{n+j}q^{3n^2+3n-j^2+1}=\frac{qJ_1}{2J_2^2}f_{1,2,1}(q^2,-q^2,q), \tag{1.10}$$

$$\sigma^{(6)}(q)=\frac{J_2}{J_1^2}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(1-q^{n+1})(-1)^nq^{n(3n+5)/2-j(j+1)/2+1}=\frac{qJ_2}{J_1^2}f_{1,2,1}(q^4,q^3,q^2), \tag{1.11}$$

$$\rho^{(6)}(q)=\frac{J_2}{J_1^2}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^nq^{3n(n+1)/2-j(j+1)/2}=\frac{J_2}{J_1^2}f_{1,2,1}(q^3,q^2,q^2). \tag{1.12}$$

In 2007, Berndt and Chan [9] found two sixth order mock theta functions, one of which appeared in [42]. They [9] established their Hecke-type double sums expressions together with some identities related to Ramanujan's mock theta functions. Very recently, Mortenson [41] discovered three new identities related to the sixth order mock theta functions.

In 2000, Gordon and McIntosh [26] discovered eight eighth order mock theta functions. Subsequently, McIntosh [34] studied three second order mock theta functions, some of which are in Ramanujan's lost notebook [42] and can also be found in Andrews' paper [2] on Mordell integrals. In [20], Hecke-type double sums expressions for those functions were provided on the lines of the Bailey pair method.

For the tenth order mock theta functions, Choi [13–16] established their Hecke-type double sums expressions with the aid of Bailey pairs and proved some identities given by Ramanujan. The tenth order mock theta functions due to Ramanujan are stated as follows:

$$\phi^{(10)}(q):=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}},\quad \psi^{(10)}(q):=\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{(q;q^2)_n}, \tag{1.13}$$

$$X^{(10)}(q):=\sum_{n=0}^{\infty}\frac{(-1)^nq^{n^2}}{(-q;q)_{2n}},\quad \chi^{(10)}(q):=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}}{(-q;q)_{2n-1}}. \tag{1.14}$$

In 2018, Mortenson [38] rewrote the Hecke-type double sums obtained by Choi into blocks of $f_{a,b,c}(x,y,q)$:

$$J_{1,2}\phi^{(10)}(q)=f_{2,3,2}(q^2,q^2,q),\qquad J_{1,2}\psi^{(10)}(q)=-q^2f_{2,3,2}(q^4,q^4,q), \tag{1.15}$$

$$\overline{J}_{1,4}X^{(10)}(q)=f_{2,3,2}(-q^3,-q^3,q^2),\quad \overline{J}_{1,4}(2-\chi^{(10)}(q))=qf_{2,3,2}(-q^{-1},-q^{-1},q^2). \tag{1.16}$$

In the survey of mock theta functions, Gordon and McIntosh [27] showed that each of the classical mock theta functions can be expressed in terms of $g_2(z,q)$ or (1.6), where

$$g_2(z,q):=\sum_{n=0}^{\infty}\frac{(-q;q)_nq^{n(n+1)/2}}{(z,q/z;q)_{n+1}}.$$

These functions are usually called universal mock theta functions. Furthermore, McIntosh [35] studied more bivariate functions. For instance,

$$K(z;q)=\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^nq^{n^2}}{(zq^2,q^2/z;q^2)_n}, \tag{1.17}$$

which can be found on page 5 of Ramanujan's lost notebook. See [37] for details. In 2015, Garvan [23] gave the Hecke-type double sums for the above three functions, such as

$$(zq^2,q^2/z,q^2;q^2)_\infty K(z;q) \tag{1.18}$$
$$=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}(-1)^m z^{n-m}q^{\frac{1}{2}(2m^2-n^2)+\frac{1}{2}(2m-n)}+\sum_{n=1}^{m}(-1)^m z^{m-n+1}q^{\frac{1}{2}(2m^2-n^2)+\frac{1}{2}(2m+n)}\right).$$

For more interesting works on mock theta functions, one may consult [4–6, 9, 10, 17, 19, 21, 27, 32, 36, 39, 42].

The object of this paper is to establish more Hecke-type double sums for mock theta functions. By constructing some new Bailey pairs and a conjugate Bailey pair and employing the Bailey transform, we obtain some trivariate identities. In particular, we derive several Hecke-type double sums expressions for some mock theta functions of orders 3, 6, and 10. Meanwhile, we derive a generalization of Garvan's identity (1.18). The main results of the paper are stated as follows:

Theorem 1.2. *We have*

$$\sum_{n=0}^{\infty}\frac{(a;q^2)_n x^n q^{n^2-n}}{(x,-q^2;q^2)_n}=\frac{(a;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1-q^{4n+2})(xq^{-2n};q^4)_n(-1)^n q^{n^2+n}}{(x;q^2)_n}$$
$$\times\sum_{r=0}^{\infty}\frac{(1+q^{2n+2r+2})(q^{2n+4}/a;q^2)_r a^r q^{2r^2+4nr+2r}}{(aq^{2n};q^2)_{r+1}}.$$

Corollary 1.3. *We have*

$$\phi^{(3)}(-q)=\frac{J_2}{J_1J_4}\left(f_{4,4,1}(q^5,q,q)+q^2f_{4,4,1}(q^7,q^3,q)\right), \tag{1.19}$$

$$\psi^{(3)}(-q)=-\frac{qJ_4}{J_2^2}f_{4,4,1}(q^6,q^2,q), \tag{1.20}$$

$$\phi^{(6)}(q)=\frac{J_1}{J_2^2}f_{1,2,1}(-q^2,q,q), \tag{1.21}$$

$$\psi^{(6)}(q)=\frac{qJ_1}{2J_2^2}f_{1,2,1}(q^2,-q^2,q), \tag{1.22}$$

$$X^{(10)}(q)=\frac{1}{J_2}\left(f_{6,6,1}(q^8,q,q)+q^2f_{6,6,1}(q^{10},q^3,q)\right), \tag{1.23}$$

$$\chi^{(10)}(q)=\frac{1}{J_2}\left(qf_{6,6,1}(q^8,q^2,q)+q^3f_{6,6,1}(q^{10},q^4,q)\right). \tag{1.24}$$

Note that (1.22) agrees with (1.10). Using Propositions 6.2 and 6.3 in [30], we can show that (1.21) is equivalent to (1.9).

Theorem 1.4. *We have*

$$\sum_{n=0}^{\infty}\frac{(a;q)_n x^{-n}q^{n(n+5)/2}}{(q;q^2)_{n+1}}=\frac{(-q^3/x,a;q)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1-xq^{2n})(-1)^n q^{\binom{n+1}{2}}}{(q;q)_n}$$

$$\times \sum_{j=0}^{\infty} \frac{(q^{-2n+3}/x^2;q^2)_j x^j q^{nj}}{(q,-q^3/x;q)_j} \sum_{r=0}^{\infty} \frac{(1-xq^{2n+2r+1})(xq^{n+1}/a,xq^n;q)_r a^r q^{r^2+2nr+r}}{(q^{n+1},aq^n;q)_{r+1}}.$$

Corollary 1.5. *We have*

$$\sigma^{(6)}(q) = \frac{qJ_2}{J_1^2} f_{1,2,1}(q^3,q^4,q^2), \tag{1.25}$$

$$\rho^{(6)}(q) = \frac{J_2}{J_1^2} f_{1,2,1}(q^3,q^2,q^2), \tag{1.26}$$

$$\psi^{(10)}(q) = \frac{q}{J_1}\left(f_{3,6,2}(q^4,q^4,q) + qf_{3,6,2}(q^5,q^6,q)\right), \tag{1.27}$$

$$\phi^{(10)}(q) = \frac{1}{J_1}\left(f_{3,6,2}(q^5,q^2,q) + qf_{3,6,2}(q^5,q^4,q)\right). \tag{1.28}$$

Note that the identities (1.25) and (1.26) are (1.11) and (1.12), respectively. See also [30, Example 1.2]. In contrast, the expressions (1.23), (1.24), (1.27), and (1.28) are new and it is not easy to show the equivalence between them and those known expressions in (1.15) and (1.16).

Theorem 1.6. *We have*

$$(aq^2,q^2/a,q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{(q^2;q^4)_n x^n q^{n^2-n}}{(x,aq^2,q^2/a;q^2)_n}$$
$$= \sum_{n=0}^{\infty} \frac{(xq^{-2n};q^4)_n(-1)^n q^{n^2+n}}{(x;q^2)_n}\left(1+\sum_{r=1}^{\infty}(-1)^r(a^r+a^{-r})q^{r^2+2nr+r}\right).$$

It should be pointed out that the above theorem is a generalization of Garvan's identity (1.18).

The rest of this paper is organized as follows. In Section 2, some preliminaries are provided. In particular, we establish some new Bailey pairs and a conjugate Bailey pair. Section 3 is devoted to the proofs of the main results. We end our paper with some concluding remarks in Section 4.

2. Preliminaries

In this section, we recall some q-series identities, and then establish some new Bailey pairs and a conjugate Bailey pair. Finally, we present some properties related to $f_{a,b,c}(x,y,q)$.

Lemma 2.1. *([25, Appendix (II.3)]) (the q-binomial theorem) For $|z|<1$,*

$$\sum_{j=0}^{\infty} \frac{(a;q)_j z^j}{(q;q)_j} = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{2.1}$$

Lemma 2.2. *([22, p. 15]) (the Rogers-Fine identity) For* $|\tau|<1$,

$$\sum_{r=0}^{\infty}\frac{(\alpha;q)_r\tau^r}{(\beta;q)_r}=\sum_{r=0}^{\infty}\frac{(\alpha;q)_r(\alpha\tau q/\beta;q)_r\beta^r\tau^rq^{r^2-r}(1-\alpha\tau q^{2r})}{(\beta;q)_r(\tau;q)_{r+1}}. \tag{2.2}$$

Lemma 2.3. *([25, Appendix (III.1)]) (the Heine transformation formula) For* $|z|<1$ *and* $|b|<1$,

$${}_2\phi_1\left(\begin{matrix}a,\ b\\ c\end{matrix};q,z\right)=\frac{(b,az;q)_\infty}{(c,z;q)_\infty}{}_2\phi_1\left(\begin{matrix}c/b,\ z\\ az\end{matrix};q,b\right). \tag{2.3}$$

Lemma 2.4. *([25, Appendix (III.10)]) For* $|de/abc|<1$ *and* $|b|<1$,

$${}_3\phi_2\left(\begin{matrix}a,\ b,\ c\\ d,\ e\end{matrix};q,\frac{de}{abc}\right)=\frac{(b,de/ab,de/bc;q)_\infty}{(d,e,de/abc;q)_\infty}{}_3\phi_2\left(\begin{matrix}d/b,\ e/b,\ de/abc\\ de/ab,\ de/bc\end{matrix};q,b\right). \tag{2.4}$$

Lemma 2.5. *([25, Appendix (III.13)]) For any nonnegative integer* n,

$${}_3\phi_2\left(\begin{matrix}q^{-n},\ b,\ c\\ d,\ e\end{matrix};q,\frac{deq^n}{bc}\right)=\frac{(e/c;q)_n}{(e;q)_n}{}_3\phi_2\left(\begin{matrix}q^{-n},\ c,\ d/b\\ cq^{1-n}/e,\ d\end{matrix};q,q\right). \tag{2.5}$$

Next, we state the definitions of Bailey pairs, conjugate Bailey pairs, and the Bailey transform.

Definition 2.6. The pair of sequences (α_n,β_n) is called a Bailey pair relative to (a,q) if (α_n,β_n) satisfies

$$\beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}.$$

Moreover, Andrews [1] proved that (α_n,β_n) forms a Bailey pair relative to (a,q) if and only if

$$\alpha_n=\frac{(1-aq^{2n})(a;q)_n(-1)^nq^{\binom{n}{2}}}{(1-a)(q;q)_n}\sum_{j=0}^{n}(q^{-n},aq^n;q)_jq^j\beta_j. \tag{2.6}$$

Definition 2.7. The pair of sequences (δ_n,γ_n) is called a conjugate Bailey pair relative to (a,q) if (δ_n,γ_n) satisfies

$$\gamma_n=\sum_{r=n}^{\infty}\frac{\delta_r}{(q;q)_{r-n}(aq;q)_{r+n}}.$$

The following lemma plays an important role in the proofs of the main results.

Lemma 2.8. *([8]) (the Bailey transform) If* (α_n,β_n) *is a Bailey pair relative to* (a,q) *and* (δ_n,γ_n) *is a conjugate Bailey pair relative to* (a,q)*, then*

$$\sum_{n=0}^{\infty}\alpha_n\gamma_n=\sum_{n=0}^{\infty}\beta_n\delta_n. \tag{2.7}$$

Lemma 2.9. *([31, Lemma 2.2]) The following pair of sequences (δ_n, γ_n) forms a conjugate Bailey pair relative to (ab, q), where*

$$\delta_n = (aq, b, q; q)_\infty \frac{(ab; q)_{2n} q^n}{(aq, b; q)_n},$$
$$\gamma_n = \frac{(1-ab)q^n}{1-abq^{2n}} \left(1 + \sum_{r=1}^{\infty} (-1)^r q^{\binom{r}{2}} \left((aq^{n+1})^r + (bq^n)^r\right)\right).$$

In addition to the above known results, we also need the following two new Bailey pairs and a conjugate Bailey pair.

Lemma 2.10. *The following pair of sequences (α_n, β_n) forms a Bailey pair relative to (q^2, q^2), where*

$$\alpha_n = \frac{(1-q^{4n+2})(xq^{-2n}; q^4)_n (-1)^n q^{n^2-n}}{(1-q^2)(x; q^2)_n},$$
$$\beta_n = \frac{x^n q^{n^2-3n}}{(q^4; q^4)_n (x; q^2)_n}. \tag{2.8}$$

Proof. Based on (2.6) and (2.8), it suffices to prove that

$$\alpha_n = \frac{(1-q^{4n+2})(-1)^n q^{n^2-n}}{1-q^2} \sum_{j=0}^{n} \frac{(q^{-2n}, q^{2n+2}; q^2)_j x^j q^{j^2-j}}{(q^2, -q^2, x; q^2)_j}. \tag{2.9}$$

Replacing q by q^2 and setting $b = q^{2n+2}$, $d = -q^2$, and $e = x$ in (2.5), and then letting $c \to \infty$ in the resulting equation, we deduce that

$$\sum_{j=0}^{n} \frac{(q^{-2n}, q^{2n+2}; q^2)_j x^j q^{j^2-j}}{(q^2, -q^2, x; q^2)_j} = \frac{1}{(x; q^2)_n} \sum_{j=0}^{n} \frac{(q^{-2n}, -q^{-2n}; q^2)_j x^j q^{2nj}}{(q^2, -q^2; q^2)_j}$$
$$= \frac{1}{(x; q^2)_n} \sum_{j=0}^{n} \frac{(q^{-4n}; q^4)_j (xq^{2n})^j}{(q^4; q^4)_j} = \frac{(xq^{-2n}; q^4)_n}{(x; q^2)_n}, \tag{2.10}$$

where the last equality follows from (2.1). Substituting (2.10) into (2.9), we see that (2.9) holds and thus complete the proof. □

Lemma 2.11. *The following pair of sequences (α_n, β_n) forms a Bailey pair relative to (a, q), where*

$$\alpha_n = \frac{(1-aq^{2n})(aq, -xq/a; q)_\infty (-1)^n q^{\binom{n}{2}}}{(q; q)_n (xq; q^2)_\infty} \sum_{j=0}^{\infty} \frac{(xq^{-2n+1}/a^2; q^2)_j a^j q^{nj}}{(q, -xq/a; q)_j},$$
$$\beta_n = \frac{x^n a^{-n} q^{\binom{n}{2}}}{(q; q)_n (xq; q^2)_n}. \tag{2.11}$$

Proof. In view of (2.6) and (2.11), it suffices to prove that

$$\alpha_n = \frac{(1-aq^{2n})(a;q)_n(-1)^n q^{\binom{n}{2}}}{(1-a)(q;q)_n} \sum_{j=0}^{n} \frac{(q^{-n}, aq^n;q)_j x^j a^{-j} q^{j(j+1)/2}}{(q,(xq)^{1/2},-(xq)^{1/2};q)_j}. \tag{2.12}$$

Replacing a, b, d, and e by q^{-n}, aq^n, $(xq)^{1/2}$, and $-(xq)^{1/2}$ in (2.4), respectively, and then letting $c \to \infty$, we find that

$$\begin{aligned} &\sum_{j=0}^{n} \frac{(q^{-n}, aq^n;q)_j x^j a^{-j} q^{j(j+1)/2}}{(q,(xq)^{1/2},-(xq)^{1/2};q)_j} \\ &= \frac{(aq^n,-xq/a;q)_\infty}{(xq;q^2)_\infty} \sum_{j=0}^{\infty} \frac{(x^{1/2}q^{-n+1/2}/a,-x^{1/2}q^{-n+1/2}/a;q)_j a^j q^{nj}}{(q,-xq/a;q)_j} \\ &= \frac{(a,-xq/a;q)_\infty}{(a;q)_n(xq;q^2)_\infty} \sum_{j=0}^{\infty} \frac{(xq^{-2n+1}/a^2;q^2)_j a^j q^{nj}}{(q,-xq/a;q)_j}. \end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.12), we see that (2.12) holds and thus complete the proof. □

Lemma 2.12. *The following pair of sequences (δ_n,γ_n) forms a conjugate Bailey pair relative to (a,q), where*

$$\delta_n = (q,b;q)_n q^n, \tag{2.14}$$

$$\gamma_n = \frac{(b;q)_\infty q^n}{(1-q^{n+1})(aq;q)_\infty} \sum_{r=0}^{\infty} \frac{(1-aq^{2n+2r+1})(aq^{n+1}/b, aq^n;q)_r b^r q^{r^2+2nr+r}}{(q^{n+2};q)_r (bq^n;q)_{r+1}}.$$

Proof. With the aid of the definition of conjugate Bailey pairs and (2.14), it suffices to prove that

$$\begin{aligned} \gamma_n &= \sum_{r=n}^{\infty} \frac{\delta_r}{(q;q)_{r-n}(aq;q)_{r+n}} \\ &= \sum_{r=0}^{\infty} \frac{\delta_{r+n}}{(q;q)_r(aq;q)_{r+2n}} \\ &= \frac{(q,b;q)_n q^n}{(aq;q)_{2n}} \sum_{r=0}^{\infty} \frac{(q^{n+1},bq^n;q)_r q^r}{(q,aq^{2n+1};q)_r}. \end{aligned} \tag{2.15}$$

Replacing a,b,c, and z by q^{n+1}, bq^n, aq^{2n+1}, and q in (2.3), respectively, we deduce that

$$\sum_{r=0}^{\infty} \frac{(q^{n+1},bq^n;q)_r q^r}{(q,aq^{2n+1};q)_r} = \frac{(q^{n+2},bq^n;q)_\infty}{(q,aq^{2n+1};q)_\infty} \sum_{r=0}^{\infty} \frac{(aq^{n+1}/b;q)_r (bq^n)^r}{(q^{n+2};q)_r}. \tag{2.16}$$

Furthermore, we derive the following identity by invoking (2.2) with $\alpha = aq^{n+1}/b$, $\beta = q^{n+2}$, and $\tau = bq^n$:

$$\sum_{r=0}^{\infty} \frac{(aq^{n+1}/b;q)_r(bq^n)^r}{(q^{n+2};q)_r} = \sum_{r=0}^{\infty} \frac{(1-aq^{2n+2r+1})(aq^{n+1}/b,aq^n;q)_r b^r q^{r^2+2nr+r}}{(q^{n+2};q)_r(bq^n;q)_{r+1}}. \tag{2.17}$$

Substituting (2.16) and (2.17) into (2.15), we see that (2.15) holds and thus complete the proof. □

Some identities containing Bailey pairs have appeared in the literature [33, 43, 44]. For example, Lovejoy [33, Equation (1.6)] found that

$$\sum_{n=0}^{\infty}(a;q)_n q^n \beta_n = \frac{(a;q)_\infty}{(a^2q,q;q)_\infty} \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(1-a^2q^{2n+2r+1})(-1)^n a^{3n} q^{n(3n+1)/2+3nr+r}}{1-aq^r}\alpha_r,$$

where (α_n, β_n) is a Bailey pair relative to (a^2, q). In this paper, inserting the conjugate Bailey pair in Lemma 2.12 into (2.7), we derive the following analog which is a key identity to prove the main theorems.

Lemma 2.13. *For a Bailey pair (α_n, β_n) relative to (a, q), we have*

$$\sum_{n=0}^{\infty}(q,b;q)_n q^n \beta_n = \frac{(b;q)_\infty}{(aq;q)_\infty} \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(1-aq^{2n+2r+1})(aq^{n+1}/b,aq^n;q)_r b^r q^{r^2+2nr+n+r}}{(q^{n+1},bq^n;q)_{r+1}}\alpha_n. \tag{2.18}$$

The following lemmas provide some properties of $f_{a,b,c}(x,y,q)$.

Lemma 2.14. *([30, Propositions 6.2]) For $x, y \in \mathbb{C}^*$,*

$$f_{a,b,c}(x,y,q) = -\frac{q^{a+b+c}}{xy} f_{a,b,c}(q^{2a+b}/x, q^{2c+b}/y, q). \tag{2.19}$$

Lemma 2.15. *We have*

$$f_{a,b,c}(x,y,q^m) = f_{ma,mb,mc}(x,y,q), \tag{2.20}$$

$$f_{a,b,c}(x,y,q) = \sum_{k=0}^{m-1}(-1)^k q^{a\binom{k}{2}} x^k f_{am^2,bm,c}\left((-1)^{m-1}q^{am(m+2k-1)/2}x^m, q^{bk}y, q\right). \tag{2.21}$$

Proof. Based on the definition of $f_{a,b,c}(x,y,q)$, the first assertion is easy to obtain.

Separating the sum with r being $mr+k$ $(k = 0, 1, \ldots, m-1)$ in (1.2), we get

$$f_{a,b,c}(x,y,q) = \sum_{k=0}^{m-1}\sum_{\mathrm{sg}(r)=\mathrm{sg}(s)} \mathrm{sg}(r)(-1)^{mr+k+s}x^{mr+k}y^s q^{a\binom{mr+k}{2}+b(mr+k)s+c\binom{s}{2}}.$$

This proves the second assertion. □

In particular, when $m=2$, we obtain that

$$f_{a,b,c}(x,y,q)=f_{4a,2b,c}(-q^{a}x^2,y,q)-xf_{4a,2b,c}(-q^{3a}x^2,q^{b}y,q). \tag{2.22}$$

The identities in the following lemma appeared in [18].

Lemma 2.16. *([18, Equations (3.12) and (3.13)]) For any nonnegative integer n, we have*

$$(-q^{-2n+1};q^4)_n=(-q;q^2)_nq^{-\binom{n+1}{2}}, \tag{2.23}$$

$$(-q^{-2n+3};q^4)_n=(-q;q^2)_nq^{-\binom{n}{2}}. \tag{2.24}$$

3. Proofs of the main results

In this section, we prove the main results.

Proof of Theorem 1.2. Inserting the Bailey pair in Lemma 2.10 into (2.18) with $(q,a)\to(q^2,q^2)$, and then replacing b by a, we complete the proof. □

Proof of Corollary 1.3. Set $a=x=-q$ in Theorem 1.2 and then make use of (2.23) to derive that

$$\begin{aligned}\phi^{(3)}(-q)&=\frac{J_2}{J_1J_4}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1+q^{2r+2n+2})(1-q^{2n+1})(-q^{-2n+1};q^4)_n(-1)^{r+n}q^{2r^2+4rn+3r+n^2+n}}{(-q;q^2)_n}\\&=\frac{J_2}{J_1J_4}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1+q^{2r+2n+2})(1-q^{2n+1})(-1)^{r+n}q^{2r^2+4rn+3r+n(n+1)/2} \qquad (3.1)\\&=\frac{J_2}{J_1J_4}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1+q^{2r+2n+2})(-1)^{r+n}q^{2r^2+4rn+3r+n(n+1)/2}\\&\quad-\frac{J_2}{J_1J_4}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1+q^{2r+2n+2})(-1)^{r+n}q^{2r^2+4rn+3r+n(n+5)/2+1}. \qquad (3.2)\end{aligned}$$

Then shifting $r\to-r-1$ and $n\to-n-1$ in the second term on the right-hand side of (3.2) yields that

$$\begin{aligned}\phi^{(3)}(-q)&=\frac{J_2}{J_1J_4}\left(\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}-\sum_{r=-\infty}^{-1}\sum_{n=-\infty}^{-1}\right)(1+q^{2r+2n+2})(-1)^{r+n}q^{2r^2+4rn+3r+n(n+1)/2}\\&=\frac{J_2}{J_1J_4}\sum_{\mathrm{sg}(r)=\mathrm{sg}(n)}\mathrm{sg}(r)(1+q^{2r+2n+2})(-1)^{r+n}q^{2r^2+4rn+3r+n(n+1)/2}\\&=\frac{J_2}{J_1J_4}\sum_{\mathrm{sg}(r)=\mathrm{sg}(n)}\mathrm{sg}(r)(-1)^{r+n}q^{4\binom{r}{2}+4rn+\binom{n}{2}+5r+n}\\&\quad+\frac{J_2}{J_1J_4}\sum_{\mathrm{sg}(r)=\mathrm{sg}(n)}\mathrm{sg}(r)(-1)^{r+n}q^{4\binom{r}{2}+4rn+\binom{n}{2}+7r+3n+2}.\end{aligned}$$

Hence, using (1.2), we prove (1.19).

Next, we prove (1.20). Applying Theorem 1.2 with $a=-q^2$ gives

$$\sum_{n=0}^{\infty}\frac{x^n q^{n^2-n}}{(x;q^2)_n}=\frac{J_4}{J_2^2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1-q^{4n+2})(xq^{-2n};q^4)_n(-1)^{r+n}q^{2r^2+4rn+4r+n^2+n}}{(x;q^2)_n}. \tag{3.3}$$

Then invoking (2.24) and (3.3) with $x=-q^3$ yields that

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+2n}}{(-q^3;q^2)_n}=\frac{(1+q)J_4}{J_2^2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1-q^{2n+1})(-1)^{r+n}q^{2r^2+4rn+4r+n(n+3)/2}. \tag{3.4}$$

Notice that after replacing (r,n) by $(-r-1,-n-1)$, we have

$$\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{r+n}q^{2r^2+4rn+4r+n(n+7)/2+1}=\sum_{r=-\infty}^{-1}\sum_{n=-\infty}^{-1}(-1)^{r+n}q^{2r^2+4rn+4r+n(n+3)/2}. \tag{3.5}$$

So, substituting (3.5) into (3.4), multiplying both sides by $q/(1+q)$, and then changing $n\to n-1$ on the left-hand side of the resulting identity, we obtain that

$$\begin{aligned}\psi^{(3)}(-q)&=-\frac{qJ_4}{J_2^2}\left(\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}-\sum_{r=-\infty}^{-1}\sum_{n=-\infty}^{-1}\right)(-1)^{r+n}q^{2r^2+4rn+4r+n(n+3)/2}\\&=-\frac{qJ_4}{J_2^2}\sum_{\mathrm{sg}(r)=\mathrm{sg}(n)}\mathrm{sg}(r)(-1)^{r+n}q^{2r^2+4rn+4r+n(n+3)/2}.\end{aligned}$$

Finally, combining (1.2) and the above identity, we derive (1.20).

To prove (1.21), setting $a=q$ in Theorem 1.2 gives

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(q;q^2)_n x^n q^{n^2-n}}{(x,-q^2;q^2)_n}=\frac{J_1}{J_2^2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}&\frac{(1+q^{2r+2n+2})(1+q^{2n+1})(xq^{-2n};q^4)_n}{(x;q^2)_n}\\&\times(-1)^n q^{2r^2+4rn+3r+n^2+n}.\end{aligned} \tag{3.6}$$

Invoke (3.6) with $x=-q$ and then apply (2.23) to arrive at

$$\begin{aligned}\phi^{(6)}(q)&=\frac{J_1}{J_2^2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1+q^{2r+2n+2})(1+q^{2n+1})(-1)^n q^{2r^2+4rn+3r+n(n+1)/2}\\&=\frac{J_1}{J_2^2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1+q^{2r+2n+2})(-1)^n q^{2r^2+4rn+3r+n(n+1)/2}\\&\quad+\frac{J_1}{J_2^2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1+q^{2r+2n+2})(-1)^n q^{2r^2+4rn+3r+n(n+5)/2+1}.\end{aligned} \tag{3.7}$$

Then replacing n by $-n-1$ and r by $-r-1$ in the second term on the right-hand side of (3.7), we derive that

$$\phi^{(6)}(q)=\frac{J_1}{J_2^2}\left(\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}-\sum_{r=-\infty}^{-1}\sum_{n=-\infty}^{-1}\right)(1+q^{2r+2n+2})(-1)^n q^{2r^2+4rn+3r+n(n+1)/2}$$

$$= \frac{J_1}{J_2^2} \sum_{\mathrm{sg}(r)=\mathrm{sg}(n)} \mathrm{sg}(r)(1+q^{2r+2n+2})(-1)^n q^{2r^2+4rn+3r+n(n+1)/2}$$
$$= \frac{J_1}{J_2^2} \left(f_{4,4,1}(-q^5,q,q) + q^2 f_{4,4,1}(-q^7,q^3,q) \right).$$

Therefore, using (2.22) with $(a,b,c,x,y)=(1,2,1,-q^2,q)$ and the above identity, we prove (1.21).

To obtain (1.22), we employ (3.6) with $x=-q^3$, multiply both sides by $q/(1+q)$ and then invoke (2.24) to find that

$$\begin{aligned}
2\psi^{(6)}(q) &= \frac{2J_1}{J_2^2} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1+q^{2r+2n+2})(-1)^n q^{2r^2+4rn+3r+n(n+3)/2+1} \\
&= \frac{J_1}{J_2^2}\left(\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} + \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\right)(1+q^{2r+2n+2})(-1)^n q^{2r^2+4rn+3r+n(n+3)/2+1} \\
&= \frac{J_1}{J_2^2}\left(\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} - \sum_{r=-\infty}^{-1}\sum_{n=-\infty}^{-1}\right)(1+q^{2r+2n+2})(-1)^n q^{2r^2+4rn+3r+n(n+3)/2+1} \\
&= \frac{J_1}{J_2^2} \sum_{\mathrm{sg}(r)=\mathrm{sg}(n)} \mathrm{sg}(r)(1+q^{2r+2n+2})(-1)^n q^{2r^2+4rn+3r+n(n+3)/2+1} \\
&= \frac{qJ_1}{J_2^2}\left(f_{4,4,1}(-q^5,q^2,q) + q^2 f_{4,4,1}(-q^7,q^4,q)\right). \qquad (3.8)
\end{aligned}$$

So, using (2.22) with $(a,b,c,x,y)=(1,2,1,-q^2,q^2)$ and (3.8), we deduce (1.22).

To prove the last two identities in the corollary, we need Theorem 1.2 with $a \to 0$, namely,

$$\sum_{n=0}^{\infty} \frac{x^n q^{n^2-n}}{(x,-q^2;q^2)_n} = \frac{1}{(q^2;q^2)_\infty} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1+q^{2r+2n+2})(1-q^{4n+2})(xq^{-2n};q^4)_n}{(x;q^2)_n} \times (-1)^{r+n} q^{3r^2+6rn+5r+n^2+n}. \qquad (3.9)$$

Utilizing (3.9) with $x=-q$ and applying (2.23), we have

$$\begin{aligned}
X^{(10)}(q) &= \frac{1}{J_2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1+q^{2r+2n+2})(1-q^{4n+2})(-1)^{r+n} q^{3r^2+6rn+5r+n(n+1)/2} \\
&= \frac{1}{J_2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1+q^{2r+2n+2})(-1)^{r+n} q^{3r^2+6rn+5r+n(n+1)/2} \\
&\quad - \frac{1}{J_2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1+q^{2r+2n+2})(-1)^{r+n} q^{3r^2+6rn+5r+n(n+9)/2+2}
\end{aligned}$$

$$= \frac{1}{J_2}\left(\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} - \sum_{r=-\infty}^{-1}\sum_{n=-\infty}^{-1}\right)(1+q^{2r+2n+2})(-1)^{r+n}q^{3r^2+6rn+5r+n(n+1)/2}$$

$$= \frac{1}{J_2}\sum_{\mathrm{sg}(r)=\mathrm{sg}(n)} \mathrm{sg}(r)(1+q^{2r+2n+2})(-1)^{r+n}q^{3r^2+6rn+5r+n(n+1)/2}.$$

Therefore, using (1.2) and the above identity, we obtain (1.23).

To derive (1.24), we set $x=-q^3$ in (3.9) and then use (2.24) to obtain that

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}} = \frac{1}{J_2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1+q^{2r+2n+2})(1-q^{2n+1})(-1)^{r+n}q^{3r^2+6rn+5r+n(n+3)/2+1}.$$

Thus,

$$\chi^{(10)}(q) = \frac{1}{J_2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1+q^{2r+2n+2})(-1)^{r+n}q^{3r^2+6rn+5r+n(n+3)/2+1}$$
$$- \frac{1}{J_2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1+q^{2r+2n+2})(-1)^{r+n}q^{3r^2+6rn+5r+n(n+7)/2+2}$$
$$= \frac{1}{J_2}\left(\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} - \sum_{r=-\infty}^{-1}\sum_{n=-\infty}^{-1}\right)(1+q^{2r+2n+2})(-1)^{r+n}q^{3r^2+6rn+5r+n(n+3)/2+1}$$
$$= \frac{1}{J_2}\sum_{\mathrm{sg}(r)=\mathrm{sg}(n)} \mathrm{sg}(r)(1+q^{2r+2n+2})(-1)^{r+n}q^{3r^2+6rn+5r+n(n+3)/2+1}. \tag{3.10}$$

Therefore, combining (1.2) and (3.10), we complete the proof. □

Proof of Theorem 1.4. Substituting the Bailey pair in Lemma 2.11 into (2.18) and then letting $(x,a,b)\to(q^2,x,a)$, we complete the proof. □

Proof of Corollary 1.5. Setting $a=-q$ in Theorem 1.4, we obtain that

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n x^{-n}q^{n(n+5)/2}}{(q;q^2)_{n+1}} = \frac{(-q^3/x,-q;q)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1-xq^{2n})(-1)^n q^{n(n+1)/2}}{(q;q)_n}$$
$$\times\sum_{j=0}^{\infty}\frac{(q^{-2n+3}/x^2;q^2)_j x^j q^{nj}}{(q,-q^3/x;q)_j}\sum_{r=0}^{\infty}\frac{(1-xq^{2n+2r+1})(x^2q^{2n};q^2)_r(-1)^r q^{r^2+2nr+2r}}{(q^{2n+2};q^2)_{r+1}}. \tag{3.11}$$

To prove (1.25), apply (3.11) with $x=q$. So,

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{n(n+3)/2}}{(q;q^2)_{n+1}} = \frac{J_2^3}{(1+q)J_1^3}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1-q^{2n+1})(-1)^{r+n}q^{r^2+2rn+2r+n(n+1)/2}}{(q;q)_n}$$
$$\times\sum_{j=0}^{\infty}\frac{(q^{-2n+1};q^2)_j q^{j(n+1)}}{(q,-q^2;q)_j}. \tag{3.12}$$

Notice that

$$\sum_{j=0}^{\infty}\frac{(q^{-2n+1};q^2)_j q^{j(n+1)}}{(q,-q^2;q)_j}=-\frac{(1+q)q^{2n+1}}{1-q^{2n+1}}\sum_{j=0}^{\infty}\frac{(q^{-2n-1};q^2)_{j+1}(1-q^{j+1})q^{j(n+1)}}{(q^2;q^2)_{j+1}}$$
$$=-\frac{(1+q)q^{2n+1}}{1-q^{2n+1}}\sum_{j=1}^{\infty}\frac{(q^{-2n-1};q^2)_j(1-q^j)q^{(j-1)(n+1)}}{(q^2;q^2)_j}$$
$$=-\frac{(1+q)q^{n}}{1-q^{2n+1}}\sum_{j=1}^{\infty}\frac{(q^{-2n-1};q^2)_j(1-q^j)q^{j(n+1)}}{(q^2;q^2)_j}$$
$$=-\frac{(1+q)q^{n}}{1-q^{2n+1}}\left(\sum_{j=1}^{\infty}\frac{(q^{-2n-1};q^2)_j q^{j(n+1)}}{(q^2;q^2)_j}-\sum_{j=1}^{\infty}\frac{(q^{-2n-1};q^2)_j q^{j(n+2)}}{(q^2;q^2)_j}\right)$$
$$=-\frac{(1+q)q^{n}}{1-q^{2n+1}}\left(\sum_{j=0}^{\infty}\frac{(q^{-2n-1};q^2)_j q^{j(n+1)}}{(q^2;q^2)_j}-\sum_{j=0}^{\infty}\frac{(q^{-2n-1};q^2)_j q^{j(n+2)}}{(q^2;q^2)_j}\right)$$
$$=-\frac{(1+q)q^{n}}{1-q^{2n+1}}\left(\frac{(q^{-n};q^2)_\infty}{(q^{n+1};q^2)_\infty}-\frac{(q^{-n+1};q^2)_\infty}{(q^{n+2};q^2)_\infty}\right), \tag{3.13}$$

where the last step follows by the q-binomial theorem (2.1). Define

$$L_n:=\frac{(q^{-n};q^2)_\infty}{(q^{n+1};q^2)_\infty}-\frac{(q^{-n+1};q^2)_\infty}{(q^{n+2};q^2)_\infty}. \tag{3.14}$$

Then combining (3.12), (3.13), and (3.14) yields that

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{n(n+3)/2}}{(q;q^2)_{n+1}}=-\frac{J_2^3}{J_1^3}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{r+n}q^{r^2+2rn+2r+n(n+3)/2}}{(q;q)_n}L_n. \tag{3.15}$$

According to the parity of n, we consider the following two cases for L_n. For even n, replacing n by $2n$ in (3.14), we have

$$L_{2n}=-\frac{(q^{-2n+1};q^2)_\infty}{(q^{2n+2};q^2)_\infty}=\frac{(q;q)_{2n}(-1)^{n+1}q^{-n^2}J_1}{J_2^2}. \tag{3.16}$$

Similarly, for odd n, replacing n by $2n+1$ in (3.14) yields that

$$L_{2n+1}=\frac{(q^{-2n-1};q^2)_\infty}{(q^{2n+2};q^2)_\infty}=\frac{(q;q)_{2n+1}(-1)^{n+1}q^{-n^2-2n-1}J_1}{J_2^2}. \tag{3.17}$$

Thus, substituting (3.16) and (3.17) into (3.15) and then multiplying both sides of the resulting identity by q, we derive that

$$\sigma^{(6)}(q)=\frac{qJ_2}{J_1^2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{r+n}q^{r^2+4rn+2r+n^2+3n}-\frac{qJ_2}{J_1^2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{r+n}q^{r^2+4rn+4r+n^2+3n+1}$$

$$
\begin{aligned}
&= \frac{qJ_2}{J_1^2}\left(\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} - \sum_{r=-\infty}^{-1}\sum_{n=-\infty}^{-1}\right)(-1)^{r+n}q^{r^2+4rn+2r+n^2+3n} \\
&= \frac{qJ_2}{J_1^2}\sum_{\mathrm{sg}(r)=\mathrm{sg}(n)}\mathrm{sg}(r)(-1)^{r+n}q^{r^2+4rn+2r+n^2+3n} \\
&= \frac{qJ_2}{J_1^2}f_{2,4,2}(q^3,q^4,q).
\end{aligned}
$$

This proves (1.25) upon using (2.20).

Next, invoking (3.11) with $x=q^2$, we deduce that

$$
\rho^{(6)}(q) = \frac{J_2^3}{J_1^3}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1-q^{2r+2n+3})(-1)^{r+n}q^{r^2+2rn+2r+n(n+1)/2}}{(q;q)_n}L'_n, \tag{3.18}
$$

where

$$
L'_n := \sum_{j=0}^{\infty}\frac{(q^{-2n-1};q^2)_j q^{j(n+2)}}{(q^2;q^2)_j} = \frac{(q^{-n+1};q^2)_\infty}{(q^{n+2};q^2)_\infty}. \tag{3.19}
$$

Here the last step follows by (2.1). It is easy to see that when n is any odd positive integer, the rightmost side of (3.19) is equal to 0. Hence, we only need to consider the even case for n. By (3.16), we have

$$
L'_{2n} = -L_{2n} = \frac{(q;q)_{2n}(-1)^n q^{-n^2}J_1}{J_2^2}. \tag{3.20}
$$

Thus, combining (3.18), (3.19), and (3.20) gives

$$
\rho^{(6)}(q) = \frac{J_2}{J_1^2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1-q^{2r+4n+3})(-1)^{r+n}q^{r^2+4rn+2r+n^2+n}.
$$

This implies

$$
\begin{aligned}
2\rho^{(6)}(q) &= \frac{J_2}{J_1^2}\left(\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} + \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\right)(1-q^{2r+4n+3})(-1)^{r+n}q^{r^2+4rn+2r+n^2+n} \\
&= \frac{J_2}{J_1^2}\left(\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} - \sum_{r=-\infty}^{-1}\sum_{n=-\infty}^{-1}\right)(1-q^{2r+4n+3})(-1)^{r+n}q^{r^2+4rn+2r+n^2+n} \\
&= \frac{J_2}{J_1^2}\sum_{\mathrm{sg}(r)=\mathrm{sg}(n)}\mathrm{sg}(r)(1-q^{2r+4n+3})(-1)^{r+n}q^{r^2+4rn+2r+n^2+n} \\
&= \frac{J_2}{J_1^2}\left(f_{2,4,2}(q^3,q^2,q) - q^3f_{2,4,2}(q^5,q^6,q)\right) \\
&= \frac{2J_2}{J_1^2}f_{2,4,2}(q^3,q^2,q).
\end{aligned} \tag{3.21}
$$

Here the last equality follows by (2.19). Hence, we complete the proof of (1.26) by using (2.20) and (3.21).

To prove the last two identities (1.27) and (1.28), we let $a \to 0$ in Theorem 1.4. Hence,

$$\sum_{n=0}^{\infty} \frac{x^{-n}q^{n(n+5)/2}}{(q;q^2)_{n+1}} = \frac{(-q^3/x;q)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(1-xq^{2n})(-1)^n q^{\binom{n+1}{2}}}{(q;q)_n}$$
$$\times \sum_{j=0}^{\infty} \frac{(q^{-2n+3}/x^2;q^2)_j x^j q^{nj}}{(q,-q^3/x;q)_j} \sum_{r=0}^{\infty} \frac{(1-xq^{2n+2r+1})(xq^n;q)_r(-1)^r x^r q^{3r^2/2+3nr+3r/2}}{(q^{n+1};q)_{r+1}}. \quad (3.22)$$

Applying (3.22) with $x = q$ yields that

$$\sum_{n=0}^{\infty} \frac{q^{n(n+3)/2}}{(q;q^2)_{n+1}} = \frac{J_2^2}{(1+q)J_1^2} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1+q^{r+n+1})(1-q^{2n+1})}{(q;q)_n}$$
$$\times (-1)^{r+n} q^{r(3r+5)/2+3rn+n(n+1)/2} \sum_{j=0}^{\infty} \frac{(q^{-2n+1};q^2)_j q^{j(n+1)}}{(q,-q^2;q)_j}$$
$$= -\frac{J_2^2}{J_1^2} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1+q^{r+n+1})(-1)^{r+n} q^{r(3r+5)/2+3rn+n(n+3)/2}}{(q;q)_n} L_n, \quad (3.23)$$

where the last step follows by (3.13) and L_n is defined in (3.14). Substituting (3.16) and (3.17) into (3.23), we arrive at

$$\sum_{n=0}^{\infty} \frac{q^{n(n+3)/2}}{(q;q^2)_{n+1}} = \frac{1}{J_1} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} (1+q^{r+2n+1})(-1)^{r+n} q^{r(3r+5)/2+6rn+n^2+3n}$$
$$- \frac{1}{J_1} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} (1+q^{r+2n+2})(-1)^{r+n} q^{r(3r+11)/2+6rn+n^2+3n+1}$$
$$= \frac{1}{J_1} \sum_{\operatorname{sg}(r)=\operatorname{sg}(n)} \operatorname{sg}(r)(1+q^{r+2n+1})(-1)^{r+n} q^{r(3r+5)/2+6rn+n^2+3n}$$
$$= \frac{1}{J_1}\left(f_{3,6,2}(q^4,q^4,q) + q f_{3,6,2}(q^5,q^6,q)\right). \quad (3.24)$$

Therefore, multiplying q on both sides of (3.24) and changing $n \to n-1$ on the left-hand side, we prove (1.27).

To prove (1.28), set $x = q^2$ in (3.22). So,

$$\phi^{(10)}(q) = \frac{J_2^2}{J_1^2} \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1-q^{2r+2n+3})(1+q^{n+1})(-1)^{r+n} q^{r(3r+7)/2+3rn+n(n+1)/2}}{(q;q)_n}$$
$$\times \sum_{j=0}^{\infty} \frac{(q^{-2n-1};q^2)_j q^{j(n+2)}}{(q^2;q^2)_j}$$

$$= \frac{J_2^2}{J_1^2}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1-q^{2r+2n+3})(1+q^{n+1})(-1)^{r+n}q^{r(3r+7)/2+3rn+n(n+1)/2}}{(q;q)_n}L'_n, \tag{3.25}$$

where L'_n is defined in (3.19). Since $L'_n = 0$ when n is any positive odd integer, inserting (3.20) into (3.25) leads to

$$\begin{aligned}
\phi^{(10)}(q) &= \frac{1}{J_1}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1-q^{2r+4n+3})(1+q^{2n+1})(-1)^{r+n}q^{r(3r+7)/2+6rn+n^2+n} \\
&= \frac{1}{J_1}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\left((1-q^{2r+6n+4})+(q^{2n+1}-q^{2r+4n+3})\right)(-1)^{r+n}q^{r(3r+7)/2+6rn+n^2+n} \\
&= \frac{1}{J_1}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1-q^{2r+6n+4})(-1)^{r+n}q^{r(3r+7)/2+6rn+n^2+n} \\
&\quad + \frac{1}{J_1}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(1-q^{2r+2n+2})(-1)^{r+n}q^{r(3r+7)/2+6rn+n^2+3n+1} \\
&= \frac{1}{J_1}\sum_{\mathrm{sg}(r)=\mathrm{sg}(n)}\mathrm{sg}(r)(-1)^{r+n}q^{r(3r+7)/2+6rn+n^2+n} \\
&\quad + \frac{1}{J_1}\sum_{\mathrm{sg}(r)=\mathrm{sg}(n)}\mathrm{sg}(r)(-1)^{r+n}q^{r(3r+7)/2+6rn+n^2+3n+1}.
\end{aligned}$$

Thus, we complete the proof of (1.28) upon using (1.2). □

Proof of Theorem 1.6. In Lemma 2.9, replace q by q^2 and then set $b = q^2/a$ to obtain the following conjugate Bailey pair relative to (q^2, q^2):

$$\delta_n = (aq^2, q^2/a, q^2; q^2)_\infty \frac{(q^2;q^2)_{2n}q^{2n}}{(aq^2, q^2/a; q^2)_n}, \tag{3.26}$$

$$\gamma_n = \frac{(1-q^2)q^{2n}}{1-q^{4n+2}}\left(1+\sum_{r=1}^{\infty}(-1)^r q^{r^2-r}\left((aq^{2n+2})^r + (q^{2n+2}/a)^r\right)\right). \tag{3.27}$$

Then combining (3.26), (3.27), and Lemma 2.10, and applying the Bailey transform, we complete the proof. □

Proof of (1.18). Setting $x = -q$ in Theorem 1.6, and then making use of (2.23), we obtain that

$$\begin{aligned}
&(aq^2, q^2/a, q^2; q^2)_\infty \sum_{n=0}^{\infty}\frac{(q;q^2)_n(-1)^n q^{n^2}}{(aq^2, q^2/a; q^2)_n} \\
&= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{r+n}a^{-r}q^{r^2+2nr+r+n(n+1)/2} + \sum_{r=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{r+n}a^r q^{r^2+2nr+r+n(n+1)/2}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m a^{n-m} q^{(m-n)^2+2n(m-n)+(m-n)+n(n+1)/2} \\
&\quad + \sum_{m=1}^{\infty}\sum_{n=0}^{m-1}(-1)^m a^{m-n} q^{(m-n)^2+2n(m-n)+(m-n)+n(n+1)/2} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m a^{n-m} q^{m^2+m-n(n+1)/2} + \sum_{m=1}^{\infty}\sum_{n=0}^{m-1}(-1)^m a^{m-n} q^{m^2+m-n(n+1)/2}, \qquad (3.28)
\end{aligned}
$$

where the second equality follows by setting $m = r + n$. So, Garvan's identity (1.18) can be derived by shifting n to $n-1$ in the second term on the right-hand side of (3.28). □

4. Concluding remarks

We may express the same Hecke-type double sums in different forms such as (1.1) and (1.2). In this paper, we express all the results in Corollaries 1.3 and 1.5 in terms of $f_{a,b,c}(x,y,q)$. We may also express them in other forms which might be useful for future investigation. For example, in the proof of (1.19), shifting r to $r-n$ in (3.1) yields that

$$
\begin{aligned}
\phi^{(3)}(-q) &= \frac{J_2}{J_1J_4}\sum_{n=0}^{\infty}\sum_{r=n}^{\infty}(1+q^{2r+2})(1-q^{2n+1})(-1)^r q^{2r^2+3r-n(3n+5)/2} \\
&= \frac{J_2}{J_1J_4}\sum_{r=0}^{\infty}\sum_{n=0}^{r}(1+q^{2r+2})(1-q^{2n+1})(-1)^r q^{2r^2+3r-n(3n+5)/2} \\
&= \frac{J_2}{J_1J_4}\sum_{r=1}^{\infty}\sum_{n=1}^{r}(1+q^{2r})(1-q^{2n-1})(-1)^{r-1} q^{2r^2-r-n(3n-1)/2}, \qquad (4.1)
\end{aligned}
$$

where we derive the last equality by setting $r \to r-1$ and $n \to n-1$. Since

$$
\begin{aligned}
\sum_{n=1}^{r}(1-q^{2n-1})q^{-n(3n-1)/2} &= \sum_{n=1}^{r} q^{-n(3n-1)/2} - \sum_{n=1}^{r} q^{-n(3n-5)/2-1} \\
&= \sum_{n=1}^{r} q^{-n(3n-1)/2} - \sum_{n=-r+1}^{0} q^{-n(3n-1)/2} = \sum_{n=-r+1}^{r} \mathrm{sg}'(n) q^{-n(3n-1)/2}, \qquad (4.2)
\end{aligned}
$$

where $\mathrm{sg}'(n) = 1$ if $n > 0$ and $\mathrm{sg}'(n) = -1$ otherwise, combining (4.1) and (4.2), we have

$$
\phi^{(3)}(-q) = \frac{J_2}{J_1J_4}\sum_{r=1}^{\infty}\sum_{n=-r+1}^{r}\mathrm{sg}'(n)(1+q^{2r})(-1)^{r-1}q^{2r^2-r-n(3n-1)/2}. \qquad (4.3)
$$

Of course, once we have a representation of Hecke-type double sums like (4.3), we can also rewrite it in the form of $f_{a,b,c}(x,y,q)$. For instance, from (4.3), we back to the step

(4.1). We have

$$
\begin{aligned}
\phi^{(3)}(-q) &= \frac{J_2}{J_1J_4}\sum_{r=1}^{\infty}\sum_{n=1}^{r}(1+q^{2r})(-1)^{r-1}q^{2r^2-r-n(3n-1)/2} \\
&\quad - \frac{J_2}{J_1J_4}\sum_{r=1}^{\infty}\sum_{n=1}^{r}(1+q^{2r})(-1)^{r-1}q^{2r^2-r-n(3n-5)/2-1} \\
&= \frac{J_2}{J_1J_4}\sum_{r=1}^{\infty}\sum_{n=1}^{r}(1+q^{2r})(-1)^{r-1}q^{2r^2-r-n(3n-1)/2} \\
&\quad - \frac{J_2}{J_1J_4}\sum_{r=1}^{\infty}\sum_{n=-r+1}^{0}(1+q^{2r})(-1)^{r-1}q^{2r^2-r-n(3n-1)/2} \\
&= \frac{J_2}{J_1J_4}\sum_{n=1}^{\infty}\sum_{r=n}^{\infty}(1+q^{2r})(-1)^{r-1}q^{2r^2-r-n(3n-1)/2} \\
&\quad - \frac{J_2}{J_1J_4}\sum_{n=-\infty}^{0}\sum_{r=-n+1}^{\infty}(1+q^{2r})(-1)^{r-1}q^{2r^2-r-n(3n-1)/2}.
\end{aligned}
$$

For the first part on the right-hand side of the above identity, we replace (n,r) by $(n+1, r+n+1)$, and for the second part, we replace (n,r) by $(n+1,-r-n-1)$. Thus,

$$
\begin{aligned}
\phi^{(3)}(-q) = \frac{J_2}{J_1J_4}&\left(\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} - \sum_{r=-\infty}^{-1}\sum_{n=-\infty}^{-1}\right)(1+q^{2r+2n+2}) \\
&\times(-1)^{r+n}q^{2r^2+4rn+3r+n(n+1)/2},
\end{aligned}
$$

which implies (1.19).

In 2019, Garvan [24] found some Hecke-type double sums for the seventh order mock theta functions

$$
\mathcal{F}_0^{(7)}(q) := \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^{n+1};q)_n}, \quad \mathcal{F}_1^{(7)}(q) := \sum_{n=1}^{\infty}\frac{q^{n^2}}{(q^{n};q)_n}, \quad \mathcal{F}_2^{(7)}(q) := \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^{n+1};q)_{n+1}}. \tag{4.4}
$$

Arguing similarly as above, we can express Garvan's identities [24, Theorem 1.2] in terms of $f_{a,b,c}(x,y,q)$:

$$
\begin{aligned}
\mathcal{F}_0^{(7)}(q) &= \frac{1}{J_1}\left(f_{1,3,2}(q^4,-q^4,q^3) + qf_{1,3,2}(q^5,-q^7,q^3)\right. \\
&\qquad \left.-qf_{1,3,2}(q^4,-q^{10},q^3) - q^2f_{1,3,2}(q^5,-q^{13},q^3)\right), && (4.5) \\
\mathcal{F}_1^{(7)}(q) &= \frac{q}{J_1}\left(f_{1,3,2}(q^4,-q^8,q^3) + qf_{1,3,2}(q^5,-q^{11},q^3)\right. \\
&\qquad \left.-qf_{1,3,2}(q^7,-q^8,q^3) - q^3f_{1,3,2}(q^8,-q^{11},q^3)\right), && (4.6) \\
\mathcal{F}_2^{(7)}(q) &= \frac{1}{J_1}\left(f_{1,3,2}(q^4,-q^5,q^3) + qf_{1,3,2}(q^5,-q^8,q^3)\right.
\end{aligned}
$$

$$-qf_{1,3,2}(q^7,-q^7,q^3)-q^3f_{1,3,2}(q^8,-q^{10},q^3)\big). \tag{4.7}$$

Note that Hickerson [29, Theorem 2.0] derived that

$$\mathcal{F}_0^{(7)}(q)=\frac{1}{J_1}f_{3,4,3}(q^2,q^2,q),$$
$$\mathcal{F}_1^{(7)}(q)=\frac{q}{J_1}f_{3,4,3}(q^4,q^4,q),$$
$$\mathcal{F}_2^{(7)}(q)=\frac{1}{J_1}f_{3,4,3}(q^3,q^3,q).$$

It would be interesting to find more properties of $f_{a,b,c}(x,y,q)$ to show the equivalence of these two types of expressions.

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