Erdös-Gallai-type results for conflict-free connection of graphs*

Meng Ji[1], Xueliang Li[1,2]

[1]Center for Combinatorics and LPMC

Nankai University, Tianjin 300071, China

jimengecho@163.com, lxl@nankai.edu.cn

[2]School of Mathematics and Statistics, Qinghai Normal University

Xining, Qinghai 810008, China

Abstract

A path in an edge-colored graph is called *a conflict-free path* if there exists a color used on only one of its edges. An edge-colored graph is called *conflict-free connected* if there is a conflict-free path between each pair of distinct vertices. The *conflict-free connection number* of a connected graph G, denoted by $cfc(G)$, is defined as the smallest number of colors that are required to make G conflict-free connected. In this paper, we obtain Erdös-Gallai-type results for the conflict-free connection numbers of graphs.

Keywords: conflict-free connection coloring; conflict-free connection number; Erdös-Gallai-type result.

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1 Introduction

All graphs mentioned in this paper are simple, undirected and finite. We follow book [1] for undefined notation and terminology. Let $P_1 = v_1v_2\cdots v_s$ and $P_2 = v_sv_{s+1}\cdots v_{s+t}$ be two paths. We denote $P = v_1v_2\cdots v_sv_{s+1}\cdots v_{s+t}$ by $P_1 \odot P_2$. Coloring problems are important subjects in graph theory. The hypergraph version of

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conflict-free coloring was first introduced by Even et al. in [7]. A hypergraph H is a pair $H = (X, E)$ where X is the set of vertices, and E is the set of nonempty subsets of X, called hyper-edges. The conflict-free coloring of hypergraphs was motivated to solve the problem of assigning frequencies to different base stations in cellular networks, which is defined as a vertex coloring of H such that every hyper-edge contains a vertex with a unique color.

Later on, Czap et al. in [6] introduced the concept of *conflict-free connection colorings* of graphs motivated by the conflict-free colorings of hypergraphs. A path in an edge-colored graph G is called a *conflict-free path* if there is a color appearing only once on the path. The graph G is called *conflict-free connected* if there is a conflict-free path between each pair of distinct vertices of G. The minimum number of colors required to make a connected graph G conflict-free connected is called the *conflict-free connection number* of G, denoted by $cfc(G)$. If one wants to see more results, the reader can refer to [3, 4, 5, 6]. For a general connected graph G of order n, the conflict-free connection number of G has the bounds $1 \leq cfc(G) \leq n-1$. When equality holds, $cfc(G) = 1$ if and only if $G = K_n$ and $cfc(G) = n-1$ if and only if $cfc(G) = K_{1,n-1}$.

The Erdös-Gallai-type problem is an interesting problem in extremal graph theory, which was studied in [9, 10, 11, 12] for rainbow connection number $rc(G)$; in [8] for proper connection number $pc(G)$; in [2] for monochromatic connection number $mc(G)$. We will study the Erdös-Gallai-type problem for the conflict-free number $cfc(G)$ in this paper.

2 Auxiliary results

At first, we need some preliminary results.

Lemma 2.1 [6] Let u, v be distinct vertices and let $e = xy$ be an edge of a 2-connected graph. Then there is a $u - v$ path in G containing the edge e.

For a 2-edge connected graph, the authors [5] presented the following result:

Theorem 2.2 [5] If G is a 2−edge connected graph, then $cfc(G) = 2$.

For a tree T, there is a sharp lower bound:

Theorem 2.3 [4] Let T be a tree of order n. Then $cfc(T) \geq cfc(P_n) = \lceil \log_2 n \rceil$.

Lemma 2.4 *Let G be a connected graph and $H = G - B$, where B denotes the set of the cut-edges of G. Then $cfc(G) \leq \max\{2, |B|\}$.*

Proof. If $B{=}\varnothing$, then by Theorem 2.2, $cfc(G){=}2$. If $|B| \geq 1$, then all the blocks are non-trivial in each component of $G - B$. Now we give G a conflict-free coloring: assign one edge with color 1 and the remaining edges with color 2 in each block of each component of $G - B$; for the edges $e \in B$, we assign each edge with a distinct color from $\{1, 2, \cdots, |B|\}$.

Now we check every pair of vertices. Let u and v be arbitrary two vertices. Consider first the case that u and v are in the same component of $G - B$. If u and v are in the same block, by Lemma 2.1 there is a conflict-free $u - v$ path. If u, v are in different blocks, let $P = P_1 \odot P_2 \odot \cdots \odot P_r$ be a $u - v$ path, where P_i $(i \in [r])$ is the path in each block of the component. Then we can choose a conflict-free path in one block, say P_1, and choose a monochromatic path with color 2 in each block of the remaining blocks, say P_i $(2 \leq i \leq r - 1)$, clearly, P is a conflict-free $u - v$ path. Now consider the case that u and v are in distinct components of $G - B$. If there exists one cut-edge e with color $c \notin \{1, 2\}$, then there is a conflict-free $u - v$ path since the color used on e is unique. If there does not exist cut-edge with color $c \notin \{1, 2\}$, then suppose that there is only one cut-edge $e = xy$ with color 1, without loss of generality, let u, x be in a same component and v, y be in a same component. We choose a monochromatic $u - x$ path P_1 with color 2 and choose a monochromatic $v - y$ path P_2 with 2, then $P = P_1xyP_2$ is a conflict-free $u - v$ path. If there is only one cut-edge $e = st$ colored by 2, without loss of generality, then we say u, s are in the same component and t, v in a same component, we choose a monochromatic $u - s$ path P_1 and a conflict-free $t - v$ path P_2 in each component. Then $P = P_1stP_2$ is a conflict-free $u - v$ path. If there are exactly two cut-edges $e_1 = st$ and $e_2 = xy$ colored by 1 and 2, respectively, without loss of generality, we say that u, s are in a same component, t, x are in a same component and y, v are in a same component. Then we choose a monochromatic u, s path P_1, t, x path P_2 and y, v path P_3 in the three components, respectively, with color 2. Hence, $P = P_1stP_2xyP_3$ is a conflict-free $u - v$ path. So, we have $cfc(G) \leq \max\{2, |B|\}$. □

Lemma 2.5 *Let G be a connected graph of order n with k cut-edges. Then*

$$|E(G)| \leq \tbinom{n}{k} + k$$

.

Proof. Clearly, it holds for $k = 0$. Assuming that $k \geq 1$. Let G be a maximal graphs with k cut-edges. Let B be the set of all the bridges. And let $G - B$ be the graph by deleting all the cut-edges. Let $C_1, C_2, \cdots, C_{k+1}$ be the components of $G - B$ and n_i be the orders of C_i. Then $E(G) = \sum_{i=1}^{k+1} \binom{n_i}{2} + k$. Let C_i and C_j be two components of $G - B$ with $1 < n_i \leq n_j$. Now we construct a graph G' by moving a vertex v from C_i to C_j, replace v with an arbitrary vertex in $V(C_k) \setminus v$ for the cut-edges incident with v, add the edges between v and the vertices in C_j, and delete the edges between v and the vertices in C_i, where v is not adjacent to the vertices of C_i. Now we have $|E(G')| = \sum_{s=1 \neq i,j}^{k+1} \binom{n_s}{2} + \binom{n_i - 1}{2} + \binom{n_j + 1}{2} + k = \sum_{s=1 \neq i,j}^{k+1} \binom{n_s}{2} + \binom{n_i}{2}$- $n_i - 1 + \binom{n_j}{2} + n_j + k = |E(G)| + n_j - n_i + 1 > |E(G)|$. When we do repetitively the operation, we have $|E(G)| \leq \binom{n}{k} + k$. □

3 Main results

Now we consider the Erdös-Gallai-type problems for $cfc(G)$. There are two types, see below.

Problem 3.1 *For each integer k with $2 \leq k \leq n - 1$, compute and minimize the function $f(n, k)$ with the following property: for each connected graph G of order n, if $|E(G)| \geq f(n, k)$, then $cfc(G) \leq k$.*

Problem 3.2 *For each integer k with $2 \leq k \leq n - 1$, compute and maximize the function $g(n, k)$ with the following property: for each connected graph G of order n, if $|E(G)| \leq g(n, k)$, then $cfc(G) \geq k$.*

Clearly, there are two parameters which are equivalent to $f(n, k)$ and $g(n, k)$ respectively. For each integer k with $2 \leq k \leq n - 1$, let $s(n, k) = \max\{|E(G)| : |V(G)| = n, cfc \geq k\}$ and $t(n, k) = \min\{|E(G)| : |V(G)| = n, cfc \leq k\}$. By the definitions, we have $g(n, k) = t(n, k - 1) - 1$ and $f(n, k) = s(n, k + 1) + 1$.

Using Lemma 2.4 we first solve Problem 3.1.

Theorem 3.3 $f(n, k) = \binom{n-k-1}{2} + k + 2$ *for* $2 \leq k \leq n - 1$.

Proof. At first, we show the following claims.

Claim 1: For $k \geq 2$, $f(n, k) \leq \binom{n-k-1}{2} + k + 2$.

Proof of Claim 1: We need to prove that for any connected graph G, if $E(G) \geq \binom{n-k-1}{2} + k + 2$, then $cfc(G) \leq k$. Suppose to the contrary that $cfc(G) \geq k + 1$. By Lemma 2.4, we have $|B| \geq k + 1$. By Lemma 2.5, $E(G) \leq \binom{n-k-1}{2} + k + 1$, which is a contradiction.

Claim 2: For $k \geq 2$, $f(n,k) \geq \binom{n-k-1}{2} + k + 2$.

Proof of Claim 2: We construct a graph G_k by identifying the center vertex of a star S_{k+2} with an arbitrary vertex of K_{n-k-1}. Clearly, $E(G_k) = \binom{n-k-1}{2} + k + 1$. Since $cfc(S_{k+2}) = k + 1$, then $cfc(G_k) \geq k + 1$. It is easy to see that $cfc(G_k) = k + 1$. Hence, $f(n,k) \geq \binom{n-k-1}{2} + k + 2$.

The conclusion holds from Claims 1 and 2. □

Now we come to the solution for Problem 3.2, which is divided as three cases.

Lemma 3.4 *For $k = 2$, $g(n,2) = \binom{n}{2} - 1$.*

Proof. Let G be a complete graph of order n. The number of edges in G is $\binom{n}{2}$, *i.e.*, $E(G) = \binom{n}{2}$. Clearly, when $g(n,2) = \binom{n}{2} - 1$ for every G, $cfc(G) \geq 2$. □

Lemma 3.5 *For every integer k with $3 \leq k < \lceil \log_2 n \rceil$, $g(n,k) = n - 1$.*

Proof. We first give an upper bound of $t(n,k)$. Let C_n be a cycle. Then $t(n,k) \leq n$ since $cfc(C_n) = 2 \leq k$. And then, we prove that $t(n,k) = n$. Suppose $t(n,k) \leq n - 1$. Let P_n be a path with size $n - 1$. Since $cfc(P_n) = \lceil \log_2 n \rceil$ by Theorem 2.3, it contradicts the condition the $k < \lceil \log_2 n \rceil$. So $t(n,k) = n$. By the relation that $g(n,k) = t(n,k-1) - 1$, we have $g(n,k) = n - 1$. □

Lemma 3.6 *For $k \geq \lceil \log_2 n \rceil$, $g(n,k)$ does not exist.*

Proof. Let P_n be a path. Then we have $t(n,k) \leq n - 1$ since $cfc(P_n) = \lceil \log_2 n \rceil$. And since $t(n,k) \geq n - 1$, it is clear that $t(n,k) = n - 1$. Since every graph G is connected, $g(n,k) \geq n - 1$. By the relation that $g(n,k) = t(n,k-1) - 1$, we have $g(n,k) = n - 2$ for $k \geq \lceil \log_2 n \rceil$, which contradicts the connectivity of graphs. □

Combining Lemmas 3.4, 3.5 and 3.6, we get the solution for Problem 3.2.

Theorem 3.7 *For k with $2 \leq k \leq n - 1$,*

$$g(n,k) = \begin{cases} \binom{n}{2} - 1, & k = 2 \\ n - 1, & 3 \leq k < \lceil \log_2 n \rceil \\ \textit{does not exist}, & \lceil \log_2 n \rceil \leq k \leq n - 1. \end{cases}$$

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