Gallai-Ramsey number for K_4

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Abstract

Given a graph H, the k-colored Gallai-Ramsey number $gr_k(K_3 : H)$ is defined to be the minimum integer n such that every k-coloring (using all k colors) of the complete graph on n vertices contains either a rainbow triangle or a monochromatic copy of H. Recently, Fox et al. [J. Fox, A. Grinshpun, and J. Pach. The Erdős-Hajnal conjecture for rainbow triangles. J. Combin. Theory Ser. B, 111:75-125, 2015.] conjectured the value of the Gallai-Ramsey numbers for complete graphs. We verify this conjecture for the first open case, where $H = K_4$.

1 Introduction

In this work, we consider colorings of only the edges of graphs. Providing a general bound on the classical Ramsey numbers, a classical result of Erdős and Szekeres is stated as follows.

Theorem 1 ([5]). *Every graph on n vertices contains either a clique or an independent set of order at least $\frac{1}{2}\log n$.*

Although many years have passed since this result and some small improvements have been made, this fundamentally still stands as one of the best general bounds on Ramsey numbers. When certain subgraphs are forbidden, it was conjectured by Erdős and Hajnal that this result could be greatly strengthened.

Conjecture 2 ([4]). *For any fixed graph H, there exists a number $\epsilon = \epsilon(H) > 0$ such that, every graph on n vertices which does not contain H as an induced subgraph, contains either a clique or an independent set of order at least n^ϵ.*

There have been many results related to this conjecture, particularly proving special cases for a fixed graph H. For a list of these, we refer to the recent survey [2]. Erdős and Hajnal also proposed the following multicolored version of this conjecture.

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Conjecture 3 ([4])**.** *For every fixed k-coloring χ of a complete graph, there is an $\epsilon = \epsilon(\chi) > 0$ such that, every k-coloring of a complete graph on n vertices without a copy of χ, contains a set of order at least n^ϵ which uses only $k-1$ colors.*

Even more specifically, Hajnal conjectured the following special case.

Conjecture 4 ([12])**.** *There is an $\epsilon > 0$ such that, every 3-coloring of a complete graph on n vertices without a rainbow triangle, contains a set of order at least n^ϵ which uses only 2 colors.*

Fox et al. proved this conjecture in the following stronger form.

Theorem 5 ([6])**.** *Every rainbow triangle free 3-coloring of a complete graph of order n contains a set of order $\Omega(n^{1/3}\log^2 n)$ which uses only 2 colors, and this bound is tight up to a constant factor.*

In their proof, Fox et al. used the following result of Gallai, which provides a strong structure on edge-colorings of complete graphs containing no rainbow triangle.

Theorem 6 ([1, 9, 11])**.** *In any edge-coloring of a complete graph with no rainbow triangle, there exists a partition of the vertices into at least two parts (called a* Gallai partition *or* G-partition *for short) such that, there are at most two colors on the edges between the parts, and only one color on the edges between each pair of parts.*

In light of this result, we say that a colored complete graph with no rainbow triangle is a *Gallai coloring* (or *G-coloring* for short).

Closely related to their results, Fox et al. also posed a conjecture about monochromatic complete graphs. In order to concisely state their conjecture, we make the following definition. Given a graph H, the (k-*colored*) *Gallai-Ramsey number* $gr_k(K_3 : H)$ is defined to be the minimum integer n such that every k-coloring (using all k colors) of the complete graph on n vertices contains either a rainbow triangle or a monochromatic copy of H.

We refer to the survey of rainbow generalizations of Ramsey Theory [7, 8] for more information on this topic and results involving Gallai-Ramsey numbers.

Recall that the *Ramsey number* $r(p, q)$ is the minimum integer n such that, for every coloring of the edges of the complete graph on n vertices, using red and blue, there is either a red clique of order p, or a blue clique of order q. In particular, we write $r(p) = r(p, p)$. We are now able to state the conjecture.

Conjecture 7 ([6])**.** *For $k \geq 1$ and $p \geq 3$,*

$$gr_k(K_3 : K_p) = \begin{cases} (r(p)-1)^{k/2}+1 & \text{if } k \text{ is even,} \\ (p-1)(r(p)-1)^{(k-1)/2}+1 & \text{if } k \text{ is odd.} \end{cases}$$

The case where $p = 3$ was actually verified in 1983 by Chung and Graham [3]. A simplified proof was given by Gyárfás et al. [10].

Theorem 8 ([3, 10])**.** *For $k \geq 1$,*

$$gr_k(K_3 : K_3) = \begin{cases} 5^{k/2}+1 & \text{if } k \text{ is even,} \\ 2 \cdot 5^{(k-1)/2}+1 & \text{if } k \text{ is odd.} \end{cases}$$

In this paper, we prove the following result, the first open case of Conjecture 7.

Theorem 9. *For $k \geq 1$,*

$$gr_k(K_3 : K_4) = \begin{cases} 17^{k/2}+1 & \text{if } k \text{ is even,} \\ 3 \cdot 17^{(k-1)/2}+1 & \text{if } k \text{ is odd.} \end{cases}$$

2 Proof of Theorem 9

Define the *refined k-colored Gallai-Ramsey number* $gr_k(K_3 : H_1, H_2, \dots, H_k)$ to be the minimum number of vertices n such that, every k-coloring of the complete graph on n vertices contains either a rainbow triangle, or a copy of H_i in color i, for some i. Since we will generally be working only with K_4 and K_3, for an integer s with $0 \leq s \leq k$, we use the following shorthand notation

$$gr_k(K_3 : sK_4, (k-s)K_3) = gr_k(K_3 : K_4, K_4, \dots, K_4, K_3, K_3, \dots, K_3)$$

where we look for K_4 in any of the first s colors or K_3 in any of the remaining $k-s$ colors.

In order to prove Theorem 9, we actually prove the following refined version. Theorem 9 follows as a corollary to Theorem 10 by choosing $s = k$.

Theorem 10. *Let $k \geq 1$, and s be an integer with $0 \leq s \leq k$. Then*

$$gr_k(K_3 : sK_4, (k-s)K_3) = g(k, s)$$

where

$$g(k,s) = \begin{cases} 17^{s/2} \cdot 5^{(k-s)/2} + 1 & \text{if } s \text{ and } (k-s) \text{ are both even,} \\ 2 \cdot 17^{s/2} \cdot 5^{(k-s-1)/2} + 1 & \text{if } s \text{ is even and } (k-s) \text{ is odd,} \\ 3 \cdot 17^{(k-1)/2} + 1 & \text{if } s = k \text{ and } s \text{ is odd,} \\ 8 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-1)/2} + 1 & \text{if } s \text{ and } (k-s) \text{ are both odd,} \\ 16 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-2)/2} + 1 & \text{if } s < k, \text{ and } s \text{ is odd, and } (k-s) \text{ is even.} \end{cases}$$

For the sake of notation, we define the functions g_j for $j \in \{1, 2, \dots, 5\}$ to be

$$\begin{aligned} g_1(k,s) &= 17^{s/2} \cdot 5^{(k-s)/2}, \\ g_2(k,s) &= 2 \cdot 17^{s/2} \cdot 5^{(k-s-1)/2}, \\ g_3(k,s) &= 3 \cdot 17^{(k-1)/2}, \\ g_4(k,s) &= 8 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-1)/2}, \\ g_5(k,s) &= 16 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-2)/2}, \end{aligned}$$

where by convention, $g_3(k, s)$ is defined only for $s = k$.

Proof. We first prove the lower bound of Theorem 10 by construction. For this construction, we will use the sharpness examples from classical Ramsey results. For $i, j \in \{3, 4\}$, let $H_{i,j}$ be a sharpness example of order $r(i, j) - 1$. In particular, $|H_{4,4}| = 17$, $|H_{4,3}| = 8$, and $|H_{3,3}| = 5$. We construct our sharpness example by taking *blow-ups* of these graphs, that is, replacing each vertex with a particular graph and replacing each edge with a complete bipartite graph in the same color.

Let G_0 be a single vertex and we iteratively construct G-colored graphs G_i using i colors forbidding the appropriate monochromatic subgraphs for the first i colors in the statement. By induction, suppose we have constructed G_i. If $i = k$, then the construction is completed. Otherwise, we consider the following cases.

- If $i \leq s - 2$, construct G_{i+2} by making 17 copies of G_i and inserting each in place of a vertex in a blow-up of $H_{4,4}$ using colors $i + 1$ and $i + 2$.
- If $i = s - 1$ and $k = s$, then construct G_{i+1} by making 3 copies of G_i and inserting each in place of a vertex in a blow-up of K_3 using color $i + 1$.
- If $i = s - 1$ and $k > s$, then construct G_{i+2} by making 8 copies of G_i and inserting each in place of a vertex in a blow-up of $H_{4,3}$ using colors $i + 1$ and $i + 2$.

- If $i \geq s$ and $i = k-1$, then construct G_{i+1} by making 2 copies of G_i and inserting each in place of a vertex in a blow-up of K_2 using color $i+1$.

- If $i \geq s$ and $i \leq k-2$, then construct G_{i+2} by making 5 copies of G_i and inserting each in place of a vertex in a blow-up of $H_{3,3}$ using colors $i+1$ and $i+2$.

Note that, if k is even, then we successively obtain the graphs $G_0, G_2, G_4, \ldots, G_k$; and if k is odd, then we successively obtain the graphs $G_0, G_2, G_4, \ldots, G_{k-1}, G_k$. In particular, i is always even in the above five iterative procedures. By construction, it is clear that G_k is a G-coloring and it contains no copy of K_4 in any of the first s colors and no copy of K_3 in any of the remaining $k-s$ colors. The order of G_k then follows the theorem statement as

$$|G_k| = \begin{cases} g_1(k,s) & \text{if } s \text{ and } (k-s) \text{ are both even,} \\ g_2(k,s) & \text{if } s \text{ is even and } (k-s) \text{ is odd,} \\ g_3(k,s) & \text{if } s = k \text{ and } s \text{ is odd,} \\ g_4(k,s) & \text{if } s \text{ and } (k-s) \text{ are both odd,} \\ g_5(k,s) & \text{if } s < k, \text{ and } s \text{ is odd, and } (k-s) \text{ is even.} \end{cases}$$

For the upper bound, we prove the desired result by induction on $k+s$. The case $k=1$ is trivial, and the case $k=2$ follows from the classical Ramsey numbers $r(3,3)=6$, $r(4,3)=9$ and $r(4,4)=18$. The case $s=0$ is Theorem 8. Now let $0 < s \leq k$ with $k \geq 3$, and suppose that Theorem 10 holds for all $k'+s' < k+s$. Let G be a G-colored complete graph on $g(k,s)$ vertices and suppose G contains no monochromatic copy of K_4 in any of the first s colors and no monochromatic copy of K_3 in any of the remaining $k-s$ colors. By Theorem 6, there is a G-partition of the vertices of G such that, there is only one color on all edges between each pair of parts, and there are only two colors in total on all edges between the parts. Choose a G-partition with the smallest number of parts, let $t \geq 2$ be the number of parts in this G-partition, and let G_i be the parts of this partition for $1 \leq i \leq t$. Let red and blue be the two colors between the parts. Choosing one vertex w_i from each part G_i of the partition yields a 2-colored *reduced graph* R of G by setting $R = G[\{w_1, w_2, \ldots, w_t\}]$. Since $r(4,4)=18$, we know that $t \leq 17$ so there are at most seventeen parts in the partition. We first suppose $t \in \{2,3\}$.

Claim 11. *If $t \in \{2,3\}$, then the desired result holds.*

Proof. If $t=3$, then the G-partition is a blow-up of a triangle using at most 2 colors, say with at least two red edges and perhaps one blue edge. Then there is a part G_i of the G-partition with all edges to the other parts being red. This means that G can be G-partitioned into G_i and $G \setminus G_i$, contradicting the assumption that the G-partition of G was chosen with t as small as possible. Note here that we identify the graph G with its vertex set $V(G)$ whenever no confusion arises. Hence, we may assume $t=2$. Suppose the color on the edges between the two parts of the partition is red.

If red is one of the last $k-s$ colors (so $s<k$), then to avoid creating a red triangle, there can be no red edge within either part. By induction, this means that both parts of the G-partition have

order at most $gr_{k-1}(K_3 : sK_4, (k-s-1)K_3) - 1$, and we get

$$\begin{aligned}
|G| &\leq 2[gr_{k-1}(K_3 : sK_4, (k-s-1)K_3) - 1] \\
&= \begin{cases} 4 \cdot 17^{s/2} \cdot 5^{(k-s-2)/2} & \text{if } s \text{ and } (k-s) \text{ are both even,} \\ 2 \cdot 17^{s/2} \cdot 5^{(k-s-1)/2} & \text{if } s \text{ is even and } (k-s) \text{ is odd,} \\ 6 \cdot 17^{(k-2)/2} & \text{if } s = k-1 \text{ and } s \text{ is odd,} \\ 32 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-3)/2} & \text{if } s < k-1, \text{ and } s \text{ and } (k-s) \text{ are both odd,} \\ 16 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-2)/2} & \text{if } s \text{ is odd and } (k-s) \text{ is even} \end{cases} \\
&< \begin{cases} g_1(k,s)+1 & \text{if } s \text{ and } (k-s) \text{ are both even,} \\ g_2(k,s)+1 & \text{if } s \text{ is even and } (k-s) \text{ is odd,} \\ g_4(k,s)+1 & \text{if } s \text{ and } (k-s) \text{ are both odd,} \\ g_5(k,s)+1 & \text{if } s \text{ is odd and } (k-s) \text{ is even} \end{cases} \\
&= g(k,s),
\end{aligned}$$

for a contradiction.

If red is one of the first s colors, then to avoid creating a red K_4 in G, there cannot be a red edge in both parts so suppose there is at least one red edge in G_1. Then there are no red edges in G_2 and there is certainly no red triangle in G_1, meaning that by induction, we get

$$\begin{aligned}
|G| &= |G_1| + |G_2| \\
&\leq [gr_k(K_3 : (s-1)K_4, (k-s+1)K_3) - 1] + [gr_{k-1}(K_3 : (s-1)K_4, (k-s)K_3) - 1] \\
&= \begin{cases} 8 \cdot 17^{(k-2)/2} + 3 \cdot 17^{(k-2)/2} \\ \qquad \text{if } s = k \text{ and } k \text{ is even,} \\ 8 \cdot 17^{(s-2)/2} \cdot 5^{(k-s)/2} + 16 \cdot 17^{(s-2)/2} \cdot 5^{(k-s-2)/2} \\ \qquad \text{if } s < k \text{ and } s \text{ and } (k-s) \text{ are both even,} \\ 16 \cdot 17^{(s-2)/2} \cdot 5^{(k-s-1)/2} + 8 \cdot 17^{(s-2)/2} \cdot 5^{(k-s-1)/2} \\ \qquad \text{if } s \text{ is even and } (k-s) \text{ is odd,} \\ 2 \cdot 17^{(k-1)/2} + 17^{(k-1)/2} \\ \qquad \text{if } s = k \text{ and } s \text{ is odd,} \\ 17^{(s-1)/2} \cdot 5^{(k-s+1)/2} + 2 \cdot 17^{(s-1)/2} \cdot 5^{(k-s-1)/2} \\ \qquad \text{if } s \text{ and } (k-s) \text{ are both odd} \\ 2 \cdot 17^{(s-1)/2} \cdot 5^{(k-s)/2} + 17^{(s-1)/2} \cdot 5^{(k-s)/2} \\ \qquad \text{if } s < k, \text{ and } s \text{ is odd, and } (k-s) \text{ is even} \end{cases} \\
&< \begin{cases} g_1(k,s)+1 & \text{if } s \text{ and } (k-s) \text{ are both even,} \\ g_2(k,s)+1 & \text{if } s \text{ is even and } (k-s) \text{ is odd,} \\ g_3(k,s)+1 & \text{if } s = k \text{ and } k \text{ is odd,} \\ g_4(k,s)+1 & \text{if } s \text{ and } (k-s) \text{ are both odd} \\ g_5(k,s)+1 & \text{if } s < k, \text{ and } s \text{ is odd, and } (k-s) \text{ is even} \end{cases} \\
&= g(k,s),
\end{aligned}$$

for a contradiction, completing the proof of Claim 11. □

By Claim 11, we may assume $4 \leq t \leq 17$. Let

$$V_r = \{w_i \in V(R) : G_i \text{ contains a red edge}\},$$
$$V_b = \{w_i \in V(R) : G_i \text{ contains a blue edge}\},$$

and $t = p_2 + p_1 + p_0$, where

$$p_2 = |V_r \cap V_b|, \quad p_1 = |V_r \triangle V_b|, \quad p_0 = |V(R) \setminus (V_r \cup V_b)|.$$

For each vertex $w_i \in V(R)$, let $d_r(w_i)$ and $d_b(w_i)$ denote its red and blue degrees respectively within R. Then $d(w_i) = d_r(w_i) + d_b(w_i) = t - 1$. Since we chose a G-partition with the smallest number of parts, we immediately see the following fact.

Fact 1. *For all $w_i \in V(R)$, we have $d_r(w_i), d_b(w_i) \geq 1$.*

If $w_i, w_j \in V_r$ and w_iw_j is a red edge in R, then by taking a red edge from each of G_i and G_j, we clearly have a red K_4. A similar observation holds for blue, and thus we have the following fact.

Fact 2. *The induced subgraph $R[V_r]$ is a blue clique, and $R[V_b]$ is a red clique.*

Clearly if w_i is in a red triangle in R, and $w_i \in V_r$, then G contains a red K_4. A similar observation holds for blue. Thus we have the following fact.

Fact 3. *If w_i is in a red triangle in R, then $w_i \notin V_r$. If w_i is in a blue triangle in R, then $w_i \notin V_b$.*

We now consider cases based on where red and blue are in the list relative to the first s colors.

Case 1. *Red and blue are both among the latter $k - s$ colors.*

This means that in G, there is no red triangle or blue triangle, which implies $s < k$. Since $r(3, 3) = 6$, we have $4 \leq t \leq 5$. By Fact 1, it follows that G_i contains no red edges and no blue edges for every i. This means that G_i is colored with $k-2$ colors available and within G_i, and there is no K_4 in one of the first s colors, and no K_3 in one of the remaining $k-s-2$ colors. Since $k-s \equiv k-s-2 \pmod 2$ and $g_j(k-2, s)/g_j(k, s) = \frac{1}{5}$ for $j \in \{1, 2, 4, 5\}$, we have

$$\frac{g(k-2, s) - 1}{g(k, s) - 1} \leq \max\left\{\frac{g_j(k-2, s)}{g_j(k, s)} : j \in \{1, 2, 4, 5\}\right\} = \frac{1}{5}.$$

Then by the induction hypothesis, we obtain

$$|G_i| \leq gr_{k-2}(K_3 : sK_4, (k-s-2)K_3) = g(k-2, s) - 1$$

and

$$\begin{aligned} |G| = \sum_{i=1}^{t} |G_i| &\leq \sum_{i=1}^{t} \big(g(k-2, s) - 1\big) \\ &\leq \sum_{i=1}^{t} \frac{1}{5}\big(g(k, s) - 1\big) = \frac{t}{5}\big(g(k, s) - 1\big) \leq g(k, s) - 1, \end{aligned}$$

a contradiction, completing the proof of Case 1.

Case 2. *Red is among the first s colors while blue is among the remaining $k - s$ colors.*

This means that in G, there is no red K_4 or blue triangle, which again implies $s < k$. By Fact 1, no G_i can have any blue edge in this case, implying that $|V_b| = 0$, $p_2 = |V_r \cap V_b| = 0$, and $p_1 = |V_r|$. Since $r(4, 3) = 9$, we have $4 \leq t \leq 8$. We first prove a key inequality for $|G|$.

First suppose G_i contains no red edges. Then G_i is colored with $k-2$ colors available and within G_i, there is no K_4 in one of the first $s-1$ colors, and no K_3 in one of the remaining $k-s-1$ colors. On the other hand, by the definition of $g(k,s)$, if $k \neq s$, we have

$$\frac{g(k-2,s-1)-1}{g(k,s)-1} = \begin{cases} \frac{g_4(k-2,s-1)}{g_1(k,s)} = \frac{8}{5\cdot 17} & \text{if } s \text{ and } k-s \text{ are both even,} \\ \frac{g_1(k-2,s-1)}{g_4(k,s)} = \frac{1}{8} & \text{if } s \text{ and } k-s \text{ are both odd,} \\ \frac{g_2(k-2,s-1)}{g_5(k,s)} = \frac{1}{8} & \text{if } s<k, \text{ and } s \text{ is odd, and } k-s \text{ is even,} \\ \frac{g_5(k-2,s-1)}{g_2(k,s)} = \frac{16}{2\cdot 5\cdot 17} & \text{if } s \text{ is even, and } k-s \text{ is odd, and } k-s\geq 3, \\ \frac{g_3(k-2,s-1)}{g_2(k,s)} = \frac{3}{2\cdot 17} & \text{if } s \text{ is odd and } k-s=1 \end{cases}$$
$$\leq \frac{1}{8}.$$

Therefore we have, by the induction hypothesis,

$$|G_i| \leq gr_{k-2}(K_3 : (s-1)K_4, (k-s-1)K_3) \leq g(k-2,s-1)-1 \leq \frac{1}{8}\big(g(k,s)-1\big) \tag{1}$$

Next suppose G_i contains at least one red edge. Then G_i is colored with $k-1$ colors available and within G_i, there is no K_4 in one of the first $s-1$ colors (which excludes red), and no K_3 in one of the remaining $k-s$ colors (which includes red, by Fact 1). Therefore since

$$\frac{g_5(k-1,s-1)}{g_1(k,s)} = \frac{16}{5\cdot 17} < \frac{5}{16}, \qquad \frac{g_4(k-1,s-1)}{g_2(k,s)} = \frac{8}{2\cdot 17} < \frac{5}{16},$$

$$\frac{g_2(k-1,s-1)}{g_4(k,s)} = \frac{2}{8}, \quad \text{and} \quad \frac{g_1(k-1,s-1)}{g_5(k,s)} = \frac{5}{16},$$

we get, by induction

$$|G_i| \leq gr_{k-1}(K_3 : (s-1)K_4, (k-s)K_3) - 1 \leq \frac{5}{16}\left[g(k,s)-1\right]. \tag{2}$$

Therefore by Inequalities (1) and (2), we get the key inequality

$$|G| \leq \left(p_1\frac{5}{16} + p_0\frac{1}{8}\right)\left[g(k,s)-1\right]. \tag{3}$$

Therefore, if we show that

$$p_1\frac{5}{16} + p_0\frac{1}{8} \leq 1, \tag{4}$$

then we obtain the contradiction $|G| < g(k,s)$. Thus for the remainder of Case 2, it suffices to prove Inequality (4).

Claim 12. $p_1 = |V_r| \leq 2$.

Proof. If $p_1 = |V_r| \geq 3$, then by Fact 2, $R[V_r]$, and thus G, contains a blue triangle, which is a contradiction. □

If $t \leq 5$, then with Claim 12, we get

$$p_1 \frac{5}{16} + p_0 \frac{1}{8} = \frac{2t + 3p_1}{16} \leq 1,$$

as required. We may therefore assume $t \geq 6$. First, assume $t \in \{7, 8\}$.

Claim 13. *If $t \in \{7, 8\}$, then $p_1 = |V_r| = 0$.*

Proof. In order to prove Claim 13, we show that every vertex $w_i \in V(R)$ is contained in a red triangle, from which the claim follows from Fact 3. Indeed, suppose that w_i is not in a red triangle. Then since there is no blue triangle, $d_r(w_i) \leq 2$ because otherwise the red neighborhood of w_i either has a red edge or contains a blue triangle. By the same logic, since there is no red K_4 and no blue triangle, we get $d_b(w_i) \leq 3$. This means that

$$d(w_i) = d_r(w_i) + d_b(w_i) \leq 5$$

so $t \leq 6$, a contradiction, completing the proof of Claim 13. □

By Claim 13, we get

$$p_1 \frac{5}{16} + p_0 \frac{1}{8} = \frac{t}{8} \leq 1.$$

Finally, let $t = 6$.

Claim 14. *If $t = 6$, then $p_1 = |V_r| \leq 1$.*

Proof. If every w_i is contained in a red triangle, then the claim follows from Fact 3. Thus without loss of generality, suppose w_1 is not contained in a red triangle. Since there is no blue triangle, we have $d_r(w_1) = 2$ and $d_b(w_1) = 3$. Let w_2 and w_3 be the neighbors of w_1 via red edges, and w_4, w_5 and w_6 be the neighbors of w_1 via blue edges. By assumption, w_2w_3 is blue and since there is no blue triangle, we see that the set $\{w_4, w_5, w_6\}$ induces a red triangle. Since there is no red K_4, each of w_2 and w_3 cannot have three red edges to $\{w_4, w_5, w_6\}$. If w_2 and w_3 each have exactly two red edges to $\{w_4, w_5, w_6\}$, then w_1 is the only vertex that is not in a red triangle, so Fact 3 implies $p_1 = |V_r| \leq 1$. Thus, at least one of w_2 or w_3 has at most one red edge to $\{w_4, w_5, w_6\}$ so there are at most 3 red edges between $\{w_2, w_3\}$ and $\{w_4, w_5, w_6\}$. Since there is no blue triangle, there are also at most three blue edges between $\{w_2, w_3\}$ and $\{w_4, w_5, w_6\}$. Putting these two facts together, we see that there are exactly three edges of each color between $\{w_2, w_3\}$ and $\{w_4, w_5, w_6\}$.

Without loss of generality, assume that w_2w_4, w_3w_5, and w_3w_6 are red and the remaining edges between the two sets are blue. Then each of $\{w_3, w_4, w_5, w_6\}$ is contained in a red triangle, leaving behind only w_1 and w_2 not in any red triangles. Since the edge w_1w_2 is red and there is no red K_4, at most one of G_1 or G_2 can contain any red edges. This shows that $p_1 = |V_r| \leq 1$, completing the proof of Claim 14. □

By Claim 14, we get

$$p_1 \frac{5}{16} + p_0 \frac{1}{8} = \frac{12 + 3p_1}{16} \leq \frac{15}{16} < 1,$$

thus completing the proof of Case 2.

Case 3. *Red and blue are both among the first s colors.*

This means that in G, there is no red K_4 or blue K_4. Recall that $4 \leq t \leq 17$. We first derive a similar inequality to Inequality (3) in Case 2.

First suppose G_i contains no red and no blue edges. This means that G_i is colored with $k - 2$ colors available and within G_i, there is no K_4 in one of the first $s - 2$ colors, and no K_3 in one of the remaining $k - s$ colors. We have

$$\frac{g_j(k-2, s-2) - 1}{g_j(k, s) - 1} = \frac{1}{17},$$

for all $j \in \{1, 2, \ldots, 5\}$ which, by induction, means that

$$|G_i| \le gr_{k-2}(K_3 : (s-2)K_4, (k-s)K_3) - 1 = \frac{1}{17}\left[g(k,s) - 1\right]. \tag{5}$$

Next suppose G_i contains no blue edges but contains red edges. This means that G_i is colored with $k-1$ colors available and within G_i, there is no K_4 in one of the first $s-2$ colors (which excludes red), and no K_3 in one of the remaining $k-s+1$ colors (which includes red, by Fact 1). Since

$$\frac{g_2(k-1, s-2)}{g_1(k,s)} = \frac{g_5(k-1, s-2)}{g_4(k,s)} = \frac{2}{17},$$

$$\frac{g_1(k-1, s-2)}{g_2(k,s)} = \frac{g_4(k-1, s-2)}{g_5(k,s)} = \frac{5/2}{17}, \text{ and } \frac{g_4(k-1, k-2)}{g_3(k,k)} = \frac{8/3}{17},$$

we see that, by induction

$$|G_i| \le gr_{k-1}(K_3 : (s-2)K_4, (k-s+1)K_3) - 1 \le \frac{8/3}{17}\left[g(k,s) - 1\right]. \tag{6}$$

The same inequality holds if G_i contains no red edges but contains blue edges.

Finally suppose G_i contains both red and blue edges. This means that G_i is colored with all k colors available and within G_i, there is no K_4 in one of the first $s-2$ colors (which excludes both red and blue), and no K_3 in one of the remaining $k-s+2$ colors (which includes both red and blue, by Fact 1). Since $g_j(k, s-2)/g_j(k,s) = \frac{5}{17}$ for $j \in \{1, 2, 4, 5\}$ and $g_5(k, k-2)/g_3(k,k) = \frac{16/3}{17}$, we have $\frac{g(k,s-2)-1}{g(k,s)-1} \le \frac{16/3}{17}$ and

$$|G_i| \le gr_k(K_3 : (s-2)K_4, (k-s+2)K_3) - 1 \le \frac{16/3}{17}[g(k,s) - 1]. \tag{7}$$

Combining Inequalities (5), (6) and (7), we obtain the key inequality

$$|G| \le \left(p_2 \frac{16/3}{17} + p_1 \frac{8/3}{17} + p_0 \frac{1}{17}\right)\left[g(k,s) - 1\right].$$

As in Case 2, if we can show that

$$p_2 \frac{16/3}{17} + p_1 \frac{8/3}{17} + p_0 \frac{1}{17} \le 1, \tag{8}$$

then we will arrive at a contradiction that $|G| < g(k,s)$. Thus for the remainder of the proof, it suffices to show Inequality (8).

Next we derive two facts. Within the red neighborhood of w_i in R, there can be no red triangle since otherwise we would have a red K_4 in G. There can also be no blue K_4 within this neighborhood so that means the red neighborhood of w_i (and similarly the blue neighborhood) has at most $r(4,3) - 1 = 8$ vertices. Formally, we obtain the following fact.

Fact 4. *For all $w_i \in V(R)$, we have $d_r(w_i), d_b(w_i) \le 8$.*

If $d_r(w_i) \ge 4$ for some $w_i \in V(R)$, then the red neighborhood of w_i certainly must contain at least one red edge since otherwise, if all edges were blue, we would have a blue K_4. Thus w_i is in a red triangle in R. A similar observation holds with the roles of red and blue switched. Thus from Fact 3, we obtain the following fact.

Fact 5. *If $d_r(w_i) \ge 4$ then $w_i \notin V_r$, and if $d_b(w_i) \ge 4$ then $w_i \notin V_b$.*

Next, we prove two claims.

Claim 15. $p_2 = |V_r \cap V_b| \le 1$.

Proof. If we have $w_i, w_j \in V_r \cap V_b$, then by Fact 2, $w_i, w_j \in V_r$ implies that the edge w_iw_j is blue in R, while $w_i, w_j \in V_b$ implies that w_iw_j is red. This is a contradiction. □

Claim 16. $|V_r| + |V_b| \le 4$.

Proof. Suppose first that there is a vertex $w_i \in V_r \cap V_b$. Then by Fact 3, w_i is contained in neither a red triangle nor a blue triangle within R. By Fact 2, any vertex of $V_r \setminus \{w_i\}$ must be a blue neighbor of w_i in R, and since the blue neighborhood of w_i induces a red clique in R, again Fact 2 implies that there can only be at most one vertex in $V_r \setminus \{w_i\}$. This means that $|V_r| \le 2$, and similarly, $|V_b| \le 2$.

Thus, we may assume $V_r \cap V_b = \emptyset$. We next claim that $|V_r| \le 3$ and $|V_b| \le 3$. If $|V_r| \ge 4$, then by Fact 2, the subgraph of R induced on the vertices of V_r contains a blue K_4, a contradiction. Thus $|V_r| \le 3$, and symmetrically $|V_b| \le 3$.

Now suppose that $|V_r| = |V_b| = 3$. If there exists a vertex $w_i \in V_r$ with at least two red neighbors in V_b, then by Fact 2, w_i is in a red triangle in R, and this contradicts Fact 3. Thus, there can be at most one red edge from each vertex in V_r to V_b, and similarly, at most one blue edge from each vertex in V_b to V_r, for a total of at most 6 edges. But R has 9 edges between V_r and V_b, a contradiction. Finally suppose $|V_r| = 3$ and $|V_b| = 2$. Then again, there can be at most one red edge from each vertex of V_r to V_b, and at most one blue edge from each vertex of V_b to V_r, for a total of at most 5 edges, while R has 6 edges between V_r and V_b, another contradiction. Symmetrically we cannot have $|V_r| = 2$ and $|V_b| = 3$, thus completing the proof of Claim 16. □

We now consider subcases based on the value of t.

Subcase 3.1. $13 \le t \le 17$.

By Fact 4, we have $d_r(w_i), d_b(w_i) \le 8$ so this means that $d_b(w_i), d_r(w_i) \ge 4$ for all $w_i \in V(R)$. This means that G_i contains no red or blue edges for all i. Thus $p_2 = p_1 = 0$, $p_0 = t$, and

$$p_2\frac{16/3}{17} + p_1\frac{8/3}{17} + p_0\frac{1}{17} = \frac{t}{17} \le 1,$$

as required.

Subcase 3.2. $4 \le t \le 10$.

By Claim 15, we have $p_2 \le 1$. First suppose $p_2 = 1$. Then if $t \ge 8$, every vertex $w_i \in V(R)$ must have at least 4 edges in one color and, by Fact 5, every set G_i is missing either red or blue, contradicting the assumption that $p_2 = 1$. Thus, we have $4 \le t \le 7$. By Claim 16, since $p_2 = 1$, we have $p_1 = |V_r| + |V_b| - 2p_2 \le 2$. Thus,

$$p_2\frac{16/3}{17} + p_1\frac{8/3}{17} + p_0\frac{1}{17} = \frac{13/3 + 5p_1/3 + t}{17} \le \frac{44/3}{17} < 1.$$

Next suppose $p_2 = 0$ so by Claim 16, $p_1 \le 4$. This means that

$$p_2\frac{16/3}{17} + p_1\frac{8/3}{17} + p_0\frac{1}{17} = \frac{5p_1/3 + t}{17} \le \frac{50/3}{17} < 1.$$

Subcase 3.3. $t \in \{11, 12\}$.

Note that $p_2 = 0$ by Fact 5, since for all i, either $d_r(w_i) \ge 4$ or $d_b(w_i) \ge 4$. If $d_r(w_i), d_b(w_i) \ge 4$ for all i, then by Fact 5, we know that $p_1 = 0$, and so $p_0 = t$. We get

$$p_2\frac{16/3}{17} + p_1\frac{8/3}{17} + p_0\frac{1}{17} = \frac{t}{17} \le \frac{12}{17} < 1.$$

Thus by Fact 4, we may assume that there is a vertex in $V(R)$, without loss of generality, say w_1 with $(d_r(w_1), d_b(w_1), t) \in \{(7,3,11), (8,2,11), (8,3,12)\}$. Let H and F be the subgraphs of R induced on the red and blue neighborhoods of w_1, so $(|H|, |F|, t) \in \{(7,3,11), (8,2,11), (8,3,12)\}$.

If there is a vertex in H with at least 4 incident red edges within H, then all edges among those red neighbors must be blue, making a blue K_4 for a contradiction. This means that every vertex of H has at most 3 incident red edges within H, and so at least $|H| - 4 \geq 3$ incident blue edges within H. If a vertex $v \in V(H)$ is not contained in a blue triangle within H, then there is a red triangle among the blue neighbors of v in H which, together with w_1, form a red K_4 for a contradiction. This means that every vertex in H is contained in a blue triangle in H, and so by Fact 3, for each $w_i \in V(H)$, we have $w_i \notin V_b$.

Similarly, if there is a vertex in H with at least 6 incident blue edges within H, then there is either a red or a blue triangle among those blue neighbors (by $r(3,3) = 6$), which would make a red or blue K_4 in R, a contradiction. Thus, each vertex in H has at least $|H| - 6 \geq 1$ incident red edge within H, meaning that every vertex in H is contained in a red triangle in R. Thus by Fact 3, for each $w_i \in V(H)$, we have $w_i \notin V_r$.

With $w \notin (V_r \cup V_b)$ for all $w \in V(H)$, we get that $p_0 \geq |H|$. If $(d_r(w_1), d_b(w_1), t) = (|H|, |F|, t) = (8,2,11)$, then $p_0 \geq |H| = 8$ and $p_1 = 11 - p_0 \leq 3$. We get

$$p_2 \frac{16/3}{17} + p_1 \frac{8/3}{17} + p_0 \frac{1}{17} = \frac{11 + 5p_1/3}{17} \leq \frac{16}{17} < 1.$$

Otherwise, we have $(d_r(w_1), d_b(w_1), t) = (|H|, |F|, t) \in \{(7,3,11), (8,3,12)\}$. If there is a blue edge within F, then w_1 is contained in a blue triangle and w_1 was already in a red triangle (using any red edge within H) so in this case, we get $w_1 \notin (V_r \cup V_b)$ by Fact 3, and so $p_0 \geq |H| + 1$. Otherwise F induces a red triangle. In order to avoid creating a red K_4, each vertex in H has at least one blue edge to F, meaning that there are a total of at least $|H| \geq 7$ blue edges between H and F. By the pigeonhole principle, there is a vertex $u \in V(F)$ with at least 3 blue edges to H. If those blue neighbors of u in H contain no blue edges, then they, along with w_1, induce a red K_4 so there must be a blue edge so u is contained in both a red triangle and a blue triangle. Thus $u \notin (V_r \cup V_b)$ by Fact 3, again implying that $p_0 \geq |H| + 1$. Thus $p_1 \leq t - |H| - 1 = 3$, and we get

$$p_2 \frac{16/3}{17} + p_1 \frac{8/3}{17} + p_0 \frac{1}{17} = \frac{5p_1/3 + t}{17} \leq 1,$$

completing the proof of this subcase, and the proof of Case 3.

This completes the proof of Theorem 10. □

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