Upper bounds for the total rainbow connection of graphs*

Hui Jiang, Xueliang Li, Yingying Zhang

Center for Combinatorics and LPMC-TJKLC

Nankai University, Tianjin 300071, China

E-mail: jhuink@163.com; lxl@nankai.edu.cn; zyydlwyx@163.com

Abstract

A total-colored graph is a graph G such that both all edges and all vertices of G are colored. A path in a total-colored graph G is a *total rainbow path* if its edges and internal vertices have distinct colors. A total-colored graph G is *total-rainbow connected* if any two vertices of G are connected by a total rainbow path of G. The *total rainbow connection number* of G, denoted by $trc(G)$, is defined as the smallest number of colors that are needed to make G total-rainbow connected. These concepts were introduced by Liu et al. Notice that for a connected graph G, $2diam(G) - 1 \leq trc(G) \leq 2n - 3$, where $diam(G)$ denotes the diameter of G and n is the order of G. In this paper we show, for a connected graph G of order n with minimum degree δ, that $trc(G) \leq 6n/(\delta + 1) + 28$ for $\delta \geq \sqrt{n-2} - 1$ and $n \geq 291$, while $trc(G) \leq 7n/(\delta + 1) + 32$ for $16 \leq \delta \leq \sqrt{n-2} - 2$ and $trc(G) \leq 7n/(\delta + 1) + 4C(\delta) + 12$ for $6 \leq \delta \leq 15$, where $C(\delta) = e^{\frac{3\log(\delta^3+2\delta^2+3)-3(\log 3-1)}{\delta-3}} - 2$. Thus, when δ is in linear with n, the total rainbow number $trc(G)$ is a constant. We also show that $trc(G) \leq 7n/4 - 3$ for $\delta = 3$, $trc(G) \leq 8n/5 - 13/5$ for $\delta = 4$ and $trc(G) \leq 3n/2 - 3$ for $\delta = 5$. Furthermore, an example from Caro et al. shows that our bound can be seen tight up to additive factors when $\delta \geq \sqrt{n-2} - 1$.

Keywords: total-colored graph; total rainbow connection; minimum degree; 2-step dominating set.

AMS subject classification 2010: 05C15, 05C40, 05C69, 05D40.

*Supported by NSFC No.11371205 and PCSIRT.

1 Introduction

In this paper, all graphs considered are simple, finite and undirected. We refer to book [2] for undefined notation and terminology in graph theory. Let G be a connected graph on n vertices with minimum degree δ. A path in an edge-colored graph G is a *rainbow path* if its edges have different colors. An edge-colored graph G is *rainbow connected* if any two vertices of G are connected by a rainbow path of G. The *rainbow connection number*, denoted by $rc(G)$, is defined as the smallest number of colors required to make G rainbow connected. Chartrand et al. [6] introduced these concepts. Notice that $rc(G) = 1$ if and only if G is a complete graph and that $rc(G) = n - 1$ if and only if G is a tree. Moreover, $diam(G) \leq rc(G) \leq n-1$. A lot of results on the rainbow connection have been obtained; see [13, 14].

From [4] we know that to compute the number $rc(G)$ of a connected graph G is NP-hard. So, to find good upper bounds is an interesting problem. Krivelevich and Yuster [11] obtained that $rc(G) \leq 20n/\delta$. Caro et al. [3] obtained that $rc(G) \leq \frac{\ln \delta}{\delta} n(1 + o_\delta(1))$. Finally, Chandran et al. [5] got the following benchmark result.

Theorem 1. *[5] For every connected graph G of order n and minimum degree δ, $rc(G) \leq 3n/(\delta + 1) + 3$.*

The concept of rainbow vertex-connection was introduced by Krivelevich and Yuster in [11]. A path in a vertex-colored graph G is a *vertex-rainbow path* if its internal vertices have different colors. A vertex-colored graph G is *rainbow vertex-connected* if any two vertices of G are connected by a vertex-rainbow path of G. The *rainbow vertex-connection number*, denoted by $rvc(G)$, is defined as the smallest number of colors required to make G rainbow vertex-connected. Observe that $diam(G) - 1 \leq rvc(G) \leq n - 2$ and that $rvc(G) = 0$ if and only if G is a complete graph. The problem of determining the number $rvc(G)$ of a connected graph G is also NP-hard; see [7, 8]. There are a few results about the upper bounds of the rainbow vertex-connection number. Krivelevich and Yuster [11] proved that $rvc(G) \leq 11n/\delta$. Li and Shi [12] improved this bound and showed the following results.

Theorem 2. *[12] For a connected graph G of order n and minimum degree δ, $rvc(G) \leq 3n/4 - 2$ for $\delta = 3$, $rvc(G) \leq 3n/5 - 8/5$ for $\delta = 4$ and $rvc(G) \leq n/2 - 2$ for $\delta = 5$. For sufficiently large δ, $rvc(G) \leq (b \ln \delta)n/\delta$, where b is any constant exceeding 2.5.*

Theorem 3. *[12] A connected graph G of order n with minimum degree δ has $rvc(G) \leq 3n/(\delta + 1) + 5$ for $\delta \geq \sqrt{n-1} - 1$ and $n \geq 290$, while $rvc(G) \leq 4n/(\delta + 1) + 5$ for $16 \leq \delta \leq \sqrt{n-1} - 2$ and $rvc(G) \leq 4n/(\delta + 1) + C(\delta)$ for $6 \leq \delta \leq 15$, where $C(\delta) = e^{\frac{3\log(\delta^3+2\delta^2+3)-3(\log 3-1)}{\delta-3}} - 2$.*

Recently, Liu et al. [16] proposed the concept of total rainbow connection. A *total-colored graph* is a graph G such that both all edges and all vertices of G are colored. A path in a total-colored graph G is a *total rainbow path* if its edges and internal vertices have distinct colors. A total-colored graph G is *total-rainbow connected* if any two vertices of G are connected by a total rainbow path of G. The *total rainbow connection number*, denoted by $trc(G)$, is defined as the smallest number of colors required to make G total-rainbow connected. It is easy to observe that $trc(G) = 1$ if and only if G is a complete graph. Moreover, $2diam(G) - 1 \leq trc(G) \leq 2n - 3$. The following proposition gives an upper bound of the total rainbow connection number.

Proposition 1. [16] Let G be a connected graph on n vertices and q vertices having degree at least 2. Then, $trc(G) \leq n - 1 + q$, with equality if and only if G is a tree.

From Theorems 1 and 3, one can see that $rc(G)$ and $rvc(G)$ are bounded by a function of the minimum degree δ, and that when δ is in linear with n, then both $rc(G)$ and $rvc(G)$ are some constants. In this paper, we will use the same idea in [12] to obtain upper bounds for the number $trc(G)$, which are also functions of δ and imply that when δ is in linear with n, then $trc(G)$ is a constant.

2 Main results

Let G be a connected graph on n vertices with minimum degree δ. Denote by $Leaf(G)$ the maximum number of leaves among all spanning trees of G. If $\delta = 3$, then $Leaf(G) \geq n/4 + 2$, which was proved by Linial and Sturtevant [15]. Griggs and Wu in [9], and Kleitman and West in [10] showed that $Leaf(G) \geq 2n/5 + 8/5$ for $\delta = 4$. Moreover, Griggs and Wu [9] showed that if $\delta = 5$, then $Leaf(G) \geq n/2 + 2$. For sufficiently large δ, Kleitman and West in [10] proved that $Leaf(G) \geq (1 - b \ln \delta/\delta)n$, where b is any constant exceeding 2.5. From these results, we can get the following results.

Theorem 4. *For a connected graph G of order n with minimum degree δ, $trc(G) \leq 7n/4 - 3$ for $\delta = 3$, $trc(G) \leq 8n/5 - 13/5$ for $\delta = 4$ and $trc(G) \leq 3n/2 - 3$ for $\delta = 5$. For sufficiently large δ, $trc(G) \leq (1 + b \ln \delta/\delta)n - 1$, where b is any constant exceeding 2.5.*

Proof. We can choose a spanning tree T with the maximum number of leaves. Denote ℓ the maximum number of leaves. Then color all non-leaf vertices and all edges of T with $2n - \ell - 1$ colors, each receiving a distinct color. Hence, $trc(G) \leq 2n - \ell - 1$. □

Theorem 5. *For a connected graph G of order n with minimum degree δ, $trc(G) \leq 6n/(\delta + 1) + 28$ for $\delta \geq \sqrt{n - 2} - 1$ and $n \geq 291$, while $trc(G) \leq 7n/(\delta + 1) + 32$ for*

$16 \le \delta \le \sqrt{n-2}-2$ *and* $trc(G) \le 7n/(\delta+1) + 4C(\delta) + 12$ *for* $6 \le \delta \le 15$, *where* $C(\delta) = e^{\frac{3\log(\delta^3+2\delta^2+3)-3(\log 3-1)}{\delta-3}} - 2$.

Remark 1. The same example mentioned in [3] can show that our bound is tight up to additive factors when $\delta \ge \sqrt{n-2}-1$.

In order to prove Theorem 5, we need some lemmas.

Lemma 1. *[11] If* G *is a connected graph of order* n *with minimum degree* δ*, then it has a connected spanning subgraph with minimum degree* δ *and with less than* $n(\delta+1/(\delta+1))$ *edges.*

Given a graph G, a set $D \subseteq V(G)$ is called a 2*-step dominating set* of G if every vertex of G which is not dominated by D has a neighbor that is dominated by D. A 2-step dominating set S is k*-strong* if every vertex which is not dominated by S has at least k neighbors that are dominated by S. If S induces a connected subgraph of G, then S is called a *connected* k*-strong* 2*-step dominating set.*

Lemma 2. *[12] If* G *is a connected graph of order* n *with minimum degree* $\delta \ge 2$*, then* G *has a connected* $\delta/3$*-strong 2-step dominating set* S *whose size is at most* $3n/(\delta+1)-2$.

Lemma 3. *[1] (Lovász Local Lemma) Let* $A_1, A_2, ..., A_n$ *be the events in an arbitrary probability space. Suppose that each event* A_i *is mutually independent of a set of all the other events* A_j *but at most* d*, and that* $P[A_i] \le p$ *for all* $1 \le i \le n$*. If* $ep(d+1) < 1$*, then* $Pr[\bigwedge_{i=1}^{n} \bar{A}_i] > 0$.

Now we are ready to prove Theorem 5.

Proof of Theorem 5: The proof goes similarly for the main result of [12]. We are given a connected graph G of order n with minimum degree δ. Suppose that G has less than $n(\delta+1/(\delta+1))$ edges by Lemma 1. Let S denote a connected $\delta/3$-strong 2-step dominating set of G. Then, we have $|S| \le 3n/(\delta+1)-2$ by Lemma 2. Let $N^k(S)$ denote the set of all vertices at distance exactly k from S. We give a partition to $N^1(S)$ as follows. First, let H be a new graph constructed on $N^1(S)$ with edge set $E(H) = \{uv : u, v \in N^1(S), uv \in E(G)$ or $\exists\ w \in N^2(S)$ such that uwv is a path of $G\}$. Let Z be the set of all isolated vertices of H. Moreover, there exists a spanning forest F of $V(H)\backslash Z$. Finally, choose a bipartition defined by this forest, denoted by X and Y. Partition $N^2(S)$ into three subsets: $A = \{u \in N^2(S) : u \in N(X) \cap N(Y)\}$, $B = \{u \in N^2(S) : u \in N(X)\backslash N(Y)\}$ and $C = \{u \in N^2(S) : u \in N(Y)\backslash N(X)\}$; see Figure 1(a).

Case 1. $\delta \ge \sqrt{n-2}-1$.

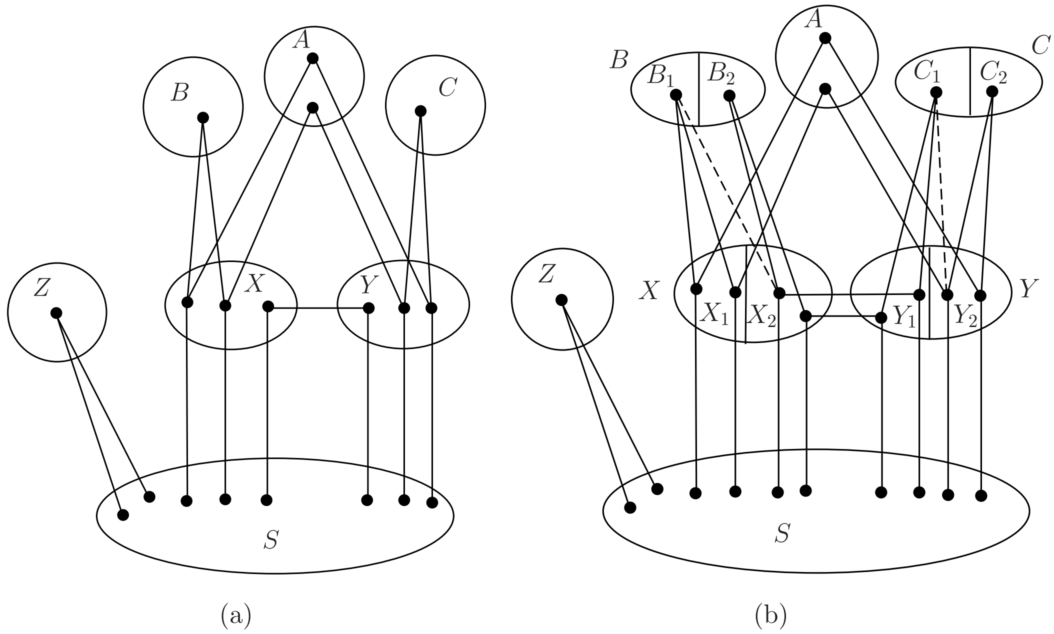

Figure 1: Illustration in the proof of Theorem 5

Next we give a coloring to the edges and vertices of G. Let $k = 2|S| - 1$ and T be a spanning tree of $G[S]$. Color the edges and vertices of T with k distinct colors such that $G[S]$ is total rainbow connected. Assign every $[X, S]$ edge with color $k + 1$, every $[Y, S]$ edge with color $k + 2$ and every edge in $N^1(S)$ with color $k + 3$. Since the minimum degree $\delta \geq 2$, every vertex in Z has at least two neighbors in S. Color one edge with $k+1$ and all others with $k + 2$. Assign every $[A, X]$ edge with color $k + 3$, every $[A, Y]$ edge with color $k+4$ and every vertex of A with color $k+5$. We assign seven new colors from $\{i_1, i_2, ..., i_7\}$ to the vertices of X such that each vertex of X chooses its color randomly and independently from all other vertices of X. Similarly, we assign another seven colors to the vertices of Y. Assign seven colors from $\{j_1, j_2, ..., j_7\}$ to the edges between B and X as follows: for every vertex $u \in B$, let $N_X(u)$ denote the set of all neighbors of u in X; for every vertex $u' \in N_X(u)$, if we color u' with i_t ($t \in \{1, 2, ..., 7\}$), then color uu' with j_t. In a similar way, we assign seven new colors to the edges between C and Y. All other edges and vertices of G are uncolored. Thus, the number of all colors we used is

$$k + 33 = 2|S| - 1 + 33 \leq 2\left(\frac{3n}{\delta + 1} - 2\right) - 1 + 33 = \frac{6n}{\delta + 1} + 28.$$

We have the following claim for any $u \in B$ (C).

Claim 1. For any $u \in B$ (C), we have a coloring for the vertices in X (Y) with seven colors such that there exist two neighbors u_1 and u_2 in $N_X(u)$ $(N_Y(u))$ that receive different colors. Hence, the edges uu_1 and uu_2 are also colored differently.

Notice that for every vertex $v \in X$, v has two neighbors in $S \cup A \cup Y$. Moreover, $(\delta+1)^2 \geq n-2$. Thus, v has less than $(\delta+1)^2$ neighbors in B. For every vertex $u \in B$, u has at least $\delta/3$ neighbors in X since S is a connected $\delta/3$-strong 2-step dominating set of G. Let A_u denote the event that $N_X(u)$ receives at least two distinct colors. Fix a set $X(u) \subset N_X(u)$ with $|X(u)| = \lceil \delta/3 \rceil$. Let B_u denote the event that all vertices of $X(u)$ are colored the same. Hence, $Pr[B_u] \leq 7^{-\lceil \delta/3 \rceil + 1}$. Moreover, the event B_u is independent of all other events B_v for $v \neq u$ but at most $((\delta+1)^2 - 1)\lceil \delta/3 \rceil$ of them. Since $e \cdot 7^{-\lceil \delta/3 \rceil + 1}(((\delta+1)^2 - 1)\lceil \delta/3 \rceil + 1) < 1$, for all $\delta \geq \sqrt{n-2} - 1$ and $n \geq 291$, we have $Pr[\bigwedge_{u \in B} \bar{B_u}] > 0$ by Lemma 3. Therefore, $Pr[A_u] > 0$.

We will show that G is total-rainbow connected. Take any two vertices u and w in $V(G)$. If they are all in S, there is a total rainbow path connecting them in $G[S]$. If one of them is in $N^1(S)$, say u, then u has a neighbor u' in S. Thus, $uu'Pw$ is a required path, where P is a total rainbow path in $G[S]$ connecting u' and w. If one of them is in $X \cup Z$, say u, and the other is in $Y \cup Z$, say w, then u has a neighbor u' in S and w has a neighbor w' in S. Hence, $uu'Pw'w$ is a required path, where P is a total rainbow path connecting u' and w' in $G[S]$. If they are all in X, then there exists a $u' \in Y$ such that u and u' are connected by a single edge or a total rainbow path of length two. We know that u' and w are total-rainbow connected. Therefore, u and w are connected by a total rainbow path. If one of them is in $A \cup B$, say u, and the other is in $A \cup C$, say w, then u has a neighbor u' in X, and w has a neighbor w' in Y. Thus, they are total-rainbow connected. If they are all in B, by Claim 1 u has two neighbors u_1 and u_2 in X such that u_1, u_2, uu_1 and uu_2 are colored differently. Similarly, we also have that w has two neighbors w_1 and w_2 in X such that w_1, w_2, ww_1 and ww_2 are colored differently. Hence, u and w are total-rainbow connected. We can check that u and w are total-rainbow connected in all other cases.

Case 2. $6 \leq \delta \leq \sqrt{n-2} - 2$.

We partition X into two subsets X_1 and X_2. For any $u \in X$, if u has at least $(\delta+1)^2$ neighbors in B, then $u \in X_1$; otherwise, $u \in X_2$. Similarly, we partition Y onto two subsets Y_1 and Y_2. Note that $|X_1 \cup Y_1| \leq n/(\delta+1)$ since G has less than $n(1 + 1/(\delta+1))$ edges. Partition B into two subsets B_1 and B_2. For any $u \in B$, if u has at least one neighbor in X_1, then $u \in B_1$; otherwise, $u \in B_2$. In a similar way, we partition C into two subsets C_1 and C_2; see Figure 1(b).

For $16 \leq \delta \leq \sqrt{n-2} - 2$, assume that $C(\delta) = 5$; for $6 \leq \delta \leq 15$, assume that $C(\delta) = e^{\frac{3\log(\delta^3+2\delta^2+3)-3(\log 3-1)}{\delta-3}} - 2$. Now we give a coloring to the edges and vertices of G. Let $k = 2|S| - 1$ and T be a spanning tree of $G[S]$. Color the edges and vertices of T with k distinct colors. Assign every $[X, S]$ edge with color $k+1$, every $[Y, S]$ edge with color $k+2$ and every edge in $N^1(S)$ with color $k+3$. Since every vertex in Z has at least two neighbors in S, color one edge with $k+1$ and all others with $k+2$. Assign every

$[A, X]$ edge with color $k+3$, every $[A, Y]$ edge with color $k+4$ and every vertex of A with color $k+5$. Assign distinct colors to each vertex of $X_1 \cup Y_1$ and $C(\delta)+2$ new colors from $\{i_1, i_2, ..., i_{C(\delta)+2}\}$ to the vertices of X_2 such that each vertex of X_2 chooses its color randomly and independently from all other vertices of X_2. Similarly, we assign $C(\delta)+2$ new colors to the vertices of Y_2. For every vertex $v \in B_1$, if v has at least two neighbors in X_1, color one edge with $k+6$ and all others with $k+7$; if v has only one neighbor in X_1, then it has another neighbor in X_2 since S is a connected $\delta/3$-strong 2-step dominating set. Thus, color the edge incident with X_1 with $k+6$ and all edges incident with X_2 with $k+7$. We assign $C(\delta)+2$ colors from $\{j_1, j_2, ..., j_{C(\delta)+2}\}$ to the edges between B_2 and X_2. For every vertex $u \in B_2$, let $N_{X_2}(u)$ denote all the neighbors of u in X_2. For every vertex $u' \in N_{X_2}(u)$, if we color u' with i_t ($t \in \{1, 2, ..., C(\delta)+2\}$), then color uu' with j_t. In a similar way, we assign another $C(\delta)+4$ colors to the edges between C and Y. All other edges and vertices of G are uncolored. Hence, the number of all colors we used is

$$k+|X_1 \cup Y_1|+4C(\delta)+17 \leq 2\left(\frac{3n}{\delta+1}-2\right)-1+\frac{n}{\delta+1}+4C(\delta)+17=\frac{7n}{\delta+1}+4C(\delta)+12.$$

We have the following claim for any $u \in B_2$ (C_2).

Claim 2. For any $u \in B_2$ (C_2), we have a coloring for the vertices in X_2 (Y_2) with $C(\delta)+2$ colors such that there exist two neighbors u_1 and u_2 in $N_{X_2}(u)$ ($N_{Y_2}(u)$) that receive different colors. Thus, the edges uu_1 and uu_2 are also colored differently.

Notice that every vertex u of B_2 has at least $\delta/3$ neighbors in X_2 since S is a connected $\delta/3$-strong 2-step dominating set of G. Let A_u denote the event that $N_{X_2}(u)$ receives at least two distinct colors. Fix a set $X_2(u) \subset N_{X_2}(u)$ with $|X_2(u)| = \lceil \delta/3 \rceil$. Let B_u denote the event that all vertices of $X_2(u)$ are colored the same. Therefore, $Pr[B_u] \leq (C(\delta)+2)^{-\lceil \delta/3 \rceil+1}$. Moreover, the event B_u is independent of all other events B_v for $v \neq u$ but at most $((\delta+1)^2-1)\lceil \delta/3 \rceil$ of them. Since $e \cdot (C(\delta)+2)^{-\lceil \delta/3 \rceil+1}(((\delta+1)^2-1)\lceil \delta/3 \rceil+1) < 1$, we have $Pr[\bigwedge_{u \in B_2} \bar{B}_u] > 0$ by Lemma 3. Hence, we have $Pr[A_u] > 0$.

Similarly, we can check that G is also total-rainbow connected.

The proof is now complete. □

Acknowledgement. The authors are very grateful to the referees for their helpful comments and suggestions.

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