# On (strong) proper vertex-connection of graphs*

Hui Jiang, Xueliang Li, Yingying Zhang, Yan Zhao

Center for Combinatorics and LPMC-TJKLC

Nankai University, Tianjin 300071, China

E-mail: jhuink@163.com; lxl@nankai.edu.cn;

zyydlwyx@163.com; zhaoyan2010@mail.nankai.edu.cn

**Abstract**

A path in a vertex-colored graph is a *vertex-proper path* if any two internal adjacent vertices differ in color. A vertex-colored graph is *proper vertex $k$-connected* if any two vertices of the graph are connected by $k$ disjoint vertex-proper paths of the graph. For a $k$-connected graph $G$, the *proper vertex $k$-connection number* of $G$, denoted by $pvc_k(G)$, is defined as the smallest number of colors required to make $G$ proper vertex $k$-connected. A vertex-colored graph is *strong proper vertex-connected*, if for any two vertices $u, v$ of the graph, there exists a vertex-proper $u$-$v$ geodesic. For a connected graph $G$, the *strong proper vertex-connection number* of $G$, denoted by $spvc(G)$, is the smallest number of colors required to make $G$ strong proper vertex-connected. These concepts are inspired by the concepts of rainbow vertex $k$-connection number $rvc_k(G)$, strong rainbow vertex-connection number $srvc(G)$, and proper $k$-connection number $pc_k(G)$ of a $k$-connected graph $G$. Firstly, we determine the value of $pvc(G)$ for general graphs and $pvc_k(G)$ for some specific graphs. We also compare the values of $pvc_k(G)$ and $pc_k(G)$. Then, sharp bounds of $spvc(G)$ are given for a connected graph $G$ of order $n$, that is, $0 \leq spvc(G) \leq n-2$. Moreover, we characterize the graphs of order $n$ such that $spvc(G) = n-2, n-3$, respectively. Finally, we study the relationship among the three vertex-coloring parameters, namely, $spvc(G)$, $srvc(G)$ and the chromatic number $\chi(G)$ of a connected graph $G$.

**Keywords**: vertex-coloring, proper vertex connection, strong proper vertex connection.

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# 1 Introduction

In this paper, all graphs considered are simple, finite and undirected. We refer to the book [2] for undefined notation and terminology in graph theory. For simplicity, a set of internally vertex-disjoint paths will be called *disjoint*. A path in an edge-colored graph is a *rainbow path* if its edges have different colors. An edge-colored graph is *rainbow $k$-connected* if any two vertices of the graph are connected by $k$ disjoint rainbow paths of the graph. For a $k$-connected graph $G$, the *rainbow $k$-connection number* of $G$, denoted by $rc_k(G)$, is defined as the smallest number of colors required to make $G$ rainbow $k$-connected. These concepts were first introduced by Chartrand et al. in [4, 5]. Since then, a lot of results on the rainbow connection have been obtained; see [12, 13] for surveys of related results.

As a natural counterpart of the concept of rainbow $k$-connection, the concept of rainbow vertex $k$-connection was first introduced by Krivelevich and Yuster in [8] for $k = 1$, and then by Liu et al. in [14] for general $k$. A path in a vertex-colored graph is a *vertex-rainbow path* if its internal vertices have different colors. A vertex-colored graph is *rainbow vertex $k$-connected* if any two vertices of the graph are connected by $k$ disjoint vertex-rainbow paths of the graph. For a $k$-connected graph $G$, the *rainbow vertex $k$-connection number* of $G$, denoted by $rvc_k(G)$, is defined as the smallest number of colors required to make $G$ rainbow vertex $k$-connected. There are many results on this topic, we refer to [6, 7, 11].

In 2011, Borozan et al. [3] introduced the concept of proper $k$-connection of graphs. A path in an edge-colored graph is a *proper path* if any two adjacent edges differ in color. An edge-colored graph is *proper $k$-connected* if any two vertices of the graph are connected by $k$ disjoint proper paths of the graph. For a $k$-connected graph $G$, the *proper $k$-connection number* of $G$, denoted by $pc_k(G)$, is defined as the smallest number of colors required to make $G$ proper $k$-connected. Note that

$$1 \leq pc_k(G) \leq \min\{\chi'(G), rc_k(G)\}, \tag{1}$$

where $\chi'(G)$ denotes the edge-chromatic number. Recently, the case for $k = 1$ has been studied by Andrews et al. [1] and Laforge et al. [9].

Inspired by the concepts above, now we introduce the concept of proper vertex $k$-connection. A path in a vertex-colored graph is a *vertex-proper path* if any two internal adjacent vertices differ in color. A vertex-colored graph is *proper vertex $k$-connected* if any two vertices of the graph are connected by $k$ disjoint vertex-proper paths of the graph. For a $k$-connected graph $G$, the *proper vertex $k$-connection number* of $G$, denoted by $pvc_k(G)$, is defined as the smallest number of colors required to make $G$ proper vertex $k$-connected. We write $pvc(G)$ for $pvc_1(G)$, and similarly, $rc(G), rvc(G)$ and $pc(G)$ for $rc_1(G), rvc_1(G)$

and $pc_1(G)$. We have $pvc(G) \geq 1$ if $G$ is a noncomplete graph. For a complete graph $G$, we set $pvc(G) = 0$, this is done because we want to be in accordance with the convention that $rvc(G) = 0$. For $k \geq 2$, by definition we have $pvc_k(G) \geq 1$ if $G$ is a $k$-connected graph. It is easy to see that

$$0 \leq pvc_k(G) \leq \min\{\chi(G), rvc_k(G)\}, \qquad (2)$$

where $\chi(G)$ denotes the chromatic number of $G$. By Brooks' theorem [2], if $G$ is a connected graph and is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta$, and so $pvc_k(G) \leq \Delta$, where $\Delta$ denotes the maximum degree of $G$.

# 2 Proper vertex $k$-connection

In this section, we determine the value of $pvc(G)$ for general graphs and $pvc_k(G)$ when $G$ is a cycle, a wheel, and a complete multipartite graph. Moreover, we show that $pc_k(G) \geq pvc_k(G)$ for $k = 1$ and provide an example graph $G$ such that $pc_k(G) > pvc_k(G)$ for $k \geq 2$.

## 2.1 Proper vertex-connection number $pvc(G)$

From the definition of $pvc(G)$, the following results are immediate. Recall that the *diameter* of a connected graph $G$, denoted by $diam(G)$, is the maximum of the distances among pairs of vertices of $G$.

**Proposition 1.** *Let $G$ be a nontrivial connected graph. Then*

(*a*) $pvc(G) = 0$ *if and only if $G$ is a complete graph;*

(*b*) $pvc(G) = 1$ *if and only if* $diam(G) = 2$.

For the case that $diam(G) \geq 3$, we have the following theorem.

**Theorem 1.** *Let $G$ be a nontrivial connected graph. Then, $pvc(G) = 2$ if and only if $diam(G) \geq 3$.*

*Proof.* The necessity can be verified by Proposition 1.

Now we prove its sufficiency. Since $diam(G) \geq 3$, we have that $pvc(G) \geq 2$ and then we just need to prove that $pvc(G) \leq 2$. Let $T$ be a spanning tree of $G$. For a vertex $v \in V(T)$, let $e_T(v)$ denote the eccentricity of $v$ in $T$, i.e., the maximum of the distances between $v$ and the other vertices in $T$. Let $V_i = \{u \in V(T) : d_T(u, v) = i\}$, where $0 \leq i \leq e_T(v)$. Hence $V_0 = \{v\}$. Define a 2-coloring of the vertices of $T$ as follows: If $i$ is odd, color the vertices of $V_i$ with color 1; otherwise, color the vertices of $V_i$ with color 2. It is easy to check that for any two vertices $x$ and $y$ in $G$, there is a vertex-proper path connecting them. Thus, $pvc(G) \leq 2$, and therefore, $pvc(G) = 2$. □

## 2.2 Proper vertex $k$-connection of some specific graphs

In this subsection, we shall determine the value of $pvc_k(G)$ for some specific graphs. Let $\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}$ denote the vertex-connectivity of $G$. Note that $pvc_k(G)$ is well defined if and only if $1 \leq k \leq \kappa(G)$. We start with the case that $G$ is a cycle of order $n$, denoted by $C_n$. Observe that $\kappa(C_n) = 2$. Clearly, $pvc(C_3) = 0$, $pvc(C_4) = pvc(C_5) = 1$, and $pvc(C_n) = 2$ for $n \geq 6$. Moreover, we have the following results for $pvc_2(C_n)$.

**Theorem 2.** *$pvc_2(C_3) = 1$, $pvc_2(C_n) = 2$ for $n \geq 4$ even, and $pvc_2(C_n) = 3$ for $n \geq 5$ odd.*

*Proof.* The assertion can be easily verified for $C_3$. Now, let $n \geq 4$. We consider two cases, depending on the parity of $n$.

Case 1. $n$ is even. By (2), we have that $pvc_2(C_n) \leq \chi(C_n) = 2$. If one colors the vertices of $C_n$ with one color, then we do not have two vertex-proper paths between any two adjacent vertices. Hence, $pvc_2(C_n) = 2$ for $n \geq 4$ even.

Case 2. $n$ is odd. Similarly from (2), it follows that $pvc_2(C_n) \leq \chi(C_n) = 3$. Assume that $C_n = v_1v_2 \cdots v_nv_1$ $(n \geq 5)$. If we have a vertex-coloring for $C_n$ with two colors, then there must exist two adjacent vertices, say $v_1$ and $v_2$, colored the same. However, there do not have two vertex-proper paths between $v_n$ and $v_3$. Thus, $pvc_2(C_n) = 3$ for $n \geq 5$ odd. □

A graph obtained from $C_n$ by joining a new vertex $v$ to every vertex of $C_n$ is the *wheel* $W_n$. The vertex $v$ is the center of $W_n$. Note that $\kappa(W_n) = 3$. Obviously, $pvc(W_3) = 0$ and $pvc(W_n) = 1$ for $n \geq 4$.

**Theorem 3.**

$(a)$ *$pvc_2(W_3) = 1$ and $pvc_2(W_n) = pvc(C_n)$ for $n \geq 4$.*

$(b)$ *$pvc_3(W_3) = 1$ and $pvc_3(W_n) = pvc_2(C_n)$ for $n \geq 4$.*

*Proof.* $(a)$ The assertion can be easily verified for $W_3$. Now, let $n \geq 4$. Take a proper vertex connected coloring for the cycle $C_n$ in $W_n$ with $pvc(C_n)$ colors and then color the center with any used color. Clearly, $W_n$ is proper vertex 2-connected. Thus, $pvc_2(W_n) \leq pvc(C_n)$. On the other hand, consider a vertex-coloring for $W_n$ with fewer than $pvc(C_n)$ colors. Then, there exist two vertices $u, v$ in the cycle $C_n$ of $W_n$ such that we do not have a vertex-proper $u$-$v$ path along the cycle. Hence, there is at most one vertex-proper $u$-$v$ path in $W_n$ (using the center of $W_n$). Thus, $pvc_2(W_n) \geq pvc(C_n)$.

$(b)$ This can be proved by a similar way as the proof of Theorem 3$(a)$. □

For the complete graph $K_n$, we have that $pvc(K_n) = 0$ and $pvc_2(K_n) = pvc_3(K_n) = \cdots = pvc_{n-1}(K_n) = 1$. Let $K_{n_1,n_2}$ denote the complete bipartite graph, where $2 \leq n_1 \leq n_2$. Clearly, $\kappa(K_{n_1,n_2}) = n_1$. Then, we have $pvc(K_{n_1,n_2}) = 1$ and $pvc_k(K_{n_1,n_2}) = 2$ for $2 \leq k \leq n_1$. Let $G = K_{n_1,\ldots,n_t}$ be a complete multipartite graph, where $1 \leq n_1 \leq \cdots \leq n_t$ with $t \geq 3$ and $n_t \geq 2$. In [14], Liu, Mestre and Sousa determined the rainbow vertex $k$-connection number of $K_{n_1,\ldots,n_t}$. By the same method as the proof of Theorem 4 in [14] and the fact that $pvc_k(G) \leq rvc_k(G)$, we deduce the following results.

**Theorem 4.** *Let $1 \leq n_1 \leq \cdots \leq n_t$, where $t \geq 3$,$n_t \geq 2$ and $m = \sum_{i=1}^{t-1} n_i$.*

*(a) If $1 \leq k \leq m-2$, then we have the following:*

*(i) $pvc_k(K_{n_1,\ldots,n_t}) = 1$ if $1 \leq k \leq m - n_{t-1} + 1$.*

*(ii) $pvc_k(K_{n_1,\ldots,n_t}) = 2$ if $m - n_{t-1} + 2 \leq k \leq m-2$.*

*(b) (i) $pvc_{m-1}(K_{n_1,\ldots,n_t}) = 1$ if $n_{t-1} \leq 2$.*

*(ii) $pvc_{m-1}(K_{n_1,\ldots,n_t}) = 2$ if $n_{t-1} \geq 3$ and we do not have $n_t = n_{t-1} = n_{t-2}$ odd.*

*(iii) $pvc_{m-1}(K_{n_1,\ldots,n_t}) = 3$ if $n_t = n_{t-1} = n_{t-2} \geq 3$ are odd.*

*(c) (i) $pvc_m(K_{n_1,\ldots,n_t}) = 1$ if $n_{t-1} = 1$.*

*(ii) $pvc_m(K_{n_1,\ldots,n_t}) = 2$ if $2 \leq n_{t-1} \leq n_t - 2$.*

*(iii) $pvc_m(K_{n_1,\ldots,n_t}) = 2$ if $n_{t-1} = n_t - 1 \geq 2$ and $n_{t-2} \leq 2$, or $n_{t-1} = n_t \geq 2$ and $n_{t-2} = 1$.*

*(iv) $pvc_m(K_{n_1,\ldots,n_t}) = 3$ if $n_{t-1} = n_t - 1$ and $n_{t-2} \geq 3$, or $n_{t-1} = n_t \geq 3, n_{t-2} \geq 2$ and we do not have $n_t = n_{t-1} = n_{t-2} = n_{t-3} = 4$ and $t \geq 4$.*

*(v) $pvc_m(K_{n_1,\ldots,n_t}) = 4$ if $t \geq 4$ and $n_t = n_{t-1} = n_{t-2} = n_{t-3} = 4$.*

*(vi) $pvc_m(K_{n_1,\ldots,n_t}) = s$ if $n_t = n_{t-1} = \cdots = n_{t-s+1} = 2$ and $n_{t-s} = n_{t-s-1} = \cdots = n_1 = 1$, for $1 \leq s \leq t$.*

## 2.3 Comparing $pc_k(G)$ and $pvc_k(G)$

In [8], Krivelevich and Yuster compared the values of $rc(G)$ and $rvc(G)$. They observed that one of $rc(G)$ and $rvc(G)$ cannot be bounded in terms of the other, by providing example graphs $G$ where $rc(G)$ is much larger than $rvc(G)$, and vice versa. In [14], Liu et al. compared the values of $rc_k(G)$ and $rvc_k(G)$, similarly.

Here, we will compare the values of $pc_k(G)$ and $pvc_k(G)$. Note that $pc(G) = 1$ if and only if $G$ is a complete graph. In addition, by Proposition 1 we have the following assertion: if $diam(G) = 1$, then $pc(G) = 1$ and $pvc(G) = 0$; if $diam(G) = 2$, then $pc(G) \geq 2$ and $pvc(G) = 1$; if $diam(G) \geq 3$, then $pc(G) \geq 2$ and $pvc(G) = 2$. Thus, we have $pc(G) \geq pvc(G)$. For $k \geq 2$, the following theorem shows that there exists an example graph $G$ such that $pc_k(G) > pvc_k(G)$.

**Theorem 5.** *Let $G$ be a complete bipartite graph with classes $U$ and $V$, where $U = \{u_1, ..., u_t\}$ and $V = \{v_1, ..., v_k\}$ $(2 \leq k < t)$. Then, $pc_k(G) = t$ and $pvc_k(G) = 2$.*

*Proof.* Clearly, $pvc_k(G) = 2$. Next we just need to prove $pc_k(G) = t$. Since $G$ is a complete bipartite graph, it follows that $\chi'(G) = \Delta$. Moreover, $pc_k(G) \leq \chi'(G)$ in (1) and $\Delta = t$. Then, $pc_k(G) \leq t$. If one colors the edges of $G$ with fewer than $t$ colors, then there exist two edges of $\{v_1u_i : u_i \in U\}$, say $v_1u_1$ and $v_1u_2$, colored the same. Thus, we can not have $k$ disjoint proper paths between $u_1$ and $u_2$. Therefore, $pc_k(G) = t$. □

We observe that $pc_k(G) \geq pvc_k(G)$ for $k = 1$. Moreover from Theorem 5, we find an example such that $pc_k(G) > pvc_k(G)$ for $k \geq 2$. Note that $pc_2(G) = pvc_2(G)$ if $G$ is a cycle of order $n \geq 4$. However, we cannot show whether there exists a graph $G$ such that $pc_k(G) < pvc_k(G)$. Thus, we pose the following problem.

**Problem 1.** *Let $k \geq 2$. Does it hold that $pc_k(G) \geq pvc_k(G)$ for any connected graph $G$?*

# 3 Strong proper vertex-connection

In [10], Li et al. introduced the concept of strong rainbow vertex-connection. A vertex-colored graph is *strong rainbow vertex-connected*, if for any two vertices $u, v$ of the graph, there exists a vertex-rainbow $u$-$v$ geodesic, i.e., a $u$-$v$ path of length $d(u, v)$. For a connected graph $G$, the *strong rainbow vertex-connection number* of $G$, denoted by $srvc(G)$, is the smallest number of colors required to make $G$ strong rainbow vertex-connected.

A natural idea is to introduce the concept of the strong proper vertex-connection. A vertex-colored graph is *strong proper vertex-connected*, if for any two vertices $u, v$ of the graph, there exists a vertex-proper $u$-$v$ geodesic. For a connected graph $G$, the *strong proper vertex-connection number* of $G$, denoted by $spvc(G)$, is the smallest number of colors required to make $G$ strong proper vertex-connected. For a noncomplete graph $G$, we have $spvc(G) \geq 1$. Similar reason to the parameters $rvc(G)$ and $pvc(G)$, we set $spvc(G) = 0$ if $G$ is a complete graph. It is easy to see that if $G$ is a nontrivial connected graph, then

$$0 \leq pvc(G) \leq spvc(G) \leq \min\{\chi(G), srvc(G)\}. \qquad (3)$$

The following results on $spvc(G)$ are immediate from definition.

**Proposition 2.** *Let $G$ be a nontrivial connected graph of order $n$. Then*

(*a*) *$spvc(G) = 0$ if and only if $G$ is a complete graph;*

(*b*) *$spvc(G) = 1$ if and only if $diam(G) = 2$.*

It is easy to obtain the following consequences.

**Observation 1.**

(1) $spvc(P_3) = 1$ and $spvc(P_n) = 2$ for $n \geq 4$;

(2) $spvc(C_4) = spvc(C_5) = 1$, $spvc(C_n) = 2$ for $n \geq 6$ even, and $spvc(C_n) = 3$ for $n \geq 7$ odd;

(3) $spvc(K_{s,t}) = 1$ for $s \geq 2$ and $t \geq 1$;

(4) $spvc(K_{n_1,n_2,\ldots,n_k}) = 1$ for $k \geq 3$ and $(n_1, n_2, \ldots, n_k) \neq (1, 1, \ldots, 1)$;

(5) $spvc(W_n) = 1$ for $n \geq 4$.

In this section, sharp upper and lower bounds of $spvc(G)$ are given for a connected graph $G$ of order $n$, that is, $0 \leq spvc(G) \leq n - 2$. We also characterize the graphs of order $n$ such that $spvc(G) = n - 2, n - 3$, respectively. Furthermore, we investigate the relationship among the three vertex-coloring parameters, namely $spvc(G), srvc(G)$ and $\chi(G)$ of a connected graph $G$.

### 3.1 Bounds and characterization of extremal graphs

The problem of finding general bounds of $srvc(G)$ has been solved completely by Li et al. [10].

**Lemma 1.** *[10] Let $G$ be a connected graph of order $n$ $(n \geq 3)$. Then $0 \leq srvc(G) \leq n-2$. Moreover, the bounds are sharp.*

**Lemma 2.** *[10] Let $G$ be a nontrivial connected graph of order $n$. Then $srvc(G) = n - 2$ if and only if $G = P_n$.*

**Theorem 6.** *Let $G$ be a nontrivial connected graph of order $n$. Then $0 \leq spvc(G) \leq n-2$. Equality on the right-hand side is attained if and only if $G \in \{P_3, P_4\}$.*

*Proof.* By (3) and Lemma 1, it is obvious that $0 \leq spvc(G) \leq srvc(G) \leq n - 2$. On one hand, we know that $spvc(P_3) = 1 = n - 2$ and $spvc(P_4) = 2 = n - 2$. On the other hand, if $spvc(G) = n - 2$, then $srvc(G) = n - 2$. It follows that $G \in \{P_3, P_4\}$ from Observation 1 and Lemma 2. □

**Theorem 7.** *Let $G$ be a nontrivial connected graph of order $n$. Then $spvc(G) = n - 3$ if and only if $G$ is one of the twelve graphs in Figure 1.*

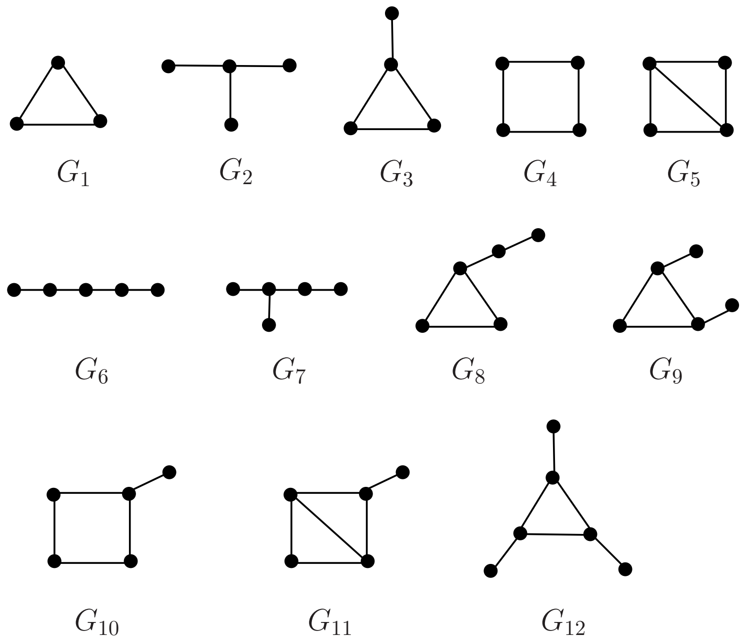


Figure 1: The twelve graphs in Theorem 7.

In order to prove Theorem 7, we need the lemma below.

**Lemma 3.** *If $G$ is a connected graph with order $n \geq 7$, then $spvc(G) < n-3$.*

*Proof.* Let $G$ be a connected graph with order $n \geq 7$ and $\Delta$ be the maximum degree of $G$.

Case 1. $\Delta = n-1$. Then, we have $diam(G) \leq 2$. By Proposition 2, it follows that $spvc(G) \leq 1 < n-3$.

Case 2. $\Delta = n-2$. Let $v$ be a vertex with the maximum degree $\Delta$ and $N(v) = \{v_1, v_2, \ldots, v_{n-2}\}$ denote its neighborhood. Let $v'$ be the only vertex not adjacent to $v$ and $N(v')$ denote its neighborhood. Color the vertices of $N(v')$ with color 1 and all other vertices with color 2. Now, we will show that for any two vertices $u$ and $w$ in $G$, there exists a vertex-proper geodesic between them. Clearly, $d(u,w) \leq 3$. Since $G$ is connected, there exists a vertex $v_i$ ($i \in \{1, 2, \ldots, n-2\}$) such that $v'$ is adjacent to $v_i$. If $d(u,w) = 3$, then $u = v'$ and $w \in N(v)$ by symmetry. Thus they are connected by a vertex-proper geodesic $uv_ivw$. For the other cases, $d(u,w) \leq 2$ and then there must be a vertex-proper $u$-$w$ geodesic. Therefore, $spvc(G) \leq 2 < n-3$.

Case 3. $\Delta = n-3$. Let $v$ be a vertex with the maximum degree $\Delta$ and $N(v) = \{v_1, v_2, \ldots, v_{n-3}\}$ denote its neighborhood. Let $v'$ and $v''$ be the vertices not adjacent to $v$.

Subcase 3.1. $N(v') \cap N(v'') \neq \emptyset$. Then, there exists a vertex $v_i$ ($i \in \{1, 2, \ldots, n-3\}$)

such that $v'$ and $v''$ are adjacent to $v_i$. Color the vertex $v_i$ with color 1 and all the other vertices with color 2. Next, we shall show that there exists a vertex-proper geodesic between any two vertices $u$ and $w$ in $G$. If $d(u,w) = 3$, then $u \in \{v', v''\}$ and $w \in N(v)$ by symmetry. Thus they are connected by a vertex-proper path $uv_ivw$. For the other cases, $d(u,w) \leq 2$. Therefore, $spvc(G) \leq 2 < n-3$.

Subcase 3.2. $N(v') \cap N(v'') = \emptyset$. Color the vertices of $N(v')$ with color 1, the vertices of $N(v'')$ with color 2 and all the others with color 3. Firstly, suppose that $v'v''$ is an edge of $G$. If $N(v'') = \{v'\}$, then there exists a vertex $v_i$ ($i \in \{1, 2, \ldots, n-3\}$) such that $v'$ is adjacent to $v_i$. By a similar discussion of Case 2, we obtain that $G - v''$ is strong proper vertex-connected. Furthermore, the color of $v'$ which is the unique neighbor of $v''$, is distinct from others. Thus, there exists a vertex-proper geodesic between any two vertices in $G$. Similarly, the case that $|N(v'')| \geq 2$ can be proved. Now, assume that $v'$ is not adjacent to $v''$. If there is an edge between $N(v')$ and $N(v'')$, say $v_1v_2$ with $v_1 \in N(v')$ and $v_2 \in N(v'')$, then $v'v_1v_2v''$ is a vertex-proper $v'$-$v''$ geodesic; otherwise, $v'v_1vv_2v''$ is a vertex-proper $v'$-$v''$ geodesic. It can be verified for any other pair of vertices in $G$ that there exists a vertex-proper geodesic between them. Hence, $spvc(G) \leq 3 < n-3$.

Case 4. $\Delta \leq n-4$. By (3), we have $spvc(G) \leq \chi(G)$. If $G$ is an odd cycle, then $\chi(G) = 3$, and so $spvc(G) \leq 3 < n-3$; otherwise, $\chi(G) \leq \Delta$ by Brook' theorem [2], and so $spvc(G) \leq \Delta < n-3$.

The proof is complete. □

Now, we are ready to prove Theorem 7.

*Proof of Theorem 7.* By Proposition 2, we obtain that $spvc(G) = n-3$ for $G = G_i$ ($1 \leq i \leq 5$). If $G = G_i$ ($6 \leq i \leq 11$), then $diam(G) \geq 3$, and so $spvc(G) \geq 2$. A 2-coloring of the vertices of $G = G_i$ ($6 \leq i \leq 11$) is shown in Figure 2 to make $G$ strong proper vertex-connected. Thus, $spvc(G) = n-3$ for $G = G_i$ ($6 \leq i \leq 11$). For the graph $G_{12}$, color the three non-leaves with distinct colors. Then, we can see that there exists a vertex-proper geodesic for any two vertices. Hence, $spvc(G_{12}) \leq 3$. However, if one colors the vertices of $G_{12}$ with two colors, there exist two non-leaves having the same color and then we can not find a vertex-proper geodesic between the corresponding pendant vertices. Thus, $spvc(G_{12}) = 3$.

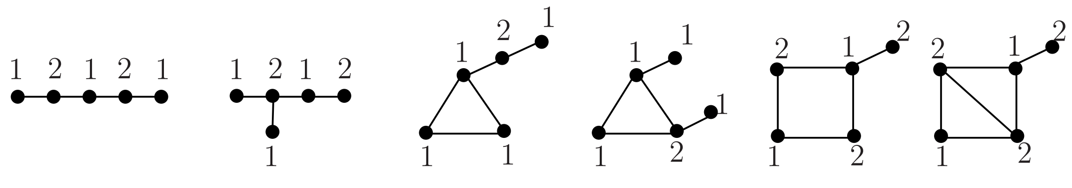

Figure 2: A 2-coloring of vertices of $G = G_i$ ($6 \leq i \leq 11$).

It remains to verify the converse. Let $G$ be a connected graph of order $n \geq 3$ such that $spvc(G) = n - 3$. By Lemma 3, we have that $n \in \{3, 4, 5, 6\}$. Firstly, suppose $n = 6$. Then, $spvc(G) = 3$. By Proposition 2, we get $diam(G) \geq 3$. Moreover, $G$ contains a cycle; otherwise, $G$ is a tree and $spvc(G) = 2$. If $G \neq G_{12}$, it follows that $G$ contains a subgraph isomorphic to one of the graphs $H_1, H_2, H_3, H_4, H_5, H_6$ in Figure 3, where a minimum vertex-coloring is also shown for each graph to make it strong proper vertex-connected. Moreover, color the remaining vertices of $G$ with any used color if there exist. It is easy to check that $G$ is strong proper vertex-connected. Thus, $spvc(G) \leq 2 < 3$, which is a contradiction. If $n = 5$, then $spvc(G) = 2$. By Proposition 2, $diam(G) \geq 3$. However, for $G \neq G_i$ $(6 \leq i \leq 11)$, the diameter of $G$ is at most two, which is a contradiction. Namely, the graphs labeled as $G_6, G_7, \ldots, G_{11}$ are all of the graphs on 5 vertices with diameter at least 3. Similarly, we deduce that $G = G_i$ $(2 \leq i \leq 5)$ for $n = 4$ and $G = G_1$ for $n = 3$. □

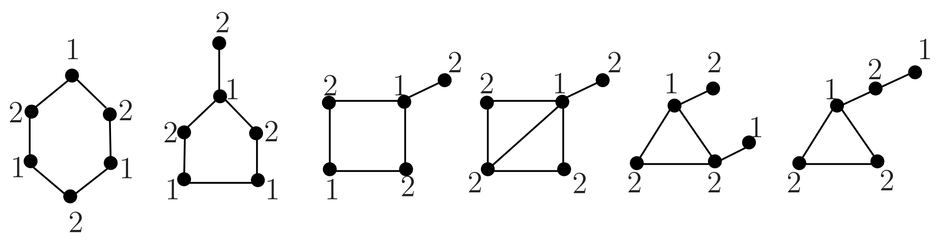


Figure 3: Subgraphs $H_1, H_2, \ldots, H_6$ in the proof of Theorem 7.

## 3.2 Relationship of $spvc(G)$, $srvc(G)$ and $\chi(G)$

By (3), if $G$ is a nontrivial connected graph with $diam(G) \geq 3$ such that $spvc(G) = a$ and $srvc(G) = b$, then $2 \leq a \leq b$. Actually, this is the only restriction on the two parameters.

**Theorem 8.** *For every pair $a, b$ of integers where $2 \leq a \leq b$, there exists a connected graph $G$ such that $spvc(G) = a$ and $srvc(G) = b$.*

*Proof.* Let $H$ be the corona $cor(K_a)$ of the complete graph $K_a$ with $V(K_a) = \{v_1, v_2, \ldots, v_a\}$ and $V(H \backslash K_a) = \{v_1^{'}, v_2^{'}, \ldots, v_a^{'}\}$, where $v_i^{'}$ is the corresponding pendant vertex of $v_i$ for $1 \leq i \leq a$. Let $F = P_{b-a}$ with $V(F) = \{w_1, w_2, \ldots, w_{b-a}\}$. Now let $G$ be the graph obtained from $H$ and $F$ by adding the edge $v_a^{'} w_1$.

Firstly, we show that $spvc(G) = a$. Define a vertex-coloring of $G$ as follows: Assign the color $j$ to $v_j$ for $1 \leq j \leq a$ and the color 1 to $v_a^{'}$. For $1 \leq k \leq b - a$, if $k$ is even, assign the color 1 to $w_k$; otherwise, assign the color 2 to $w_k$. We can see that every two vertices $x$ and $y$ are connected by a vertex-proper $x$-$y$ geodesic. Hence, $spvc(G) \leq a$. If one colors the vertices of $G$ with fewer than $a$ colors, then there must be two vertices $v_s$ and $v_t$, where $1 \leq s, t \leq a$, such that they have the same color. However, we can not find a vertex-proper geodesic between $v_s^{'}$ and $v_t^{'}$. Thus, $spvc(G) = a$.

Next, we prove that $srvc(G) = b$. Define a vertex-coloring of $G$ by assigning (1) the color $j$ to $v_j$ for $1 \leq j \leq a$, (2) the color $a+1$ to $v_a^{'}$ and (3) the color $a+1+k$ to $w_k$ for $1 \leq k \leq b-a-1$. Since all non-leaves of $G$ have distinct colors, $G$ is strong rainbow vertex-connected. Hence, $srvc(G) \leq b$. If one colors the vertices of $G$ with fewer than $b$ colors, then there must be two vertices of $\{v_1, v_2, \ldots, v_a, v_a^{'}, w_1, w_2, \ldots, w_{b-a-1}\}$ having the same color. Furthermore, the colors of $\{v_a^{'}, w_1, w_2, \ldots, w_{b-a-1}\}$ must be distinct since there is only one path between $v_a$ and $w_{b-a}$. If the colors of $v_i$ and $v_j$ $(1 \leq i, j \leq a)$ are the same, then there does not exist a vertex-rainbow geodesic between $v_i^{'}$ and $v_j^{'}$. If the colors of $v_i$ and $w_k$, where $1 \leq i \leq a-1$ and $1 \leq k \leq b-a-1$, are the same, then we can not find a vertex-rainbow geodesic between $v_i^{'}$ and $w_{k+1}$. If the colors of $v_a$ and $v_k$ $(1 \leq k \leq b-a-1)$ are the same, then there does not exist a vertex-rainbow geodesic between $v_1$ and $w_{k+1}$. Thus $srvc(G) = b$. □

We saw in (3) that if $G$ is a nontrivial connected graph with $diam(G) \geq 3$ which is not an odd cycle such that $spvc(G) = a, \chi(G) = b$ and $\Delta(G) = c$, then $2 \leq a \leq b \leq c$. In fact, this is the only restriction on the three parameters.

**Theorem 9.** *For every triple $a, b, c$ of integers where $2 \leq a \leq b \leq c$, there exists a connected graph $G$ such that $spvc(G) = a, \chi(G) = b$ and $\Delta(G) = c$.*

*Proof.* Let $H = K_b$ with $V(K_b) = \{v_1, v_2, \ldots, v_b\}$. Then, add $c-b+1$ pendant vertices, denoted by $\{v_1^1, v_1^2, \ldots, v_1^{c-b+1}\}$, to $v_1$, and a pendant vertex $v_i^1$ to $v_i$ for $2 \leq i \leq a$. Write $G$ as the resulting graph. It is easy to check that $\chi(G) = b$ and $\Delta(G) = c$.

In the following, we show that $spvc(G) = a$. Define a vertex-coloring of $G$ by assigning the color $j$ to $v_j$ for $1 \leq j \leq a$. Moreover, color the remaining vertices of $G$ with any used color. It is easy to check that every two vertices $x$ and $y$ are connected by a vertex-proper $x$-$y$ geodesic. Hence, $spvc(G) \leq a$. If one colors the vertices of $G$ with fewer than $a$ colors, then there must be two vertices $v_j$ and $v_k$ $(1 \leq j, k \leq a)$ such that they have the same color. However, we can not find a vertex-proper geodesic between $v_j^1$ and $v_k^1$. Thus $spvc(G) = a$. □

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## References


[1] E. Andrews, E. Laforge, C. Lumduanhom, P. Zhang, On proper-path colorings in graphs, *J. Combin. Math. Combin. Comput*, to apprear.

[2] J.A. Bondy, U.S.R. Murty, *Graph Theory*, GTM 244, Springer, 2008.

[3] V. Borozan, S. Fujita, A. Gerek, C. Magnant, Y. Manoussakis, L. Montero, Z. Tuza, Proper connection of graphs, *Discrete Math.* **312(17)** (2012) 2550-2560.

[4] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, Rainbow connection in graphs, *Math. Bohemica* **133** (2008) 85-98.

[5] G. Chartrand, G.L. Johns, K.A. Mckeon, P. Zhang, The rainbow connectivity of a graph, *Networks* **54(2)** (2009) 75-81.

[6] L. Chen, X. Li, M. Liu, Nordhaus-Gassum-type theorem for the rainbow vertex-connection number of a graph, *Util. Math.* **86** (2011) 335-340.

[7] L. Chen, X. Li, Y. Shi, The complexity of deterimining the rainbow vertex-connection of a graph, *Theoret. Comput. Sci.* **412(35)** (2011) 4531-4535.

[8] M. Krivelevich, R. Yuster, The rainbow connection of a graph is (at most) reciprocal to its minimum degree, *J. Graph Theory* **63** (2010) 185-191.

[9] E. Laforge, C. Lumduanhom, P. Zhang, Characterizations of graphs having large proper connection numbers, to appear in *Discuss. Math. Graph Theory.*

[10] X. Li, Y. Mao, Y. Shi, The strong rainbow vertex-connection of graphs, *Util. Math.* **93** (2014) 213-223.

[11] X. Li, Y. Shi, On the rainbow vertex-connection, *Discuss. Math. Graph Theory* **33** (2013) 307-313.

[12] X. Li, Y. Shi, Y. Sun, Rainbow connections of graphs: A survey, *Graphs Combin.* **29(1)** (2013) 1-38.

[13] X. Li, Y. Sun, *Rainbow Connections of Graphs,* Springer Briefs in Math., Springer, New York, 2012.

[14] H. Liu, Â. Mestre, T. Sousa, Rainbow vertex $k$-connection in graphs, *Discrete Appl. Math.* **162** (2013) 2549-2555.