Note on the spanning-tree packing number of lexicographic product graphs*

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Abstract

The spanning-tree packing number of a graph G is the maximum number of edge-disjoint spanning trees contained in G. In this paper, we obtain a sharp lower bound for the spanning-tree packing number of lexicographic product graphs.

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1 Introduction

All graphs in this paper are undirected, finite and simple. For any graph G of order n, the *spanning-tree packing number* of G, denoted by $\sigma(G)$, is the maximum number of edge-disjoint spanning trees contained in G. This has been used as measure of reliability of communication network, and studied by several authors, see the surveys by Palmer [9] and Ozeki and Yamashita [8]. It is worth pointing out that for a given graph G, the maximum number of edge-disjoint spanning trees in G can be found in polynomial time; see [12] (Page 879). Actually, Roskind and Tarjan [11] proposed a $O(m^2)$ time algorithm for finding the maximum number of edge-disjoint spanning trees in a graph, where m is the number of edges in the graph.

In [10], Peng and Tay determined the spanning-tree packing numbers of Cartesian products of various combinations of complete graphs, cycles, complete multipartite graphs.

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Later, Ku, Wang and Hung [5] obtained the following result: $\sigma(G\Box H) \geq \sigma(G)+\sigma(H)-1$ for two connected graphs G and H.

In this paper, we focus our attention on another graph product, called lexicographic product. The *lexicographic product* (sometimes known as composition) of two graphs G and H, written as $G \circ H$, is defined as follows: The vertex set of $G \circ H$ is $V(G) \times V(H)$; and any two distinct vertices (u, v) and (u', v') of $G \circ H$ are adjacent if and only if either $(u, u') \in E(G)$ or $u = u'$ and $(v, v') \in E(H)$. Note that, unlike the Cartesian product, the lexicographic product is a non-commutative product since $G \circ H$ is usually not isomorphic to $H \circ G$. It is easy to see that $|E(G \circ H)| = |E(H)||V(G)| + |E(G)||V(H)|^2$.

Theorem 1 *Let G and H be two connected nontrivial graphs, and let $\sigma(G) = k$, $\sigma(H) = \ell$, $|V(G)| = n_1$ $(n_1 \geq 2)$, and $|V(H)| = n_2$ $(n_2 \geq 2)$. Then*

(1) *if $kn_2 = \ell n_1$, then $\sigma(G \circ H) \geq kn_2 (= \ell n_1)$;*

(2) *if $\ell n_1 > kn_2$, then $\sigma(G \circ H) \geq kn_2 - \lceil \frac{kn_2-1}{n_1} \rceil + \ell - 1$;*

(3) *if $\ell n_1 < kn_2$, then $\sigma(G \circ H) \geq kn_2 - 2\lceil \frac{kn_2}{n_1+1} \rceil + \ell$.*

Moreover, the bounds are sharp.

2 Proof of Theorem 1

Throughout this paper, assume that G and H are two connected graphs with $V(G) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(H) = \{v_1, v_2, \ldots, v_{n_2}\}$, respectively. For $v \in V(H)$, we use $G(v)$ to denote the subgraph of $G \circ H$ induced by the vertex set $\{(u_j, v) \,|\, 1 \leq j \leq n_1\}$. Similarly, for $u \in V(G)$, we use $H(u)$ to denote the subgraph of $G \circ H$ induced by the vertex set $\{(u, v_i) \,|\, 1 \leq i \leq n_2\}$. We refer to the book [1] for graph theoretic notation and terminology not described here. In the sequel, we let $\sigma(G) = k$, $\sigma(H) = \ell$, and $T_1, T_2, \cdots, T_k$ be k edge-disjoint spanning trees in G and $T'_1, T'_2, \cdots, T'_\ell$ be ℓ edge-disjoint spanning trees in H.

The proof consists of two steps: in the first step (presented in Section 2.1), we decompose $G \circ H$ into small graphs; in the second step (presented in Section 2.2), we divide these small graphs into groups and combine the small graphs in each group into a spanning tree of $G \circ H$, thus obtaining the desired number of edge-disjoint spanning trees. After the second step, we can obtain a lower bound of $\sigma(G \circ H)$.

The details are given below.

2.1 Graph decomposition

From the definition, the lexicographic product graph $G \circ H$ is a graph obtained by replacing each vertex of G by a copy of H and replacing each edge of G by a complete bipartite graph K_{n_2,n_2}. For an edge $e = u_i u_j \in E(G)$ $(1 \le i, j \le n_1)$, the induced subgraph obtained from the edges between the vertex set $V(H(u_i)) = \{(u_i, v_1), (u_i, v_2), \cdots, (u_i, v_{n_2})\}$ and the vertex set $V(H(u_j)) = \{(u_j, v_1), (u_j, v_2), \cdots, (u_j, v_{n_2})\}$ in $G \circ H$ is a complete equipartition bipartite graph of order $2n_2$, denoted by K_e or K_{u_i,u_j}. Obviously, K_e can be decomposed into n_2 perfect matching, denoted by $M_1^e, M_2^e, \ldots, M_{n_2}^e$.

For each T_i $(1 \le i \le k)$ in G, we define a spanning subgraph $\mathcal{T}_i$ of $G \circ H$ corresponding to T_i as follows: $V(\mathcal{T}_i) = V(G \circ H)$ and $E(\mathcal{T}_i) = \{(u_p, v_s)(u_q, v_t) \,|\, u_p u_q \in E(T_i),\ u_p, u_q \in V(G),\ v_s, v_t \in V(H)\}$. We call $\mathcal{T}_i$ a *blow-up graph corresponding to T_i in G*; see Figure 1 for an example.

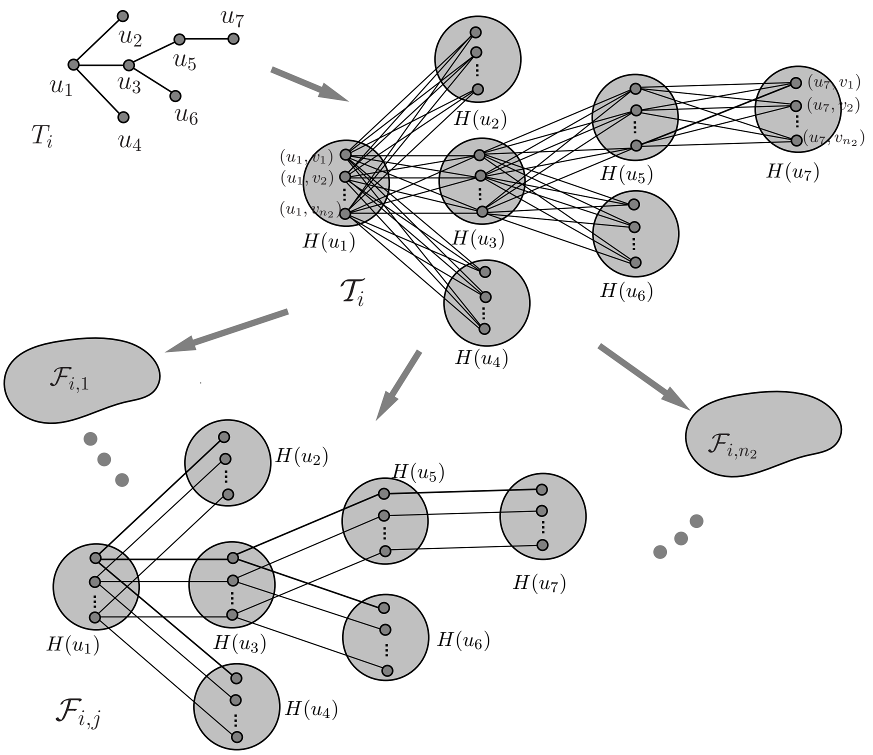

Figure 1: The blow-up graph $\mathcal{T}_i$ and parallel forest $\mathcal{F}_{i,j}$ in $G \circ H$ corresponding to T_i in G.

For each i $(1 \le i \le k)$ and j $(1 \le j \le n_2)$, we define another spanning subgraph $\mathcal{F}_{i,j}$ of $G \circ H$ corresponding to T_i in G as follows: $V(\mathcal{F}_{i,j}) = V(G \circ H)$ and $E(\mathcal{F}_{i,j}) = \bigcup_{e \in E(T_i)} M_{i,j}^e$, where $M_{i,j}^e$ is a matching of K_e ($M_{i,j}^e$ will be chosen later). We call $\mathcal{F}_{i,j}$ a *parallel forest of $G \circ H$ corresponding to the tree T_i in G*; see Figure 1 for an example.

Similarly, for a spanning tree T'_j $(1 \le j \le \ell)$ in H, we define a spanning subgraph $\mathcal{F}'_j$ of $G \circ H$ as follows: $V(\mathcal{F}'_j) = V(G \circ H)$ and $E(\mathcal{F}'_j) = \{(u, v_s)(u, v_t) \,|\, u \in V(G), v_s v_t \in E(T_j)\}$. Clearly, $\mathcal{F}'_j = \bigcup_{u_i \in V(G)} T'_j(u_i)$, where $V(T'_j(u_i)) = \{(u_i, v) \,|\, v \in V(H)\}$ and $E(T'_j(u_i)) = \{(u_i, v_s)(u_i, v_t) \,|\, u_i \in V(G), v_s v_t \in E(T'_j)\}$. We call each of $\mathcal{F}'_j$ $(1 \le i \le \ell)$ a *vertical forest of $G \circ H$ corresponding to the tree T'_j in H*. The tree $T'_j(u_i)$ is called *the isomorphic tree of T'_j $(1 \le i \le \ell)$ in $H(u_i)$*. So, for each tree T'_j of H there are n_1 edge-disjoint isomorphic trees $T'_j(u_i)$ $(1 \leq i \leq n_1)$ in $G \circ H$.

The following results are useful for our proof, which were obtained by Dirac [3]; see Laskar and Auerbach [6].

Proposition 2 [3, 6] (1) *For all even $r \geq 2$, $K_{r,r}$ is the union of its $\frac{1}{2}r$ Hamiltonian cycles.*

(2) *For all odd $r \geq 3$, $K_{r,r}$ is the union of its $\frac{1}{2}r$ Hamiltonian cycles and one perfect matching.*

For $r \geq 2$, the complete equipartition bipartite graph $K_{r,r}$ can be decomposed into $\lfloor \frac{r}{2} \rfloor$ Hamiltonian cycles for r even, or $\lfloor \frac{r}{2} \rfloor$ Hamiltonian cycles and one perfect matching for r odd. We call each Hamiltonian cycle in the decomposition a *good cycle*.

We now decompose the above blow-up graph $\mathcal{T}_i$ $(1 \leq i \leq k)$ in $G \circ H$ corresponding to T_i in G into our desired n_2 parallel forests by Proposition 2.

Lemma 3 *The blow-up graph $\mathcal{T}_i$ corresponding to the tree T_i in G can be decomposed into n_2 parallel forests corresponding to the tree T_i, say $\mathcal{F}_{i,1}, \mathcal{F}_{i,2}, \cdots, \mathcal{F}_{i,n_2}$, such that there exist $2x$ parallel forests $\mathcal{F}_{i,1}, \mathcal{F}_{i,2}, \cdots, \mathcal{F}_{i,2x}$ such that $\mathcal{F}_{i,2j-1} \cup \mathcal{F}_{i,2j}$ $(1 \leq j \leq x \leq \lfloor \frac{n_2}{2} \rfloor)$ contains exactly $n_1 - 1$ good cycles.*

Proof. We decompose $G \circ H$ as follows:

(i) for every $i \in [k]$ and $e \in T_i$, by Proposition 2, we decompose K_e into n_2 disjoint perfect matchings $M^e_{i,1}, \cdots, M^e_{i,n_2}$ such that $M^e_{i,2j+1} \cup M^e_{i,2j+2}$ is a Hamilton cycle (which we call a good cycle) for every $j \leq \lfloor n_2/2 \rfloor - 1$;

(ii) for every $i \in [k]$, we have that, for every $e = uw \in E(T_i)$, $e' = u'w' \in E(T_i)$ and $t \in [n_2]$, the matchings $\{vz : \{(u,v),(w,z)\} \in M^e_t\}$ and $\{vz : \{(u',v),(w',z)\} \in M^{e'}_t\}$ are the same.

We give the definition of $\mathcal{F}_{i,j}$ as follows: $\mathcal{F}_{i,j} = \bigcup_{e \in E(T_i)} M^e_{i,j}$, where $1 \leq j \leq \lfloor n_2/2 \rfloor$. For each $e \in E(T_i)$, $K_e \cap (\mathcal{F}_{i,2j-1} \cup \mathcal{F}_{i,2j})$ is a good cycle, where $1 \leq j \leq r$. Since $|E(T_i)| = n_1 - 1$, this implies that, for $1 \leq j \leq \lfloor n_2/2 \rfloor$, $\mathcal{F}_{i,2j-1} \cup \mathcal{F}_{i,2j}$ contains exactly $n_1 - 1$ good cycles. So all the edges of $T_i \circ H$ can be decomposed into n_2 parallel forests $\mathcal{F}_{i,1}, \mathcal{F}_{i,2}, \cdots, \mathcal{F}_{i,n_2}$ such that there exist $2x$ parallel forests $\mathcal{F}_{i,1}, \mathcal{F}_{i,2}, \cdots, \mathcal{F}_{i,2x}$ such that $\mathcal{F}_{i,2j-1} \cup \mathcal{F}_{i,2j}$ $(1 \leq j \leq x \leq \lfloor n_2/2 \rfloor)$ contains exactly $n_1 - 1$ good cycles. The proof is now complete. ∎

2.2 Graph combination

Recall that $\sigma(G) = k$ and $T_1, \cdots, T_k$ are edge-disjoint spanning trees of G (as defined in the beginning of Section 2.1) and that $\mathcal{F}_{i,j}$ $(1 \leq i \leq k,\ 1 \leq j \leq n_2)$ corresponding to

T_i in H are the parallel forests obtained by Lemma 3. Similarly, $\sigma(H) = \ell$ and $T'_1, \cdots, T'_\ell$ are edge-disjoint spanning trees of H (as defined in the beginning of Section 2.1) and that $\mathcal{F}'_j$ $(1 \leq j \leq \ell)$ are the vertical forests corresponding to T'_j of H.

After the above preparations, we now give the proof of Theorem 1.

Proof of (1): Since the union of any tree in $\{T'_j(u_i) \,|\, 1 \leq i \leq n_1, 1 \leq j \leq \ell\}$ with any parallel forest in $\{\mathcal{F}_{i,j} \,|\, 1 \leq i \leq k, 1 \leq j \leq n_2\}$ is a spanning tree of $G \circ H$, we can get $kn_2 = \ell n_1$ edge-disjoint spanning trees in $G \circ H$. Thus, $\sigma(G \circ H) \geq kn_2$ $(= \ell n_1)$. ∎

Proof of (2): Note that $\{\mathcal{F}_{i,j} \,|\, 1 \leq i \leq k, 1 \leq j \leq n_2\} \setminus \{\mathcal{F}_{k,n_2}\}$ is a set of $kn_2 - 1$ edge-disjoint parallel forests and, for $1 \leq x \leq \ell$, $\{T'_{i,j} \,|\, 1 \leq i \leq x, 1 \leq j \leq n_1\}$ is a set of xn_1 edge-disjoint trees. The union of any forest in $\{\mathcal{F}_{i,j} \,|\, 1 \leq i \leq k, 1 \leq j \leq n_2\} \setminus \{\mathcal{F}_{k,n_2}\}$ with any tree in $\{T'_{i,j} \,|\, 1 \leq i \leq x, 1 \leq j \leq n_1\}$ is a spanning tree of $G \circ H$. We set $x = \lceil \frac{kn_2-1}{n_1} \rceil$ so that $xn_1 \geq kn_2 - 1$. Since $\ell n_1 > kn_2$, it follows that $\lceil \frac{kn_2-1}{n_1} \rceil \leq \ell$ and hence $x \leq \ell$. Thus, we can obtain $kn_2 - 1$ edge-disjoint spanning tree of $G \circ H$.

Recall that we also have $\ell - x$ vertical forests $\mathcal{F}'_{x+1}, \mathcal{F}'_{x+2}, \cdots, \mathcal{F}'_\ell$. We now find some spanning trees of $G \circ H$ from the union of $\mathcal{F}_{k,n_2}$ and the above $\ell - x$ vertical forests. By the definition of the vertical forest $\mathcal{F}_{k,n_2}$, it is the union of n_2 vertex-disjoint trees isomorphic to T_k, say $T_{k,1}, T_{k,2}, \cdots, T_{k,n_2}$. Note that the union of any vertical forest in $\{\mathcal{F}'_{x+1}, \mathcal{F}'_{x+2}, \cdots, \mathcal{F}'_\ell\}$ and any tree in $\{T_{k,1}, T_{k,2}, \cdots, T_{k,n_2}\}$ is a spanning tree of $G \circ H$. Since $\ell - x \leq \ell \leq \lfloor \frac{n_2}{2} \rfloor \leq n_2$, we can obtain $\ell - x$ edge-disjoint spanning trees of $G \circ H$.

From the above arguments, the total number of the edge-disjoint spanning trees is at least $(kn_2 - 1) + (\ell - x)$. Thus, $\sigma(G \circ H) \geq kn_2 - 1 + \ell - x = kn_2 - \lceil \frac{kn_2-1}{n_1} \rceil + \ell - 1$. ∎

Proof of (3): Let $\mathcal{F}_{i,j}$ $(1 \leq i \leq k, 1 \leq j \leq n_2)$ be the kn_2 parallel forests in $G \circ H$ corresponding to T_i $(1 \leq i \leq k)$ in Lemma 3. Pick up $2x$ parallel forests from $\{\mathcal{F}_{i,j} \,|\, 1 \leq i \leq k, 1 \leq j \leq n_2\}$, say $\mathcal{F}_{a_1,b_1}, \mathcal{F}_{a_1,c_1}, \mathcal{F}_{a_2,b_2}, \mathcal{F}_{a_2,c_2}, \cdots, \mathcal{F}_{a_x,b_x}, \mathcal{F}_{a_x,c_x}$ where $a_i \in \{1, 2, \cdots, k\}$ $(1 \leq i \leq x)$ and $b_i, c_i \in \{1, 2, \cdots, n_2\}$ $(1 \leq i \leq x)$, such that $\mathcal{F}_{a_i,b_i} \cup \mathcal{F}_{a_i,c_i}$ $(1 \leq i \leq x)$ contains $(n_1 - 1)$ good cycles. Note that we have to choose $x \leq k\lfloor n_2/2 \rfloor$. Thus we can obtain $x(n_1 - 1)$ good cycles from the above $2x$ parallel forests. Now we still have $kn_2 - 2x$ parallel forests. Note that the union of any of these $kn_2 - 2x$ parallel forests with any of those $x(n_1 - 1)$ good cycles is a spanning subgraph of $G \circ H$, which contains a spanning tree of $G \circ H$. If $x(n_1 - 1) \geq kn_2 - 2x$, then we can obtain $kn_2 - 2x$ edge-disjoint spanning trees of $G \circ H$. We set $x = \lceil \frac{kn_2}{n_1+1} \rceil$ so that x is the smallest possible integer satisfying $x(n_1 - 1) \geq kn_2 - 2x$. Since $kn_2 > \ell n_1$, it follows that $x = \lceil \frac{kn_2}{n_1+1} \rceil \geq 1$. Since $x \leq k\lfloor n_2/2 \rfloor$, we need to show that $\lceil \frac{kn_2}{n_1+1} \rceil \leq k\lfloor \frac{n_2}{2} \rfloor$. Therefore, it suffices to prove that $\frac{kn_2+n_1}{n_1+1} \leq k\frac{n_2-1}{2}$, that is, $k(n_1 - 1)(n_2 - 1) - 2n_1 - 2k \geq 0$. Since $k \leq \lfloor \frac{n_1}{2} \rfloor$, we need to show that $k(n_1 - 1)(n_2 - 1) - 3n_1 \geq 0$. Since $k \geq 1$ and $n_1 \geq 2$, it follows that $k(n_1-1)(n_2-1)-3n_1 = (n_1-1)(k(n_2-1)-3)-3 \geq k(n_2-1-3)-3 \geq n_2-1-3-3 \geq 0$ for $n_2 \geq 7$. So, the above inequality holds for $n_2 \geq 7$, as desired. One can check that the equality $\lceil \frac{kn_2}{n_1+1} \rceil \leq k\lfloor \frac{n_2}{2} \rfloor$ also holds for $2 \leq n_2 \leq 6$. Thus we get $kn_2 - 2\lceil \frac{kn_2}{n_1+1} \rceil$ spanning tree of $G \circ H$ from the parallel forests.

From the above arguments we can see that by combining a parallel forest and a good cycle (Hamiltonian cycle) we form a spanning subgraph of size $(n_1+1)n_2$, which contains a spanning tree of $G \circ H$ of size n_1n_2-1. Clearly, some edges of such a spanning subgraph are not used in the construction of a spanning tree of $G \circ H$. Our aim is to choose some of such unused edges and combine them with all the n_1 copies $H(u_1), H(u_2), \cdots, H(u_{n_1})$ of H in $G \circ H$ to form ℓ edge-disjoint spanning trees of $G \circ H$. Without loss of generality, assume that $a_1 = 1$, $b_1 = 1$ and $c_1 = 2$. Then $\mathcal{F}_{1,1} \cup \mathcal{F}_{1,2}$ contains (n_1-1) good cycles. Let $C^e_{1,1}$ be a good cycle in $\mathcal{F}_{1,1} \cup \mathcal{F}_{1,2}$, where $e \in E(T_1)$. Suppose that $\mathcal{F}_{i,j}$ be a parallel forest that is not used to construct good cycles. Then we have the following claim.

Claim 1. For each edge $e \in E(T_1)$, there exists a subset $E^e_{1,1}$ of $E(C^e_{1,1})$ such that $|E^e_{1,1}| = n_2 - 1$ and $\mathcal{F}_{i,j} \cup E^e_{1,1}$ is a spanning tree of $G \circ H$.

Proof of Claim 1: Let u and w denote the endpoints of e. Recall that the parallel forest $\mathcal{F}_{i,j}$ consists of n_2 vertex-disjoint isomorphic trees, each containing exactly one vertex of $H(u)$ for every $u \in V(H)$. Let $R_1, \cdots, R_{n_2}$ be such trees, and let P be the path joining u and w in T_i and $\mathcal{P} = \{P_{(u,v)(w,z)} : P_{(u,v)(w,z)}$ is the path joining (u,v) and (w,z) in R_i for $i \in [n_2]\}$. Then $\mathcal{P}$ consists of n_2 isomorphic paths. The connected components of the graph formed by $\mathcal{P} \cup M^e_{1,j}$ consist of a collection of disjoint cycles, say $C_1, \cdots, C_m$. For every $i \in [m]$, let f_i denote an arbitrary edge in C_i and let $D_i = C_i \setminus \{f_i\}$. Since the spanning subgraph of K_e with edge set $M^e_{1,1} \cup M^e_{1,2}$ is connected (it is a Hamilton cycle), there is a set of $S^e \subseteq M^e_{1,2}$ of size $m-1$ such that the $S^e \cup \bigcup_{i=1}^m D_i$ is connected. Thus, by defining $E^e_{1,1} = (S^e \cup M^e_{1,1} \setminus \{f_i : i \in [m]\}$, we have that $|E^e_{1,1}| = n_2 - 1$ and $E^e_{1,1} \cup \mathcal{F}_{i,j}$ is a spanning tree of $G \circ H$. ∎

From Claim 1, for each good cycle $C^e_{1,1}$ in $\mathcal{F}_{1,1} \cup \mathcal{F}_{1,2}$, we can find a subset $E^e_{1,1}$ of $E(C^e_{1,1})$ such that $|E^e_{1,1}| = n_2 - 1$ and $\mathcal{F}_{i,j} \cup E^e_{1,1}$ is a spanning tree of $G \circ H$, where $\mathcal{F}_{i,j}$ is a parallel forest that was not used in the construction of good cycles. For the good cycle $C^e_{1,1}$ of $\mathcal{F}_{1,1} \cup \mathcal{F}_{1,2}$ where $e \in E(T_1)$, we define a set $\overline{E}^e_{1,1}$ of edges as follows: if it was used in the construction of a spanning tree of $G \circ H$, then $\overline{E}^e_{1,1} = E(C^e_{1,1}) \setminus E^e_{1,1}$; otherwise, $\overline{E}^e_{1,1} = E(C^e_{1,1})$. Then $|\overline{E}^e_{1,1}| \geq n_2 + 1 \geq \ell$.

We are now in a position to combine some edges of the set $\bigcup_{e \in E(T_1)} \overline{E}^e_{1,1}$ of edges with all the n_1 copies $H(u_1), H(u_2), \cdots, H(u_{n_1})$ of H in $G \circ H$ to form ℓ edge-disjoint spanning trees of $G \circ H$. Since $\sigma(H) = \ell$, there exist ℓ edge-disjoint spanning trees in H, say $T'_1, T'_2, \cdots, T'_\ell$. Then there exist vertical forests $\mathcal{F}'_j = \bigcup_{u_i \in V(G)} T'_j(u_i)$ $(1 \leq j \leq \ell)$ in $G \circ H$ corresponding to T'_j, where $T'_j(u_i)$ is the isomorphic tree of T'_j. Recall that $|\overline{E}^e_{1,1}| \geq \ell$ for each edge $e \in E(T_1)$. Choose ℓ edges in $\overline{E}^e_{1,1}$, say $f^e_1, f^e_2, \cdots, f^e_\ell$. Let $E_i = \bigcup_{e \in E(T_1)} f^e_i$ $(1 \leq i \leq \ell)$. Note that any of the sets $\{E_i \,|\, 1 \leq i \leq \ell\}$ of edges with any of the vertical forests $\mathcal{F}'_1, \mathcal{F}'_2, \cdots, \mathcal{F}'_\ell$ is a spanning tree of $G \circ H$. It is clear that we can find ℓ edge-disjoint spanning trees of $G \circ H$ from the edges of $\bigcup_{e \in E(T_1)} \overline{E}^e_{1,1}$ and the n_1 copies of H in $G \circ H$.

From the above arguments, the total number of the edge-disjoint spanning trees of $G \circ H$ is at least $kn_2 - 2\lceil \frac{kn_2}{n_1+1} \rceil + \ell$. So $\sigma(G \circ H) \geq kn_2 - 2\lceil \frac{kn_2}{n_1+1} \rceil + \ell$. ∎

To show the sharpness of the above lower bounds of Theorem 1, we consider the following three examples.

Example 1. Let G and H be two connected graphs with $|V(G)| = n_1$ and $|V(H)| = n_2$ which can be decomposed into exactly k and ℓ edge-disjoint spanning trees of G and H, respectively, satisfying $kn_2 = \ell n_1$. From (1) of Theorem 1, $\sigma(G \circ H) \geq kn_2 = \ell n_1$. Since $|E(G \circ H)| = |E(H)|n_1 + |E(G)|n_2^2 = \ell(n_2-1)n_1 + k(n_1-1)n_2^2 = kn_2(n_2-1) + k(n_1-1)n_2^2 = kn_2(n_1n_2-1)$, we have $\sigma(G \circ H) \leq \frac{|E(G \circ H)|}{n_1n_2-1} = kn_2$. Then $\sigma(G \circ H) = kn_2 = \ell n_1$. So the lower bound of (1) is sharp.

Example 2. Consider the graphs $G = P_3$ and $H = K_4$. Clearly, $\sigma(G) = k = 1$, $\sigma(H) = \ell = 2$, $|V(G)| = n_1 = 3$, $|V(H)| = n_2 = 4$, $|E(G)| = 2$, $|E(H)| = 6$ and $6 = \ell n_1 > kn_2 = 4$. On one hand, we have $\sigma(G \circ H) \geq kn_2 - \lceil \frac{kn_2-1}{n_1} \rceil + \ell - 1 = 4 - 1 + 2 - \lceil \frac{4-1}{3} \rceil = 4$ by (2) of Theorem 1. On the other hand, $|E(G \circ H)| = 50$ and hence $\sigma(G \circ H) \leq \frac{|E(G \circ H)|}{n_1n_2-1} = \lfloor \frac{50}{11} \rfloor = 4$. So $\sigma(G \circ H) = 4$. So the lower bound of (2) is sharp.

Example 3. Consider the graphs $G = P_2$ and $H = P_3$. Clearly, $\sigma(G) = k = 1$, $\sigma(H) = \ell = 1$, $|V(G)| = n_1 = 2$, $|V(H)| = n_2 = 3$, $|E(G)| = 1$, $|E(H)| = 2$ and $2 = \ell n_1 < kn_2 = 3$. On one hand, $\sigma(G \circ H) \geq kn_2 - 2\lceil \frac{kn_2}{n_1+1} \rceil + \ell = 2$ by (3) of Theorem 1. On the other hand, $|E(G \circ H)| = |E(H)|n_1 + |E(G)|n_2^2 = 13$. Then $\sigma(G \circ H) \leq \frac{|E(G \circ H)|}{n_1n_2-1} = \lfloor \frac{13}{5} \rfloor = 2$. So $\sigma(G \circ H) = 2$ and the lower bound of (3) is sharp.

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