

# Rainbow $k$ -connectivity of random bipartite graphs\*

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## Abstract

A path in an edge-colored graph  $G$  is called a *rainbow path* if no two edges of the path are colored the same. The minimum number of colors required to color the edges of  $G$  such that every pair of vertices are connected by at least  $k$  internally vertex-disjoint rainbow paths is called the *rainbow  $k$ -connectivity* of the graph  $G$ , denoted by  $rc_k(G)$ . For the random graph  $G(n, p)$ , He and Liang got a sharp threshold function for the property  $rc_k(G(n, p)) \leq d$ . For the random equi-bipartite graph  $G(n, n, p)$ , Fujita et. al. got a sharp threshold function for the property  $rc_k(G(n, n, p)) \leq 3$ . They also posed the following problem: For  $d \geq 2$ , determine a sharp threshold function for the property  $rc_k(G) \leq d$ , where  $G$  is another random graph model. This paper is to give a solution to their problem in the general random bipartite graph model  $G(m, n, p)$ .

**Keywords:** rainbow  $k$ -connectivity, sharp threshold function, random bipartite graph

**AMS Subject Classification Numbers:** 05C15, 05C40, 05C80

## 1 Introduction

In this paper, unless otherwise stated, all graphs are finite, simple and undirected. For basic terminology and notation in graph theory, see [4]. Connectivity is one of the basic concepts of graph theory. Recently, the concepts of rainbow connectivity (or rainbow connection) and rainbow  $k$ -connectivity are introduced by Chartrand et. al. in [7] and [8] as a strengthening of the canonical connectivity concept. Given an edge-colored graph  $G$ , we call a path a *rainbow path* if no two edges of the path are colored the same. We call the graph  $G$  *rainbow connected* if every pair of vertices are connected by at least one rainbow path. The minimum number of colors required to make  $G$  rainbow connected is called the *rainbow connectivity*, denoted by  $rc(G)$ . In general, for an integer  $k \geq 1$ , a graph  $G$  is called *rainbow  $k$ -connected* if every pair of

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vertices of  $G$  are connected by at least  $k$  internally vertex-disjoint rainbow paths. The minimum number of colors required to make  $G$  rainbow  $k$ -connected is called the *rainbow  $k$ -connectivity*, denoted by  $rc_k(G)$ .

In addition to regarding it as a natural combinatorial concept, rainbow connectivity also has interesting applications in transferring information of high security and networking [8], [6] and [9]. The following motivation comes from [6]: Suppose we wish to route messages between any two vertices in a cellular network and require that each link on the route between the vertices is assigned with a distinct channel. We clearly wish to minimize the number of distinct channels. The minimum number is exactly the rainbow connectivity of the underlying graph. The subject has since attracted considerable interest. A great number of results about the rainbow connectivity have been obtained by the researchers. Recently, Li and Sun published a book [14] and Li et. al. wrote a survey [13] on the current status of rainbow connectivity. We refer them to the reader for details.

We will study the rainbow  $k$ -connectivity in the random graph setting [1]. Some results have been obtained in the Erdős-Rényi random graph model  $G(n, p)$ , which is a graph with  $n$  vertices where each of the  $\binom{n}{2}$  potential edges appears with probability  $p$ , independently. Random bipartite graph model is a general model for complex networks, thus in this paper, we will extend the results to the random bipartite graph  $G(m, n, p)$  with bipartition  $(U, V)$ , where  $|U| = m, |V| = n$  and for each  $u \in U$  and  $v \in V$  the edge  $uv$  appears with probability  $p$ , independently. We say that an event  $E = E(n)$  happens *almost surely* (or a.s. for brevity) if  $\lim_{n \rightarrow \infty} \Pr[E(n)] = 1$ . For a graph property  $\mathcal{P}$ , a function  $p^*(n)$  is called a *threshold function* of  $\mathcal{P}$  if <sup>†</sup>

- for every  $p(n) = o(p^*(n))$ ,  $G(n, p(n))$  almost surely does not satisfy  $\mathcal{P}$ ; and
- for every  $p'(n) = \omega(p^*(n))$ ,  $G(n, p'(n))$  almost surely satisfies  $\mathcal{P}$ .

Furthermore,  $p^*(n)$  is called a sharp threshold function of  $\mathcal{P}$  if there are two positive constants  $c$  and  $C$  such that

- for every  $p(n) \leq c \cdot p^*(n)$ ,  $G(n, p(n))$  almost surely does not satisfy  $\mathcal{P}$ ; and
- for every  $p'(n) \geq C \cdot p^*(n)$ ,  $G(n, p'(n))$  almost surely satisfies  $\mathcal{P}$ .

It is well known that all monotone graph properties have a sharp threshold function [3] and [10]. Obviously for every  $k, d$ , the property that the rainbow  $k$ -connectivity is at most  $d$  is monotone, and thus has a sharp threshold. Caro et. al. [5] proved that  $p = \sqrt{\log n/n}$  is a sharp threshold function for the property  $rc(G(n, p)) \leq 2$ . This was generalized by He and Liang [12], who proved that if  $d \geq 2$  and  $k \leq O(\log n)$ , then  $p = (\log n)^{1/d}/n^{(d-1)/d}$  is a sharp threshold

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<sup>†</sup>We use the following standard asymptotic notations: as  $n \rightarrow \infty$ ,  $f(n) = o(g(n))$  means that  $f(n)/g(n) \rightarrow 0$ ;  $f(n) = \omega(g(n))$  means that  $f(n)/g(n) \rightarrow \infty$ ;  $f(n) = O(g(n))$  means that there exists a constant  $C$  such that  $|f(n)| \leq Cg(n)$ ;  $f(n) = \Omega(g(n))$  means that there exists a constant  $c > 0$  such that  $f(n) \geq cg(n)$ ;

function for the property  $rc_k(G(n, p)) \leq d$ . Moreover, Fujita et. al. [11] proved that in the random bipartite graph  $G(n, n, p)$ ,  $p = \sqrt{\log n/n}$  is a sharp threshold function for the property  $rc_k(G(n, n, p)) \leq 3$ . They also posed some open problems, one of which is stated as follows.

**Problem 1.1** [11] *For  $d \geq 2$ , determine a sharp threshold function for the property  $rc_k(G) \leq d$ , where  $G$  is another random graph model.*

In this paper, we consider the random bipartite graph  $G(m, n, p)$ . The following results are obtained.

**Theorem 1.2** *Let  $d \geq 2$  be a fixed positive integer and  $k = k(n) \leq O(\log n)$ .*

*If  $d$  is odd, then*

$$p = (\log(mn))^{1/d} / (m^{(d-1)/(2d)} n^{(d-1)/(2d)})$$

*is a sharp threshold function for the property  $rc_k(G(m, n, p)) \leq d + 1$ , where  $m$  and  $n$  satisfy that  $pn \geq pm \geq (\log n)^4$ .*

*If  $d$  is even, then*

$$p = (\log n)^{1/d} / (m^{1/2} n^{(d-2)/(2d)})$$

*is a sharp threshold function for the property  $rc_k(G(m, n, p)) \leq d + 1$ , where  $m$  and  $n$  satisfy that there exists a small constant  $\epsilon$  with  $0 < \epsilon < 1$  such that  $pn^{1-\epsilon} \geq pm^{1-\epsilon} \geq (\log n)^4$ .*

Then, the following corollary follows immediately.

**Corollary 1.3** *Let  $d \geq 2$  be a fixed integer and  $k = k(n) \leq O(\log n)$ . Then  $p = (\log n)^{1/d} / n^{(d-1)/d}$  is a sharp threshold function for the property  $rc_k(G(n, n, p)) \leq d + 1$ .*

When  $d = 2$ , the corollary is just the result of Fujita et. al. in [11].

In the sequel, we will first show Theorem 1.2 in Section 2. Then in Section 3, we will prove a conclusion stated in Section 2, which plays a key role during our proof of Theorem 1.2.

## 2 Threshold of the rainbow $k$ -connectivity

In this section, we establish a sharp threshold function of the random bipartite graph  $G(m, n, p)$  for the property  $rc_k(G(m, n, p)) \leq d + 1$ . We distinguish two parts to prove Theorem 1.2 according to the parity of  $d$ . For brevity, let  $p_1 = (\log(mn))^{1/d} / (m^{(d-1)/(2d)} n^{(d-1)/(2d)})$  and  $p_2 = (\log n)^{1/d} / (m^{1/2} n^{(d-2)/(2d)})$ . For a fixed  $d$ , we always assume that  $p_1 n \geq p_1 m \geq (\log n)^4$  if  $d$  is odd and there exists a small constant  $\epsilon$  with  $0 < \epsilon < 1$  such that  $p_2 n^{1-\epsilon} \geq p_2 m^{1-\epsilon} \geq (\log n)^4$  if  $d$  is even. In the sequel, we fix  $\epsilon$ . Before our proof, we first recall the following fact on the diameter of a random bipartite graph.

**Theorem 2.1** [2, p.272] *Suppose that for all  $n$ ,*

$$pn \geq pm \geq (\ln n)^4$$

and that  $d$  is a fixed positive integer.

If  $d$  is odd and

$$p^d m^{(d-1)/2} n^{(d-1)/2} - \ln(mn) \rightarrow \infty,$$

or if  $d$  is even and

$$p^d m^{d/2} n^{d/2-1} - 2 \ln n \rightarrow \infty,$$

then almost every  $G(m, n, p)$  is of diameter at most  $d + 1$ .

If  $d$  is odd and

$$p^d m^{(d-1)/2} n^{(d-1)/2} - \ln(mn) \rightarrow -\infty,$$

or if  $d$  is even and

$$p^d m^{d/2} n^{d/2-1} - 2 \ln n \rightarrow -\infty,$$

then almost every  $G(m, n, p)$  is of diameter at least  $d + 2$ .

From the above theorem, we can derive that

when  $d$  is odd, for every  $c < 1$  and  $p(n) \leq c \cdot (\ln(mn))^{1/d} / (m^{(d-1)/(2d)} n^{(d-1)/(2d)})$ ,  $G(m, n, p)$  almost surely does not satisfy the property that  $\text{diam}(G(m, n, p)) \leq d + 1$  and for every  $C > 1$  and  $p(n) \geq C \cdot (\ln(mn))^{1/d} / (m^{(d-1)/(2d)} n^{(d-1)/(2d)})$ ,  $G(m, n, p)$  almost surely satisfies the property that  $\text{diam}(G(m, n, p)) \leq d + 1$ . Similarly, when  $d$  is even, for every  $c < 1$  and  $p(n) \leq c \cdot (2 \log n)^{1/d} / (m^{1/2} n^{(d-2)/(2d)})$ ,  $G(m, n, p)$  almost surely does not satisfy the property that  $\text{diam}(G(m, n, p)) \leq d + 1$  and for every  $C > 1$  and  $p(n) \geq C \cdot (2 \log n)^{1/d} / (m^{1/2} n^{(d-2)/(2d)})$ ,  $G(m, n, p)$  almost surely satisfies the property that  $\text{diam}(G(m, n, p)) \leq d + 1$ .

We also need the following key conclusion during our proof. Here we only state it but give its proof next section. Assume that  $c_0 \geq 1$  is a positive constant and  $k = k(n) \leq c_0 \log n$ . Let  $C_1 = 2^{10d} \cdot c_0$  and  $C_2 = 2^{10d} \cdot c_0 / \epsilon$ .

**Theorem 2.2** *If  $d$  is odd, then with probability at least  $1 - n^{-\Omega(1)}$ , the random bipartite graph  $G(m, n, C_1 p_1)$  satisfies the property:*

*every two distinct vertices of the same partite are connected by at least  $2^{10d} c_0 \log n$  internally vertex-disjoint paths of length exactly  $d + 1$ , and every two distinct vertices of different partites are connected by at least  $2^{10d} c_0 \log n$  internally vertex-disjoint paths of length exactly  $d$ .*

*If  $d$  is even, then with probability at least  $1 - n^{-\Omega(1)}$ , the random bipartite graph  $G(m, n, C_2 p_2)$  satisfies the property:*

*every two distinct vertices of the same partite are connected by at least  $2^{10d} c_0 \log n$  internally vertex-disjoint paths of length exactly  $d$ , and every two distinct vertices of different partites are connected by at least  $2^{10d} c_0 \log n$  internally vertex-disjoint paths of length exactly  $d + 1$ .*

Now we are ready to give the proof of Theorem 1.2.

**Part 1:  $d$  is odd.**

We consider the random bipartite  $G(m, n, p)$  with  $p_1 n \geq p_1 m \geq (\log n)^4$ . To establish a sharp threshold function for a graph property should have two-folds. They are corresponding to the following two lemmas.

**Lemma 2.3**  $rc_k(G(m, n, p)) \geq d + 2$  almost surely holds for every  $p \leq ((\ln 2)^{1/d})/2 \cdot p_1$ .

*Proof.* Let  $c_1 = ((\ln 2)^{1/d})/2$ . Obviously  $c_1 < 1$ . Since

$$p \leq c_1 p_1 = (1/2) \cdot \left( (\ln(mn))^{1/d} / (m^{(d-1)/(2d)} n^{(d-1)/(2d)}) \right),$$

by Theorem 2.1,  $\text{diam}(G(m, n, p)) \geq d + 2$  almost surely holds. By

$$\Pr[rc_k(G(m, n, p)) \geq d + 2] \geq \Pr[\text{diam}(G(m, n, p)) \geq d + 2],$$

we get that for every  $p \leq ((\ln 2)^{1/d})/2 \cdot p_1$ ,  $rc_k(G(m, n, p)) \geq d + 2$  almost surely holds.  $\blacksquare$

**Lemma 2.4**  $rc_k(G(m, n, p)) \leq d + 1$  almost surely holds for every  $p \geq C_1 \cdot p_1$ .

*Proof.* Let  $S = \{1, 2, \dots, d, d + 1\}$  be a set of  $d + 1$  distinct colors. Randomly color the edges of  $G(m, n, p)$  with colors from  $S$ . By Theorem 2.2, for every two distinct vertices  $u, v \in U$  (or  $u, v \in V$ ) there are at least  $2^{10d} c_0 \log n$  internally vertex-disjoint  $uv$ -paths of length exactly  $d + 1$ . Let  $P_1$  be such a  $uv$ -path. Under the random coloring, the probability that  $P_1$  is a rainbow path is

$$q_1 = (d + 1)! / (d + 1)^{d+1} \geq ((d + 1)/e)^{d+1} / (d + 1)^{d+1} \geq 8^{-d},$$

by Stirling's formula. Meanwhile, for every  $u \in U$  and  $v \in V$  there are also at least  $2^{10d} c_0 \log n$  internally vertex-disjoint  $uv$ -paths of length exactly  $d$ . Let  $P_2$  be such a  $uv$ -path. The probability that  $P_2$  is a rainbow path is

$$q_2 = \binom{d + 1}{d} d! / (d + 1)^d \geq ((d + 1)/e)^{d+1} / (d + 1)^d \geq 4^{-d}.$$

Let  $q = \min\{q_1, q_2\} \geq 8^{-d}$ . Fix  $u, v \in U$  (or  $u, v \in V$  or  $u \in U, v \in V$ ), we can estimate the upper bound of the probability that there are at most  $k - 1$  such  $uv$ -paths that are rainbow ones by

$$\begin{aligned} & \binom{2^{10d} c_0 \log n}{k - 1} (1 - q)^{2^{10d} c_0 \log n - (k - 1)} \\ & \leq \binom{2^{10d} c_0 \log n}{c_0 \log n} (1 - 8^{-d})^{(2^{10d} - 1) c_0 \log n} \\ & \leq \left( \frac{2^{10d} c_0 \log n \cdot e}{c_0 \log n} \right)^{c_0 \log n} \cdot 2^{-8^{-d} \cdot (2^{10d} - 1) c_0 \log n} \\ & = (2^{10d} e)^{c_0 \log n} \cdot 2^{-8^{-d} \cdot (2^{10d} - 1) c_0 \log n} \\ & = \left( \frac{e \cdot 2^{(10d + 8 - d)}}{2^{8^{-d} \cdot 2^{10d}}} \right)^{c_0 \log n} \\ & \leq n^{-100}, \end{aligned}$$

where we apply the inequality  $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$  <sup>†</sup>

<sup>†</sup>We find that if in [12] He and Liang use this inequality, their proof could be simplified significantly.

By the Union Bound, with probability at least

$$1 - \left[ \binom{m}{2} + \binom{n}{2} + \binom{m}{1} \binom{n}{1} \right] n^{-100} \geq 1 - n^{-90},$$

every two distinct vertices of  $G(m, n, p)$  have at least  $k$  internally vertex-disjoint rainbow paths connecting them. This implies that with probability at least  $1 - n^{-90}$ , the event  $rc_k(G(m, n, p)) \leq d + 1$  happens, which gives precisely we want.  $\blacksquare$

By the two lemmas above, it follows that

$$p_1 = (\log(mn))^{1/d} / (m^{(d-1)/(2d)} n^{(d-1)/(2d)})$$

is a sharp threshold function for the property  $rc_k(G(m, n, p)) \leq d + 1$ , where  $p_1 n \geq p_1 m \geq (\log n)^4$ .

**Part 2:  $d$  is even.**

Recall that  $p_2 = (\log n)^{1/d} / (m^{1/2} n^{(d-2)/(2d)})$ . We consider the random bipartite graph  $G(m, n, p)$ , where  $m$  and  $n$  satisfy that  $p_2 n^{1-\epsilon} \geq p_2 m^{1-\epsilon} \geq (\log n)^4$ . The following two lemmas imply that  $p_2$  is a sharp threshold function for the property  $rc_k(G(m, n, p)) \leq d + 1$ .

**Lemma 2.5**  $rc_k(G(m, n, p)) \geq d + 2$  almost surely holds for every  $p \leq p_2$ .

*Proof.* Let  $c_2 = 1 / ((2 \ln 2)^{1/d})$ . Obviously,  $c_2 < 1$ . Since

$$\begin{aligned} p \leq p_2 &= (\log n)^{1/d} / (m^{1/2} n^{(d-2)/(2d)}) \\ &= \left( 1 / (2 \ln 2)^{1/d} \right) \cdot \left( (2 \ln n)^{1/d} / (m^{1/2} n^{(d-2)/(2d)}) \right) \\ &= c_2 \cdot \left( (2 \ln n)^{1/d} / (m^{1/2} n^{(d-2)/(2d)}) \right), \end{aligned}$$

by Theorem 2.1,  $\text{diam}(G(m, n, p)) \geq d + 2$  almost surely holds. Then it follows that for every  $p \leq p_2$ ,  $rc_k(G(m, n, p)) \geq d + 2$  almost surely holds.  $\blacksquare$

Similar to the proof of Lemma 2.4, we can easily get the following result.

**Lemma 2.6**  $rc_k(G(m, n, p)) \leq d + 1$  almost surely holds for every  $p \geq C_2 \cdot p_2$ .

By the two lemmas above, we can conclude that

$$p_2 = (\log n)^{1/d} / (m^{1/2} n^{(d-2)/(2d)})$$

is a sharp threshold function for the property  $rc_k(G(m, n, p)) \leq d + 1$ , where  $m$  and  $n$  satisfy that  $p_2 n^{1-\epsilon} p_2 m^{1-\epsilon} \geq (\log n)^4$ .

Combining the two parts discussed above, we complete the proof of Theorem 1.2.

### 3 Proof of Theorem 2.2

In this section, we give the proof of Theorem 2.2. We also divide our proof into two parts according to the parity of  $d$ . We first give a definition. An  $(s, t)$ -ary tree with a designated root is a tree such that every non-leaf vertex of even level has exactly  $s$  children and every non-leaf vertex of odd level has exactly  $t$  children, where we assume that the root is in zero-level. Obviously, an  $(s, t)$ -ary tree and a  $(t, s)$ -ary tree of the same depth are usually different trees.

**Part 1:  $d$  is odd.**

Let  $p = C_1 p_1 = C_1 (\log(mn))^{1/d} / (m^{(d-1)/(2d)} n^{(d-1)/(2d)})$ . For every  $u \in U$  and  $S \subseteq V$  (or  $u \in V$  and  $S \subseteq U$ ), let  $X$  be the random variable counting the number of neighbors of  $u$  inside  $S$ .

**Lemma 3.1** *For every fixed  $u, S$  such that  $u \in U$ ,  $S \subseteq V$  and  $|S| \geq n/2$  for sufficiently large  $n$ ,*

$$\Pr[X \geq pn/10] \geq 1 - 2^{-\Omega(n^{1/d})}.$$

*Proof.* Denote by  $S'$  any subset of  $S$  with cardinality  $n/2$ . Let  $X_1$  be the random variable counting the number of neighbors of  $u$  inside  $S'$ . Obviously,  $X_1$  can be expressed as the sum of  $n/2$  independent random variables, each of which taking 1 with probability  $p$  and 0 with probability  $1 - p$ . Thus  $\mathbf{E}[X_1] = pn/2$ . By the Chernoff-Hoeffding Bound, we have

$$\Pr[X_1 < (1 - 4/5)pn/2] \leq \exp(-(1/2)(4/5)^2(pn/2)) = 2^{-\Omega(n^{1/d})},$$

By  $X \geq X_1$ , the event  $X \geq pn/10$  happens with probability at least  $1 - 2^{-\Omega(n^{1/d})}$ , which is precisely what we want.  $\blacksquare$

**Lemma 3.2** *For every fixed  $u, S$  such that  $u \in V$ ,  $S \subseteq U$  and  $|S| \geq m/2$  for sufficiently large  $n$ ,*

$$\Pr[X \geq pm/10] \geq 1 - n^{-\Omega(\log^3 n)}.$$

The proof is similar to that of Lemma 3.1. From Lemmas 3.1 and 3.2, it follows that  $\Pr[X \geq pn/\log n] \geq 1 - 2^{-\Omega(n^{1/d})}$  and  $\Pr[X \geq pm/\log m] \geq 1 - n^{-\Omega(\log^3 n)}$ .

**Lemma 3.3** *With probability at least  $1 - n^{-\Omega(1)}$ , every two distinct vertices of  $U$  are connected by at least  $2^{10d} c_0 \log n$  internally vertex-disjoint paths of length exactly  $d + 1$ .*

*Proof.* Fix  $u, v \in U, u \neq v$ . Consider the following process to generate a  $(pn/\log n, pm/\log m)$ -ary tree of depth  $d$  rooted at  $u$ :

Step 1. Let  $T_0 = \{u\}$ ,  $i \leftarrow 1$ , and  $T_i \leftarrow \emptyset$ .

Step 2. If  $i$  is odd, for every vertex  $w \in T_{i-1}$ , choose  $pn/\log n$  distinct neighbors of  $w$  from the set  $V \setminus (\cup_{j=0}^i T_j)$ , and add them to  $T_i$ . (Note that  $T_{i-1} \subseteq U$ ,  $T_i$  is updated every

time after the processing of a vertex  $w$ , and in fact only when  $j$  is odd  $T_j \subseteq V$ .)

If  $i$  is even, for every vertex  $w \in T_{i-1}$ , choose  $pm/\log m$  distinct neighbors of  $w$  from the set  $U \setminus (\{v\} \cup \cup_{j=0}^i T_j)$ , and add them to  $T_i$ .

Step 3. Let  $i \leftarrow i + 1$ . If  $i \leq d$  then go to Step 2, otherwise stop.

Of course, the process may fail during Step 2, since with nonzero probability no neighbor of  $w$  can be chosen as a candidate. However, we will show that with high probability the tree can be successfully constructed. Observe that when  $j$  is even,  $T_j \subseteq U$ , and when  $j$  is odd,  $T_j \subseteq V$ . Thus,  $T_{d-1} \subseteq U$ ,  $T_d \subseteq V$  and  $|T_{d-1}| = (pm/\log m)^{(d-1)/2}(pn/\log n)^{(d-1)/2}$ ,  $|T_d| = (pm/\log m)^{(d-1)/2}(pn/\log n)^{(d+1)/2}$ . At any time during the process,

$$\begin{aligned} |\{v\} \cup \bigcup_{j=0, j \text{ is even}}^i T_j| &\leq 1 + \sum_{j=0}^{d-1} |T_j| \leq d \cdot |T_{d-1}| \\ &= d \cdot (pm/\log m)^{(d-1)/2}(pn/\log n)^{(d-1)/2} \\ &= \frac{d \cdot C_1^{d-1} (\log(mn))^{(d-1)/d}}{(\log m)^{(d-1)/2} (\log n)^{(d-1)/2}} \cdot m^{(d-1)/(2d)} n^{(d-1)/(2d)} \\ &\leq m/2 \end{aligned}$$

and

$$\begin{aligned} |\bigcup_{j=0, j \text{ is odd}}^i T_j| &\leq \sum_{j=0}^d |T_j| \leq (d+1)|T_d| \\ &= (d+1) \cdot (pm/\log m)^{(d-1)/2}(pn/\log n)^{(d+1)/2} \\ &= \frac{(d+1) \cdot C_1^d \log(mn)}{(\log m)^{(d-1)/2} (\log n)^{(d+1)/2}} \cdot n \\ &\leq n/2 \end{aligned}$$

for all sufficiently large  $n$ .

By Lemmas 3.1 and 3.2, every execution of Step 2 fails with probability at most  $n^{-\Omega(\log^3 n)}$ . Since Step 2 can be executed for at most  $(d+1)(pm/\log m)^{(d-1)/2}(pn/\log n)^{(d+1)/2}$ , we obtain that, with probability at least

$$1 - (d+1) \cdot (pm/\log m)^{(d-1)/2}(pn/\log n)^{(d+1)/2} \cdot n^{-\Omega(\log^3 n)} = 1 - n^{-\Omega(\log^3 n)},$$

the process can be successfully terminated.

Now we assume that  $T$  has been successfully constructed. The number of leaves in  $T$  is exactly  $|T_d|$ . Let  $Y$  be the random variable counting the number of neighbors of  $v$  inside  $T_d$ . It is obvious that

$$\mathbf{E}[Y] = p \cdot |T_d| = \frac{C_1^{d+1} (\log(mn))^{(d+1)/d}}{(\log m)^{(d-1)/2} (\log n)^{(d+1)/2}} \cdot \frac{n^{(d+1)/(2d)}}{m^{(d-1)/(2d)}} \geq 10 \cdot n^{1/(2d)}.$$

By the Chernoff-Hoeffding Bound, we get

$$\Pr[Y < n^{1/(2d)}] \leq \exp(-(1/2)(9/10)^2 \cdot 10n^{1/(2d)}) \leq 2^{-n^{1/(4d)}}$$



For every  $w \in T_1$ , define as the vice-tree  $T_w$  of  $T$  the subtree of  $T$  of depth  $d - 1$  rooted at  $w$ . Notice that every vice-tree contains  $(pm/\log m)^{(d-1)/2}(pn/\log n)^{(d-1)/2}$  leaves. For each vice-tree  $T_w$ , let  $Z_w$  be the random variable counting the number of neighbors of  $v$  inside the set of leaves of  $T_w$ . Then we have

$$\begin{aligned} \Pr[Z_w \geq n^{1/(10d)}] &\leq \left( \frac{(pm/\log m)^{(d-1)/2}(pn/\log n)^{(d-1)/2}}{n^{1/(10d)}} \right) \cdot p^{n^{1/(10d)}} \\ &\leq \left( \frac{(pm/\log m)^{(d-1)/2}(pn/\log n)^{(d-1)/2} \cdot e}{n^{1/(10d)}} \right)^{n^{1/(10d)}} \cdot p^{n^{1/(10d)}} \\ &= \left( \frac{C_1^d \log(mn) \cdot e}{(\log m)^{(d-1)/2}(\log n)^{(d-1)/2} \cdot n^{1/(10d)}} \right)^{n^{1/(10d)}} \\ &\leq n^{-100}, \end{aligned}$$

where we apply the inequality  $\binom{n}{k} \leq (\frac{ne}{k})^k$ . By applying the Union Bound, we get

$$\Pr\left[ \bigvee_{w \in T_1} (Z_w \geq n^{1/(10d)}) \right] \leq (pn/\log n) \cdot n^{-100} \leq n^{-90}.$$

Combined with previous estimations, we derive that with probability at least

$$1 - n^{-\Omega(\log^3 n)} - 2^{-n^{1/(4d)}} - n^{-90} \geq 1 - n^{-80},$$

the following three events simultaneously happen:

1. the tree  $T$  is successfully constructed,
2.  $v$  has at least  $n^{1/(2d)}$  neighbors inside the set of leaves of  $T$ ,
3. every vice-tree  $T_w$  contains at most  $n^{1/(10d)}$  leaves that are neighbors of  $v$ .

It is clear that each neighbor  $v'$  of  $v$  inside  $T_d$  induces a  $uv$ -path of length  $d + 1$ . If two neighbors  $v'$  and  $v''$  of  $v$  belong to distinct vice-trees, then the corresponding two  $uv$ -paths are internally vertex-disjoint. When all these three events happen, we can choose  $n^{1/(2d)}/n^{1/(10d)} = n^{2/(5d)} \geq 2^{10d} c_0 \log n$  neighbors of  $v$  inside  $T_d$ , every two of which are from different vice-trees. Thus we can immediately obtain at least  $2^{10d} c_0 \log n$  internally vertex-disjoint  $uv$ -paths.

By using the Union Bound again, it then follows that, with probability at least

$$1 - \binom{m}{2} \cdot n^{-80} = 1 - n^{-\Omega(1)},$$

every two distinct vertices of  $U$  are connected by at least  $2^{10d} c_0 \log n$  internally vertex-disjoint paths of length exactly  $d + 1$ . This completes the proof of Lemma 3.3.  $\blacksquare$

**Lemma 3.4** *With probability at least  $1 - n^{-\Omega(1)}$ , every two distinct vertices of  $V$  are connected by at least  $2^{10d} c_0 \log n$  internally vertex-disjoint paths of length exactly  $d + 1$ .*

*Proof.* The proof is similar to that of Lemma 3.3. Here we only point out the differences. Fix  $u, v \in V, u \neq v$ . We first construct a  $(pm/((\log m)^{2/(d-1)}), pn/((\log n)^{2/(d-1)}))$ -ary tree of depth  $d$  rooted at  $u$ . We can easily determinate that with probability at least  $1 - n^{-\Omega(\log^3 n)}$  the tree  $T$  can be successfully constructed.  $T_d$  is just the set of leaves of  $T$  and

$$T_d = \left( pm/((\log m)^{2/(d-1)}) \right)^{(d+1)/2} \cdot \left( pn/((\log n)^{2/(d-1)}) \right)^{(d-1)/2}.$$

Let  $Y$  be the random variable counting the number of neighbors  $v$  inside  $T_d$ . It is obvious that

$$\mathbf{E}[Y] = p \cdot |T_d| = \frac{C_1^{d+1} (\log(mn))^{(d+1)/d}}{(\log m)^{(d+1)/(d-1)} \log n} \cdot \frac{m^{(d+1)/(2d)}}{n^{(d-1)/(2d)}} \geq C_1^2 (\log n)^2.$$

As before, we have

$$\begin{aligned} \mathbf{Pr}[Y < C_1 (\log n)^2] &\leq \mathbf{Pr}[Y < (1 - (C_1 - 1)/C_1) \cdot C_1^2 (\log n)^2] \\ &\leq \exp(-(1/2)((C_1 - 1)/C_1)^2 \cdot C_1^2 (\log n)^2) \\ &\leq n^{-O(\log n)} \end{aligned}$$

Notice that every vice-tree contains  $(pm/((\log m)^{2/(d-1)}))^{(d-1)/2} \cdot (pn/((\log n)^{2/(d-1)}))^{(d-1)/2} = (p^{d-1} m^{(d-1)/2} n^{(d-1)/2}) / (\log m \cdot \log n)$  leaves. For each vice-tree  $T_w$ , let  $Z_w$  be the random variable counting the number of neighbors  $v$  inside the set of leaves of  $T_w$ . Then we have

$$\begin{aligned} \mathbf{Pr}[Z_w \geq \log n] &\leq \left( \frac{(p^{d-1} m^{(d-1)/2} n^{(d-1)/2}) / (\log m \cdot \log n)}{\log n} \right) \cdot p^{\log n} \\ &\leq \left( \frac{p^d m^{(d-1)/2} n^{(d-1)/2} e}{\log m \cdot \log^2 n} \right)^{\log n} \\ &= \left( \frac{C_1^d \log(mn) \cdot e}{\log m \cdot \log^2 n} \right)^{\log n} \\ &\leq n^{-O(\log \log n)}, \end{aligned}$$

and

$$\mathbf{Pr}\left[ \bigvee_{w \in T_1} (Z_w \geq \log n) \right] \leq (pm/(\log m)^{2/(d-1)}) \cdot n^{-O(\log \log n)} = n^{-O(\log \log n)}.$$

Since  $C_1 \log^2 n / \log n = 2^{10d} c_0 \log n$ , combined with the estimations above, we derive that with probability at least

$$1 - n^{-\Omega(\log^3 n)} - n^{-O(\log n)} - n^{-O(\log \log n)} = 1 - n^{-O(\log \log n)},$$

there are at least  $2^{10d} c_0 \log n$  internally vertex-disjoint  $uv$ -paths.

Therefore, we can easily obtain that with probability at least

$$1 - \binom{n}{2} \cdot n^{-O(\log \log n)} = 1 - n^{-O(\log \log n)},$$

every two distinct vertices of  $V$  have at least  $2^{10d} c_0 \log n$  internally vertex-disjoint paths of length  $d + 1$  connecting them.  $\blacksquare$

**Lemma 3.5** *With probability at least  $1 - n^{-\Omega(1)}$ , every two distinct vertices of different partites are connected by at least  $2^{10d} c_0 \log n$  internally vertex-disjoint paths of length exactly  $d$ .*

*Proof.* Similarly, fix  $u \in U, v \in V$ . We first construct a  $(pn/10, pm/10)$ -ary tree of depth  $d - 1$  rooted at  $u$ . It can also be estimated that with probability at least  $1 - n^{-\Omega(\log^3 n)}$  the tree  $T$  can be successfully constructed.

Let  $Y$  be the random variable counting the number of neighbors of  $v$  inside  $T_{d-1}$  which is just the set of leaves of  $T$ . Then

$$\begin{aligned} |T_{d-1}| &= (pm/10)^{(d-1)/2} (pn/10)^{(d-1)/2}, \\ \mathbf{E}[Y] &= p \cdot |T_{d-1}| = 10 \cdot (C_1/10)^d \cdot \log(mn), \\ \Pr[Y < (C_1/10)^d \log(mn)] &\leq n^{-10}. \end{aligned}$$

Notice that every vice-tree contains  $(pm/10)^{(d-1)/2} (pn/10)^{(d-3)/2}$  leaves. For each vice-tree  $T_w$ , let  $Z_w$  be the random variable counting the number of neighbors  $v$  inside the set of leaves of  $T_w$ . Then we have

$$\Pr[Z_w \geq 10d] \leq \binom{(pm/10)^{(d-1)/2} \cdot (pn/10)^{(d-3)/2}}{10d} \cdot p^{10d} \leq O(n^{-5}),$$

$$\Pr\left[\bigvee_{w \in T_1} (Z_w \geq 10d)\right] \leq (pn/10) \cdot O(n^{-5}) \leq O(n^{-4}).$$

Since  $((C_1/10)^d / 10d) \log(mn) \geq 2^{10d} c_0 \log n$ , combined with the estimations above, we derive that with probability at least

$$1 - n^{-\Omega(\log^3 n)} - n^{-10} - O(n^{-4}) \geq 1 - O(n^{-3}),$$

there are at least  $2^{10d} c_0 \log n$  internally vertex-disjoint  $uv$ -paths.

It follows that with probability at least  $1 - mn \cdot O(n^{-3}) \geq 1 - n^{-1}$  every two distinct vertices of different partites have at least  $2^{10d} c_0 \log n$  internally vertex-disjoint paths of length  $d$  connecting them.  $\blacksquare$

Now we have seen that Theorem 2.2 is true for the case that  $d$  is odd.

**Part 2:  $d$  is even.**

Let  $p = C_2 p_2 = C_2 (\log n)^{1/d} / (m^{1/2} n^{(d-2)/(2d)})$ . For every  $u \in U$  and  $S \subseteq V$  (or  $u \in V$  and  $S \subseteq U$ ), let  $X$  be the random variable counting the number of neighbors of  $u$  inside  $S$ . We have the following results similar to Lemmas 3.1 and 3.2.

**Lemma 3.6** *For every fixed  $u, S$  such that  $u \in U, S \subseteq V$  and  $|S| \geq n/2$  for sufficiently large  $n$ ,*

$$\Pr[X \geq pn/10] \geq 1 - 2^{-\Omega(n^{1/d})}.$$

**Lemma 3.7** For every fixed  $u, S$  such that  $u \in V$ ,  $S \subseteq U$  and  $|S| \geq m/2$  for sufficiently large  $n$ ,

$$\Pr[X \geq pm/10] \geq 1 - n^{-\Omega(\log^3 n)}.$$

Lemmas 3.6 and 3.7 also imply that  $\Pr[X \geq pn/\log n] \geq 1 - 2^{-\Omega(n^{1/d})}$  and  $\Pr[X \geq pm/\log m] \geq 1 - n^{-\Omega(\log^3 n)}$ . The proofs of the following three lemmas are similar to those of Lemmas 3.3, 3.4 and 3.5, but the estimations are different. So we only sketch the proofs of them and list the results of their estimations.

**Lemma 3.8** With probability at least  $1 - n^{-\Omega(1)}$ , every two distinct vertices of  $U$  are connected by at least  $2^{10d}c_0 \log n$  internally vertex-disjoint paths of length exactly  $d$ .

*Proof.* Fix  $u, v \in U, u \neq v$ . We construct a  $(pn/10, pm/10)$ -ary tree of depth  $d - 1$  rooted at  $u$ . Then we can similarly give the estimations. We obtain that with probability at least

$$1 - n^{-\Omega(\log^3 n)} - n^{-10} - n^{-5} \geq 1 - n^{-4},$$

the following three events simultaneously happen:

1. the tree  $T$  is successfully constructed,
2.  $v$  has at least  $((C_2/10)^d \log n) \cdot (n/m)$  neighbors inside the set of leaves of  $T$ ,
3. every vice-tree  $T_w$  contains at most  $10d \cdot (n/m)$  leaves that are neighbors of  $v$ .

Since

$$\frac{((C_2/10)^d \log n) \cdot (n/m)}{10d \cdot (n/m)} \geq 2^{10d}c_0 \log n,$$

and

$$1 - \binom{n}{2} \cdot n^{-4} = 1 - n^{-\Omega(1)},$$

every two distinct vertices of  $U$  are connected by at least  $2^{10d}c_0 \log n$  internally vertex-disjoint paths of length exactly  $d$ . ■

**Lemma 3.9** With probability at least  $1 - n^{-\Omega(1)}$ , every two distinct vertices of  $V$  are connected by at least  $2^{10d}c_0 \log n$  internally vertex-disjoint paths of length exactly  $d$ .

*Proof.* Fix  $u, v \in V, u \neq v$ . In this case, we construct a  $(pm/10, pn/10)$ -ary tree of depth  $d - 1$  rooted at  $u$ . We can determine that with probability at least

$$1 - n^{-\Omega(\log^3 n)} - n^{-10} - O(n^{-8}) \geq 1 - O(n^{-7}),$$

the following three events simultaneously happen:

1. the tree  $T$  is successfully constructed,

2.  $v$  has at least  $(C_2/10)^d \log n$  neighbors inside the set of leaves of  $T$ ,
3. every vice-tree  $T_w$  contains at most  $(10d)/\epsilon$  leaves that are neighbors of  $v$ .

Since

$$\frac{(C_2/10)^d \log n}{(10d)/\epsilon} \geq 2^{10d} c_0 \log n,$$

and

$$1 - \binom{n}{2} \cdot O(n^{-7}) \geq 1 - n^{-\Omega(1)},$$

every two distinct vertices of  $V$  are connected by at least  $2^{10d} c_0 \log n$  internally vertex-disjoint paths of length exactly  $d$ . ■

**Lemma 3.10** *With probability at least  $1 - n^{-\Omega(1)}$ , every two distinct vertices of different partite sets are connected by at least  $2^{10d} c_0 \log n$  internally vertex-disjoint paths of length exactly  $d + 1$ .*

*Proof.* Fix  $u \in U, v \in V$ . In this case, the estimations are more complicated than the previous cases. We construct a  $(pm/\log m, pn/\log n)$ -ary tree of depth  $d$  rooted at  $v$ . We can determine that with probability at least

$$1 - n^{-\Omega(\log^3 n)} - n^{-10} - O(n^{-8}) \geq 1 - O(n^{-6}),$$

the following three events simultaneously happen:

1. the tree  $T$  is successfully constructed,
2.  $u$  has at least

$$L_1 = \frac{C_2^d (\log n)^{(d+1)/d}}{(\log m)^{d/2} (\log n)^{d/2}} \cdot \frac{n^{1/2+1/d}}{m^{1/2}}$$

neighbors inside the set of leaves of  $T$ ,

3. every vice-tree  $T_w$  contains at most

$$L_2 = \frac{C_2^d \log n}{(\log m)^{d/2-1} (\log n)^{d/2}} \cdot \frac{n}{m^{1-\epsilon/2}}$$

leaves that are neighbors of  $v$ .

Since

$$L_1/L_2 \geq 2^{10d} c_0 \log n,$$

and

$$1 - \binom{n}{2} \cdot O(n^{-6}) \geq 1 - n^{-\Omega(1)},$$

every two distinct vertices of different partite sets are connected by at least  $2^{10d} c_0 \log n$  internally vertex-disjoint paths of length exactly  $d$ . ■

Now we can see that Theorem 2.2 is also true for the case that  $d$  is even.

Combining the two parts discussed above, the proof of Theorem 2.2 is thus completed.

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