On the difference of two generalized connectivities of a graph

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Abstract

The concept of k-connectivity $\kappa'_k(G)$ of a graph G, introduced by Chartrand in 1984, is a generalization of the cut-version of the classical connectivity. Another generalized connectivity of a graph G, named the generalized k-connectivity $\kappa_k(G)$, mentioned by Hager in 1985, is a natural generalization of the path-version of the classical connectivity.

In this paper, we get the lower and upper bounds for the difference of these two parameters by showing that for a connected graph G of order n, if $\kappa'_k(G) \neq n-k+1$ where $k \geq 3$, then $0 \leq \kappa'_k(G) - \kappa_k(G) \leq n-k-1$; otherwise, $-\lfloor \frac{k}{2} \rfloor + 1 \leq \kappa'_k(G) - \kappa_k(G) \leq n-k$. Moreover, all of these bounds are sharp. Some specific study is focused for the case $k = 3$. As results, we characterize the graphs with $\kappa'_3(G) = \kappa_3(G) = t$ for $t \in \{1, n-3, n-2\}$, and give a necessary condition for $\kappa'_3(G) = \kappa_3(G)$ by showing that for a connected graph G of order n and size m, if $\kappa'_3(G) = \kappa_3(G) = t$ where $1 \leq t \leq n-3$, then $m \leq \binom{n-2}{2} + 2t$. Moreover, the unique extremal graph is given for the equality to hold.

Keywords: k-connectivity, generalized k-connectivity

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1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to book [1] for graph theoretical notation and terminology not described here. For a graph G, let $V(G)$, $E(G)$, $\overline{G}$ be the set of vertices, the set of edges, the complement of G, respectively. For $X \subseteq V(G)$, we denote by $G \setminus X$ the subgraph obtained by deleting from G the vertices of X together with the edges incident with them. We use P_n and C_m to denote a path of order n and a cycle of order m, respectively.

Connectivity is one of the most basic concepts in graph theory, both in combinatorial sense and in algorithmic sense. The classical connectivity has two equivalent definitions. The connectivity of G, written $\kappa(G)$, is the minimum size of a vertex set $X \subseteq V(G)$ such that $G \setminus X$ is disconnected or has only one vertex. This definition is called the *cut-version* definition of the connectivity. A well-known theorem of Menger provides an equivalent definition, which can be called the *path-version* definition of the connectivity. For any two distinct vertices x and y in G, the local connectivity $\kappa_G(x,y)$ is the maximum number of internally disjoint paths connecting x and y. Then $\kappa(G) = \min\{\kappa_G(x,y) | x, y \in V(G), x \neq y\}$ is defined to be the connectivity of G.

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Although there are many elegant and powerful results on connectivity in graph theory, the basic notation of classical connectivity may not be general enough to capture some computational settings and so people tried to generalize this concept. For the cut-version definition of the connectivity, we find that the above minimum vertex set does not regard to the number of components of $G \setminus X$. Two graphs with the same connectivity may have different degrees of vulnerability in the sense that the deletion of a vertex cut-set of minimum cardinality from one graph may produce a graph with considerably more components than in the case of the other graph. For example, the star $K_{1,n-1}$ and the path P_n ($n \geq 3$) are both trees of order n and therefore have connectivity 1, but the deletion of a cut-vertex from $K_{1,n-1}$ produces a graph with $n-1$ components while the deletion of a cut-vertex from P_n produces only two components. Chartrand et al. [2] generalized the cut-version definition of the connectivity as follows: For an integer k ($k \geq 2$) and a graph G of order n ($n \geq k$), the *k-connectivity* $\kappa'_k(G)$ is the smallest number of vertices whose removal from G of order n ($n \geq k$) produces a graph with at least k components or a graph with fewer than k vertices. By the definition, we clearly have $\kappa'_2(G) = \kappa(G)$. Thus, the concept of the k-connectivity could be seen as a generalization of the classical connectivity. For more details about this topic, we refer to [2, 4, 15, 16].

Another generalized connectivity of a graph G, mentioned by Hager in 1985 [6], is a natural generalization of the path-version definition of the connectivity. For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an *S-Steiner tree* or a *Steiner tree connecting S* (or simply, an *S-tree*) is a such subgraph T of G that is a tree with $S \subseteq V(T)$. Two S-trees T_1 and T_2 are said to be *internally disjoint* if $E(T_1) \cap E(T_2) = \emptyset$ and $V(T_1) \cap V(T_2) = S$. The *generalized local connectivity* $\kappa_G(S)$ is the maximum number of internally disjoint S-trees in G. For an integer k with $2 \leq k \leq n$, the *generalized k-connectivity* is defined as $\kappa_k(G) = \min\{\kappa_G(S) | S \subseteq V(G), |S| = k\}$. Thus, $\kappa_k(G)$ is the minimum value of $\kappa_G(S)$ when S runs over all the k-subsets of $V(G)$. By the definition, we clearly have $\kappa_2(G) = \kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of G. By convention, for a connected graph G with less than k vertices, we set $\kappa_k(G) = 1$, and $\kappa_k(G) = 0$ when G is disconnected. There are many results on the generalized k-connectivity, see [3, 5, 6, 7, 8, 9, 11, 12, 13, 14, 17, 18, 19, 20, 21]. The reader is also referred to a recent survey [10] on the state-of-the-art of research on the generalized k-connectivity and their applications.

Note that the generalized k-connectivity and the k-connectivity of a graph are indeed different. For example, consider the cycle C_n with $n \geq 6$. We clearly have $\kappa'_3(G) = 3$ and $\kappa_3(G) = 1$. We want to find the difference between these two parameters and the following problem is very interesting.

Problem 1.1. *Give nice bounds for $\kappa'_k(G) - \kappa_k(G)$ for $3 \leq k \leq n$.*

In this paper, we answer Problem 1.1 by giving sharp lower and upper bounds for $\kappa'_k(G) - \kappa_k(G)$ with $3 \leq k \leq n$ (Theorem 3.5).

Recall that $\kappa'_k(G)$ and $\kappa_k(G)$ are different. However, they may be equal in some cases, for example, $\kappa'_3(K_4) = \kappa_3(K_4) = 2$. So we want to know when these two parameters are equal. Can we characterize those graphs with $\kappa'_k(G) = \kappa_k(G)$? Or more simply, can we give a sufficient condition or a necessary condition for this equality to hold ? This is the following interesting problem:

Problem 1.2. *Characterize the graphs with $\kappa'_k(G) = \kappa_k(G)$ for $3 \leq k \leq n$.*

For this problem, it is quite difficult to characterize the graphs with $\kappa'_k(G) = \kappa_k(G)$ even for the case $k = 3$. In this paper, we completely characterize the graphs with $\kappa'_3(G) = \kappa_3(G) = t$ for $t \in \{1, n-3, n-2\}$ (Theorem 4.1), and also give a necessary condition for the equality $\kappa'_3(G) = \kappa_3(G)$ to hold (Theorem 4.5).

As usual, the *union* of two graphs G and H is the graph, denoted by $G \cup H$, with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Let mH be the disjoint union of m copies of a graph H.

2 Preliminaries

For a general graph G, the following results concern the bounds for $\kappa_k(G)$.

Proposition 2.1. *[13] Let k, n be two integers with $2 \leq k \leq n$. For a connected graph G of order n, $1 \leq \kappa_k(G) \leq n - \lceil \frac{k}{2} \rceil$. Moreover, the upper and lower bounds are sharp.*

From [13], we know that $\kappa_k(K_n) = n - \lceil \frac{k}{2} \rceil$.

By the definition of $\kappa'_k(G)$, the following two observations clearly hold.

Observation 2.2. *If H is a spanning subgraph of G, then $\kappa'_k(H) \leq \kappa'_k(G)$.*

Observation 2.3. *For a connected graph G of order n, we have $\kappa'_k(G) \leq n - \alpha(G)$, where $\alpha(G)$ is the independence number of G.*

We use $K_n \setminus e$ to denote the subgraph of K_n by deleting an edge e from K_n. In [8], graphs with order n and $\kappa_3(G) = n - 2$ were characterized.

Theorem 2.4. *[8] For a connected graph G of order n, $\kappa_3(G) = n - 2$ if and only if $G = K_n$ or $G = K_n \setminus e$.*

Graphs with order n and $\kappa_3(G) = n - 3$ were also characterized in [8].

Theorem 2.5. *[8] For a connected graph G of order n with $n \geq 3$, $\kappa_3(G) = n - 3$ if and only if $\overline{G} = P_4 \cup (n-4)K_1$ or $\overline{G} = P_3 \cup iP_2 \cup (n-2i-3)K_1 (i = 0, 1)$ or $\overline{G} = C_3 \cup iP_2 \cup (n-2i-3)K_1 (i = 0, 1)$ or $\overline{G} = rP_2 \cup (n-2r)K_1 (2 \leq r \leq \lfloor \frac{n}{2} \rfloor)$.*

3 Bounds for $\kappa'_k(G) - \kappa_k(G)$

For two integers s, n with $1 \leq s \leq n-1$, we define a class of graph $G(x, s)$ as follows: For each graph $G \in G(x, s)$, there exists a cut vertex x such that $G \setminus \{x\}$ contains at least s components. By definition, for any $s_1 < s_2$, we have that $G(x, s_2)$ is a subclass of $G(x, s_1)$.

Proposition 3.1. *Let k, n be two integers with $2 \leq k \leq n$. For a connected graph G of order n, $1 \leq \kappa'_k(G) \leq n - k + 1$. Moreover, $\kappa'_k(G) = 1$ if and only if $k = n$ or $G \cong G(x, k)$, and $\kappa'_k(G) = n - k + 1$ if and only if $\alpha(G) \leq k - 1$.*

Proof. By the definitions of $\kappa'_k(G)$ and $G(x, s)$, we clearly have $1 \leq \kappa'_k(G) \leq n - k + 1$, and it is not hard to show that $\kappa'_k(G) = 1$ if and only if $k = n$ or $G \cong G(x, k)$.

Now we assume $\kappa'_k(G) = n - k + 1$. Suppose $\alpha(G) \geq k$. Then we have $\kappa'_k(G) \leq n - \alpha(G) \leq n - k$ by Observation 2.3, which produces a contradiction, so we have $\alpha(G) \leq k - 1$.

If $\alpha(G) \leq k-1$, suppose $\kappa'_k(G) \leq n-k$, then there exists a set $X \subseteq V(G)$ with $|X| \leq n-k$ such that $G \setminus X$ contains at least k components. We choose one vertex in each such component, and we know that any two such vertices are nonadjacent. Then, $\alpha(G) \geq k$, which produces a contradiction. Thus, $\kappa'_k(G) = n-k+1$. □

For example, for $2 \leq k < n$ we know $K_{1,n-1} \in G(x,k)$, and we clearly have $\kappa'_k(K_{1,n-1}) = 1$ and $\kappa'_k(K_n) = n-k+1$.

The following result concerns the bounds of the difference for the two parameters $\kappa'_3(G)$ and $\kappa_3(G)$.

Theorem 3.2. *For a connected graph G of order n, we have $0 \leq \kappa'_3(G) - \kappa_3(G) \leq n-3$. Moreover, the bounds are sharp.*

Proof. We consider the lower bound first. If $n \leq \kappa'_3(G)+2$, then $n-2 \leq \kappa'_3(G) \leq n$. Thus, $\kappa_3(G) \leq \kappa_3(K_n) \leq n-2 \leq \kappa'_3(G)$. In the following, we assume $n \geq \kappa'_3(G)+3$. By definition, there exists a set $X \subseteq V(G)$ such that $G \setminus X$ contains ℓ components, say $G_1, G_2, \ldots, G_\ell$, where $\ell \geq 3$. We choose $S = \{u_i | 1 \leq i \leq 3\}$ where $u_i \in V(G_i)$. Clearly, each S-tree must contain at least one vertex in X. Then, $\kappa'_3(G) = |X| \geq \kappa_G(S) \geq \kappa_3(G)$. Thus, $\kappa'_3(G) - \kappa_3(G) \geq 0$. For the sharpness of this bound, letting $G = K_n$, we know $\kappa'_3(G) = \kappa_3(G) = n-2$.

Now we consider the upper bound. We know that $\kappa_3(G) \geq 1$ and $\kappa'_3(G) \leq n-2$ by Propositions 2.1 and 3.1. Then, $\kappa'_3(G) - \kappa_3(G) \leq n-3$. In fact, it is easy to show that $\kappa'_3(G) - \kappa_3(G) = n-3$ if and only if $\kappa'_3(G) = n-2$ and $\kappa_3(G) = 1$. Thus, we only need to find those graphs G with $\kappa'_3(G) = n-2$ and $\kappa_3(G) = 1$. For example, let G be a graph with $V(G) = \{u_i | 1 \leq i \leq n\}$ such that $u_1u_n \in E(G)$ and $V' = \{u_i | 1 \leq i \leq n-1\}$ is a clique. Then $\kappa_3(G) = 1$ and $\kappa'_3(G) = n - \alpha(G) = n-2$ by Proposition 3.1. □

For a general $k \geq 3$, if $\kappa'_k(G) \neq n-k+1$, then we can get sharp lower and upper bounds of $\kappa'_k(G) - \kappa_k(G)$.

Lemma 3.3. *For a connected graph G of order n, if $\kappa'_k(G) \neq n-k+1$ with $k \geq 3$, then $0 \leq \kappa'_k(G) - \kappa_k(G) \leq n-k-1$. Moreover, the bounds are sharp.*

Proof. Since $\kappa'_k(G) \neq n-k+1$, we have $\kappa'_k(G) \leq n-k$ by Proposition 3.1, and then $n \geq \kappa'_k(G)+k$. By definition, there exists a set $X \subseteq V(G)$ with $|X| = \kappa'_k(G)$ such that $G \setminus X$ contains ℓ components, say $G_1, G_2, \ldots, G_\ell$, where $\ell \geq k$. We choose $S = \{u_i | 1 \leq i \leq k\}$ where $u_i \in V(G_i)$. With a similar argument to that of Theorem 3.2, we can deduce that $\kappa'_k(G) - \kappa_k(G) \geq 0$. For the sharpness of this bound, we consider the graph $G \in G(x,k)$. By the definition of $G(x,k)$, we clearly have $\kappa'_k(G) = \kappa_k(G) = 1$.

Since $\kappa_k(G) \geq 1$ and $\kappa'_k(G) \leq n-k$, we have $\kappa'_k(G) - \kappa_k(G) \leq n-k-1$. For the sharpness of this bound, we consider the following example: Let G be a connected graph with vertex set $V(G) = A \cup B$ such that $A = \{u_i | 1 \leq i \leq n-k\}$ is a clique, $B = \{v_j | 1 \leq j \leq k\}$ is an independent set, and $u_1v_1, u_iv_j \in E(G)$ where $1 \leq i \leq n-k, 2 \leq j \leq k$. Since G is connected and $\delta(G) = 1$, $\kappa_k(G) = 1$. Clearly, $\alpha(G) = k$ and so $\kappa'_k(G) \leq n-k$ by Observation 2.3. It is also not hard to show that for any set $X \subseteq V(G)$ with $|X| < n-k$, the subgraph $G \setminus X$ contains at most two components, and so we have $\kappa'_k(G) \geq n-k$. Thus, $\kappa'_k(G) = n-k$, and then $\kappa'_k(G) - \kappa_k(G) = n-k-1$. □

By this lemma, we know that $\kappa_k(G) \leq \kappa'_k(G)$ if $\kappa'_k(G) < n-k+1$. However, for the case that $\kappa'_k(G) = n-k+1$, the situation is complicated. $\kappa_k(G)$ may be less than $n-k+1$, for example, let G be a connected graph with vertex set $V(G) = A \cup B$ where $A = \{u_i | 1 \leq i \leq k-1\}$ and $B = \{v_j | 1 \leq j \leq n-k+1\}$, and edge set $E(G) = \{v_{j_1}v_{j_2} | 1 \leq j_1, j_2 \leq n-k+1\} \cup \{u_iv_j | 1 \leq i \leq k-1, 1 \leq j \leq n-k\} \cup \{u_1v_{n-k+1}\}$. Clearly, we have $\alpha(G) = k-1$, and so $\kappa'_k(G) = n-k+1$ by Proposition 3.1. Choose $S = A \cup \{v_{n-k+1}\}$. It is not hard to show that $\kappa_k(G) \leq \kappa_G(S) = n-k < n-k+1 = \kappa'_k(G)$. Note that $\kappa_k(G)$ may be larger than or equal to $n-k+1$. For example, letting $G = K_n$, we have $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil \geq n-k+1 = \kappa'_k(G)$. Actually, we can get the following result.

Lemma 3.4. *For a connected graph G of order n, if $\kappa'_k(G) = n-k+1$ where $k \geq 3$, then $-\lfloor \frac{k}{2} \rfloor + 1 \leq \kappa'_k(G) - \kappa_k(G) \leq n-k$. Moreover, the bounds are sharp.*

Proof. The bounds are deduced from Proposition 2.1 and the assumption that $\kappa'_k(G) = n-k+1$. For the sharpness of the lower bound, just consider the graph K_n, since $\kappa_k(K_n) = n - \lceil \frac{k}{2} \rceil$ and $\kappa'_k(K_n) = n-k+1$, we have $\kappa'_k(K_n) - \kappa_k(K_n) = -\lfloor \frac{k}{2} \rfloor + 1$. For the sharpness of the upper bound, we consider the example which was used in the proof of Theorem 3.2: Let G be a graph with $V(G) = \{u_i | 1 \leq i \leq n\}$ such that $u_1u_n \in E(G)$ and $V' = \{u_i | 1 \leq i \leq n-1\}$ is a clique. Clearly, $\kappa_k(G) = 1$ since $\delta(G) = 1$, and we know $\alpha(G) = 2 \leq k-1$. Then, $\kappa'_k(G) = n-k+1$ by Proposition 3.1. $\square$

By Lemmas 3.3 and 3.4, we now get the sharp lower and upper bounds for $\kappa'_k(G) - \kappa_k(G)$.

Theorem 3.5. *For a connected graph G of order n, if $\kappa'_k(G) \neq n-k+1$ where $k \geq 3$, then $0 \leq \kappa'_k(G) - \kappa_k(G) \leq n-k-1$; otherwise, $-\lfloor \frac{k}{2} \rfloor + 1 \leq \kappa'_k(G) - \kappa_k(G) \leq n-k$. Moreover, all of these bounds are sharp.*

4 Graphs with $\kappa'_3(G) = \kappa_3(G)$

In this section, we will discuss the graphs with $\kappa'_3(G) = \kappa_3(G) = t$ with $1 \leq t \leq n-2$. In the following result, we completely characterize the graphs for $t \in \{1, n-3, n-2\}$.

Theorem 4.1. *For a connected graph of order n, the following assertions hold:*
(i) $\kappa'_3(G) = \kappa_3(G) = 1$ if and only if $G \in G(x, 3)$;
(ii) $\kappa'_3(G) = \kappa_3(G) = n-3$ if and only if $\overline{G} = C_3 \cup iP_2 \cup (n-2i-3)K_1 (i = 0, 1)$;
(iii) $\kappa'_3(G) = \kappa_3(G) = n-2$ if and only if $G = K_n$ or $G = K_n \setminus e$.

Proof. By the definitions of $G(x, s)$ and $\kappa'_k(G)$, it is not hard to show that $\kappa'_3(G) = 1$ if and only if $G \in G(x, 3)$. For $G \in G(x, 3)$, we also have $\kappa_3(G) = 1$, and then (i) holds. Since $\alpha(K_n \setminus e) = 2$, by Theorem 2.4 and Proposition 3.1, we have (iii).

We now consider (ii). If $\kappa'_3(G) = n-3$, then by the definition of $\kappa'_3(G)$, there exists a set $X \subseteq V(G)$ with $|X| = n-3$ such that $G \setminus X$ contains at least three components. We choose a vertex in each component. Then, any two chosen vertices are nonadjacent, and so $\alpha(G) \geq 3$.

By Theorem 2.5, $\kappa_3(G) = n-3$ if and only if $\overline{G} = P_4 \cup (n-4)K_1$ or $\overline{G} = P_3 \cup iP_2 \cup (n-2i-3)K_1 (i = 0, 1)$ or $\overline{G} = C_3 \cup iP_2 \cup (n-2i-3)K_1 (i = 0, 1)$ or $\overline{G} = rP_2 \cup (n-2r)K_1 (2 \leq r \leq \lfloor \frac{n}{2} \rfloor)$.

If $\overline{G} = P_4 \cup (n-4)K_1$ or $\overline{G} = P_3 \cup iP_2 \cup (n-2i-3)K_1 (i = 0, 1)$ or $\overline{G} = rP_2 \cup (n-2r)K_1 (2 \leq r \leq \lfloor \frac{n}{2} \rfloor)$, then it is not hard to show that $\alpha(G) = 2$ and then $\kappa'_3(G) \neq n-3$ from the above argument. Otherwise, $\overline{G} = C_3 \cup iP_2 \cup (n-2i-3)K_1 (i = 0, 1)$. Clearly, we have $\alpha(G) = 3$, and so $\kappa'_3(G) \leq n-3$ by Observation 2.3. It is not hard to show that any set $X \subseteq V(G)$ with

$|X| \leq n-4$ such that $G \setminus X$ contains exactly one component, and so $\kappa'_3(G) \geq n-3$. Thus, $\kappa'_3(G) = n-3$ if $\overline{G} = C_3 \cup iP_2 \cup (n-2i-3)K_1 (i=0,1)$. Therefore, the assertion (ii) holds. □

Lemma 4.2. *If G is a graph with at least three components, then $m \leq \binom{n-2}{2}$; the equality holds if and only if G has exactly three components such that two of them are trivial, the remaining one is a clique of order $n-2$.*

Proof. Let $G_1, G_2, \ldots, G_\ell$ be the components of G such that $n(G_1) \geq n(G_2) \geq \ldots \geq n(G_\ell)$, where $n(G_i) = |V(G_i)|$ is the order of G_i for $1 \leq i \leq \ell$. Let G' be a graph with three components, say G'_1, G'_2, G'_3, such that $V(G'_1) = \bigcup_{i=1}^{\ell-2} V(G_i)$, $V'(G_2) = V(G_{\ell-1})$, $V'(G_3) = V(G_\ell)$ and each component of G' is complete.

Clearly, G is a spanning subgraph of G' and then $m(G) \leq m(G') = \binom{n(G'_1)}{2} + \binom{n(G'_2)}{2} + \binom{n(G'_3)}{2}$, where $\sum_{i=1}^{3} n(G'_i) = n$. Furthermore, it is not hard to show that $m(G') \leq \binom{n-2}{2}$, and the equality holds if and only if $n(G'_1) = n-2, n(G'_2) = n(G'_3) = 1$. Thus, our result holds. □

For two integers t, n with $1 \leq t \leq n-3$, we define a class of graph $G(n,t)$ as follows: For each graph $G \in G(n,t)$, let $V(G) = A \cup B \cup \{v_{n-t-1}, v_{n-t}\}$ with $A = \{u_i | 1 \leq i \leq t\}$ and $B = \{v_j | 1 \leq j \leq n-t-2\}$ such that $xy \in E(G)$ for each pair $(x,y) \in (A, V(G) \setminus A)$, both A and B are cliques of G. Clearly, the size of G is $\binom{n-2}{2} + 2t$. Furthermore, the following result holds.

Lemma 4.3. $\kappa'_3(G) = \kappa_3(G) = t$ *for* $G \in G(n,t)$.

Proof. On one hand, since $n \geq t+3$ and for any set $X \subseteq V(G)$ with $|X| \leq t-1$, $G \setminus X$ is connected, we have $\kappa'_3(G) \geq t$; on the other hand, since $G \setminus A$ contains three components, we have $\kappa'_3(G) \leq t$. Thus, $\kappa'_3(G) = t$.

We choose a set $S = \{x,y,z\} \subseteq V(G)$ and consider the case that $x, y \in A$ and $z = v_{n-t}$. Without loss of generality, we can assume $x = u_1, y = u_2$. Let T_1 be the path x, y, z, T_2 be the path y, v_1, x, z, and T_i be the claw $K_{1,3}$ formed by the vertex set $\{u_i, x, y, z\}$ where $3 \leq i \leq t$. Then $\kappa_G(S) = t$ in this case. With a similar argument, we can deduce that $\kappa_G(S) \geq t$ for the remaining cases. Then, $\kappa_3(G) \geq t$. Since $\kappa_3(G) \leq \delta(G) = t$, we have $\kappa_3(G) = t$. □

By Lemma 4.3 and the fact that $\kappa'_3(K_n) = \kappa_3(K_n) = n-2$, the following result clearly holds.

Theorem 4.4. *For each integer t with $1 \leq t \leq n-2$, there exists a graph G of order n such that $\kappa'_3(G) = \kappa_3(G) = t$.*

By (ii) of Theorem 4.1, we know that $G = K_n$ or $G = K_n \setminus e$, and clearly $m(G) = \binom{n}{2}$ or $\binom{n}{2} - 1$ for $\kappa_3(G) = n-2$. The following theorem concerns the case that $\kappa_3(G) \leq n-3$ and gives a necessary condition for $\kappa'_3(G) = \kappa_3(G)$.

Theorem 4.5. *Let G be a connected graph of order n and size m, if $\kappa'_3(G) = \kappa_3(G) = t$ where $1 \leq t \leq n-3$, then $m \leq \binom{n-2}{2} + 2t$; moreover, the equality holds if and only if $G \in G(n,t)$.*

Proof. Since $\kappa'_3(G) = \kappa_3(G) = t$ and $n \geq t+3$, by definition there exists a subset $X \subseteq V(G)$ such that $G \setminus X$ contains at least three components, say $G_1, G_2, \ldots, G_\ell$, where $\ell \geq 3$. Without loss of generality, we can assume that $n(G_1) \geq n(G_2) \geq \ldots \geq n(G_\ell)$.

Let G' be a graph with vertex set $V(G') = V(G)$ such that $xy \in E(G)$ for each pair $(x,y) \in (X, V(G') \setminus X)$, $G[X]$ and G_i are complete for $1 \leq i \leq \ell$. Clearly, G is a spanning subgraph of G' and then $m(G) \leq m(G')$. By Lemma 4.2, we know that $m(G')$ reaches the maximum value if and only if $\ell = 3$, $n(G_2') = n(G_3') = 1$ and G_1' is a clique of order $n-t-2$, that is, in this case $G' \in G(n,t)$, and the sets X, $V(G_1')$ correspond to A, B in $G(n,t)$, respectively. Thus, $m(G) \leq m(G') \leq \binom{n-2}{2} + 2t$, and by Lemma 4.3, the conclusion holds. $\square$

Note that in the above theorem, we obtain a sharp upper bound for the size of G provided that $\kappa_3'(G) = \kappa_3(G) = t$ where $1 \leq t \leq n-3$, and also get the unique extremal graph $G(n,t)$. In fact, according to our argument, the bound is also sharp provided that $\kappa_3'(G) = t$ where $1 \leq t \leq n-3$. However, $G(n,t)$ may not be the unique extremal graph in this situation.

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