Proper connection number of random graphs*

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Abstract

A path in an edge-colored graph is called a proper path if no two adjacent edges of the path are colored the same. For a connected graph G, the proper connection number $pc(G)$ of G is defined as the minimum number of colors needed to color its edges, so that every pair of distinct vertices of G is connected by at least one proper path in G. In this paper, we show that almost all graphs have the proper connection number 2. More precisely, let $G(n,p)$ denote the Erdös-Rényi random graph model, in which each of the $\binom{n}{2}$ pairs of vertices appears as an edge with probability p independent from other pairs. We prove that for sufficiently large n, $pc(G(n,p)) \leq 2$ if $p \geq \frac{\log n+\alpha(n)}{n}$, where $\alpha(n) \to \infty$.

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1 Introduction

All graphs in this paper are undirected, finite and simple. We follow [4] for graph theoretical notation and terminology not defined here. Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \to \{1, 2, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may have the same color. A path of G is called a *rainbow path* if no two edges on the path have the same color. The graph G is called *rainbow connected* if for any two

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vertices of G there is a rainbow path of G connecting them. An edge-coloring of a connected graph is called a *rainbow connecting coloring* if it makes the graph rainbow connected. For a connected graph G, the *rainbow connection number* $rc(G)$ of G is the smallest number of colors that are needed in order to make G rainbow connected. This concept of rainbow connection of graphs was introduced by Chartrand et al. [7] in 2008. The interested readers can see [14, 13] for a survey on this topic.

Motivated by rainbow coloring and proper coloring in graphs, Andrews et al. [1] introduced the concept of proper-path coloring. Let G be a nontrivial connected graph with an edge-coloring. A path in G is called a *proper path* if no two adjacent edges of the path are colored the same. An edge-coloring of a connected graph G is a *proper-path coloring* if every pair of distinct vertices of G are connected by a proper path in G. For a connected graph G, the minimum number of colors that are needed to produce a proper-path coloring of G is called the *proper connection number* of G, denoted by $pc(G)$. From the definition, it follows that $1 \leq pc(G) \leq min\{rc(G), \chi'(G)\} \leq m$, where $\chi'(G)$ is the chromatic index of G and m is the number of edges of G. And it is easy to check that $pc(G) = 1$ if and only if $G = K_n$, and $pc(G) = m$ if and only if $G = K_{1,m}$. For more details we refer to [1, 5].

The study on rainbow connectivity of random graphs has attracted the interest of many researchers, see [6, 11, 12]. It is worth investigating the proper connection number of random graphs, which is the purpose of this paper. The most frequently occurring probability model of random graphs is the Erdös-Rényi random graph model $G(n,p)$ [9]. The model $G(n,p)$ consists of all graphs with n vertices in which the edges are chosen independently and with probability p. We say an event $\mathcal{A}$ happens *with high probability* if the probability that it happens approaches 1 as $n \to \infty$, i.e., $Pr[\mathcal{A}] = 1 - o_n(1)$. Sometimes, we say *w.h.p.* for short. We will always assume that n is the variable that tends to infinity.

Let G and H be two graphs on n vertices. A property P is said to be *monotone* if whenever $G \subseteq H$ and G satisfies P, H also satisfies P. For any property P of graphs and any positive integer n, define $Prob(P,n)$ to be the ratio of the number of graphs with n labeled vertices having P over the total number of graphs with these vertices. If $Prob(P,n)$ approaches 1 as n tends to infinity, then we say that almost all graphs have the property P. Similarly, for a fixed integer r, we say that almost all r-regular graphs have the property P if the ratio of the number of r-regular graphs with n labeled vertices having P over the total number of r-regular graphs with these vertices approaches 1 as n tends to infinity.

There are many results in the literature asserting that almost all graphs have some property. Here we list some of them, which are related to our study on the proper connection number of random graphs.

Theorem 1.1 *[3] Almost all graphs are connected with diameter 2.*

Theorem 1.2 *[3] For every nonnegative integer k, almost all graphs are k-connected.*

Theorem 1.3 *[16] For fixed integer $r \geq 3$, almost all r-regular graphs are Hamiltonian.*

In [5], Borozan et al. got the following result.

Theorem 1.4 *If the diameter of graph G is 2 and G is 2-connected, then $pc(G) = 2$.*

The authors in [1] proved the following result.

Theorem 1.5 *If G is not complete and has a Hamiltonian path, then $pc(G) = 2$.*

From Theorem 1.1 and Theorem 1.2 and the formula that $Pr[A \cap B] = Pr[A] + Pr[B] - Pr[A \cup B]$, it is easy to derive that almost all graphs are 2-connected with diameter 2. Hence, by Theorem 1.4, we have

Theorem 1.6 *Almost all graphs have the proper connection number 2.*

Even if we concentrate on regular graphs, from Theorem 1.3 and Theorem 1.5, we also have the following result.

Theorem 1.7 *For fixed integer $r \geq 3$, almost all r-regular graphs have the proper connection number 2.*

Next, we study the value of the proper connection number of $G(n,p)$, when p belongs to different ranges. The following theorem is a classical result on the connectedness of a random graph.

Theorem 1.8 *[9] Let $p = (\log n + a)/n$. Then*

$$Pr[G(n,p) \text{ is connected}] \to \begin{cases} e^{-e^{-a}} & \text{if } |a| = O(1), \\ 0 & \text{if } a \to -\infty, \\ 1 & \text{if } a \to +\infty. \end{cases}$$

Since the concept of proper-path coloring only makes sense when the graph is connected, we only study on the proper-path coloring of $G(n,p)$ which is w.h.p. connected. Our main result is as follows.

Theorem 1.9 *For sufficiently large n, $pc(G(n,p)) \leq 2$ if $p \geq \frac{\log n+\alpha(n)}{n}$, where $\alpha(n) \to \infty$.*

For a graph property P, a function $p(n)$ is called a threshold function of P if:

- for every $r(n) = \omega(p(n))$, $G(n, r(n))$ w.h.p. satisfies P; and
- for every $r'(n) = o(p(n))$, $G(n, r'(n))$ w.h.p. does not satisfy P.

From Theorem 1.8 and Theorem 1.9, we can obtain that the threshold for $pc(G(n,p)) = 2$ is equal to the threshold for $G(n,p)$ to be connected. We will prove Theorem 1.9 in Section 2.

2 Proof of Theorem 1.9

In order to prove the first part of Theorem 1.9, we first present a classical result on random graphs as follows.

Theorem 2.1 *[2] Let $\omega(n) \to \infty$, $p = \frac{1}{n}\{\log n + \log\log n + \omega(n)\}$. Then, w.h.p. $G(n,p)$ is Hamiltonian.*

Let $p' = \frac{1}{n}\{\log n + \log\log n + \omega(n)\}$, where $\omega(n) \to \infty$. Since Hamiltonian is a monotone property, combining with Theorem 1.5, we know that $pc(G(n,p)) = 2$ if $p' \leq p < 1$. Thus in the sequel, we assume that $p = \frac{\log n+\alpha(n)}{n}$, where $\alpha(n) = o(\log n)$, and $\alpha(n) \to \infty$.

For two disjoint vertex-subsets X and Y of G, let $e(X,Y)$ be the number of the edges with one endpoint in X and the other in Y. For vertex-subsets $U \subset S$, $N(U,S)$ is the disjoint neighbor set of U in $G[S]$, i.e., $N(U,S) = \{w \in S - U : \exists u \in U \textit{ with } \{uw\} \in G[S]\}$ and $d_S(v) = |N(v) \cap S|$ is the degree of v in S. For ease of notation, let $G \in G(n,p)$ and denote by V the vertex set of $G(n,p)$.

It is known that w.h.p. the diameter of $G(n,p)$ is asymptotically equal to $D = \frac{\log n}{\log\log n}$ [2]. We call a vertex u *large* if its degree $d(u) \geq \frac{\log n}{100}$ and *small* otherwise. Let SMALL denote the vertex-subset consisting of all the small vertices. We first give some properties of small vertices as follows.

Lemma 2.1 *The following hold w.h.p. in $G(n,p)$.*

(1) $|SMALL| \leq n^{0.1}$.

(2) No pair of small vertices are adjacent or share a common neighbor.

Proof. (1) Let $s = \lceil n^{0.1} \rceil$. Let $\mathcal{A}$ denote the event that there exists a vertex-subset S with order s such that each vertex $v \in S$ is small. Then $\mathcal{A}$ happens with probability

$$\begin{aligned}
Pr[\mathcal{A}] &\leq \binom{n}{s} \left[\sum_{k=0}^{\frac{\log n}{100}} \binom{n}{k} p^k (1-p)^{n-1-k} \right]^s \\
&\leq \left(\frac{ne}{s}\right)^s \left[\frac{\log n}{100} \left(\frac{100ne}{\log n}\right)^{\frac{\log n}{100}} \left(\frac{\log n + \alpha(n)}{n}\right)^{\frac{\log n}{100}} e^{-\frac{\log n + \alpha(n)}{n}\left(n-1-\frac{\log n}{100}\right)} \right]^s \\
&\leq \left(\frac{ne}{s} \cdot \frac{\log n}{100} (101e)^{\frac{\log n}{100}} e^{-(\log n + \alpha(n)) + \frac{\log n + \alpha(n)}{n} + \frac{\log n}{100} \cdot \frac{\log n + \alpha(n)}{n}} \right)^s \\
&\leq \left(\frac{ne}{s} \cdot \frac{\log n}{100} \cdot n^{\frac{6}{100}} \cdot n^{-1} \cdot O(1) \right)^s \\
&\leq O(n^{-0.01 \cdot s}).
\end{aligned}$$

That implies that w.h.p. $|SMALL| \leq n^{0.1}$.

(2) Let $\mathcal{B}$ denote the event that there exist two small vertices x, y and the distance between x and y is at most 2. We have

$$
\begin{aligned}
Pr[\mathcal{B}] \le & \binom{n}{2}\Bigg\{ p\left(\sum_{i=1}^{\frac{\log n}{100}} \binom{n-2}{i} p^i (1-p)^{n-2-i}\right)^2 \\
& + \binom{n-2}{1} p^2 \left(\sum_{i=1}^{\frac{\log n}{100}} \binom{n-3}{i} p^i (1-p)^{n-3-i}\right)^2 \Bigg\} \\
\le & \, n^2 \Bigg\{ \frac{\log n + \alpha(n)}{n} \left[2\left(\frac{ne}{\frac{\log n}{100}}\right)^{\frac{\log n}{100}} p^{\frac{\log n}{100}} (1-p)^{n-2-\frac{\log n}{100}} \right]^2 \\
& + (n-2)\left(\frac{\log n + \alpha(n)}{n}\right)^2 \left[2\left(\frac{ne}{\frac{\log n}{100}}\right)^{\frac{\log n}{100}} p^{\frac{\log n}{100}} (1-p)^{n-2-\frac{\log n}{100}} \right]^2 \Bigg\} \\
\le & \, n^2 \left[\frac{\log n + \alpha(n)}{n} + n\left(\frac{\log n + \alpha(n)}{n}\right)^2 \right] \left[2\binom{n}{\frac{\log n}{100}} p^{\frac{\log n}{100}} (1-p)^{n-2-\frac{\log n}{100}} \right]^2 \\
\le & \left[n(2\log n) + n(2\log n)^2 \right] \left[2\left(\frac{ne}{\frac{\log n}{100}}\right)^{\frac{\log n}{100}} p^{\frac{\log n}{100}} (1-p)^{n-2-\frac{\log n}{100}} \right]^2 \\
\le & \left[n(2\log n) + n(2\log n)^2 \right] \left[2\left(\frac{ne}{\frac{\log n}{100}}\right)^{\frac{\log n}{100}} \left(\frac{\log n}{n}\right)^{\frac{\log n}{100}} e^{\frac{\log n}{n}\left(n - \frac{\log n}{100}\right)} \right]^2 \\
\le & \left[n(2\log n) + n(2\log n)^2 \right] n^{-1.9} \\
\le & \, n^{-0.8}.
\end{aligned}
$$

□

From Lemma 2.1, we can obtain that every small vertex is adjacent to a large vertex and there is at most one small vertex among the neighbors of a large vertex. Thus, we can find a matching M consisting of $|SMALL|$ edges in G such that for every edge e in M, one endpoint of e is small and the other endpoint is large. Let $s = |M| = |SMALL|$. Denote the large vertices in M by $x_1, x_2, \ldots, x_s$ and denote the small vertices in M by $y_1, y_2, \ldots, y_s$. By the definition of M, we assume that for every $i \in \{1, 2, \cdots, s\}$, $\{x_i y_i\}$ is an edge in M. If $|V \backslash SMALL|$ is odd, then we take an arbitrary edge $\{uv\}$ disjoint from M (such an edge $\{uv\}$ must exist since $n > 2s$ and G is connected) and let $M' = M \cup \{uv\}$. Since the vertex set of M includes all the small vertices and $\{uv\}$ is disjoint from M, we can obtain that both u and v are large vertices. If $|V \backslash SMALL|$ is even, just let $M' = M$. Denote the cardinality of

M' by s', that is,

$$s' = \begin{cases} s & \text{if } |V\backslash SMALL| \text{ is even,} \\ s+1 & \text{if } |V\backslash SMALL| \text{ is odd.} \end{cases}$$

Let

$$V_1 = \begin{cases} V\backslash SMALL & \text{if } |V\backslash SMALL| \text{ is even,} \\ V\backslash (SMALL \cup \{u\}) & \text{if } |V\backslash SMALL| \text{ is odd.} \end{cases}$$

So $|V_1|$ is even.

The following is an important structural property of G, and the proof is given in Section 2.1.

Claim 2.1 *The induced subgraph $G[V_1]$ of G is w.h.p. Hamiltonian.*

Note that to prove $pc(G) \leq 2$, it suffices to give G an edge-coloring with 2 colors and verify that the edge-coloring is a proper-path coloring of G. Denote the Hamiltonian cycle of $G[V_1]$ by C. We color the edges of C consecutively and alternately with color 1 and 2, and color all the edges in M' with color 1. It is easy to get that under this partial coloring, every pair of large vertices have a proper path connecting them, and there exists a proper path connecting a vertex in $\{y_1, y_2, \ldots, y_s, u\}$ (if such u exists) with a vertex in V_1. The following claim helps us to take care of pairs of vertices in $\{y_1, y_2, \ldots, y_s, u\}$. We will proof that claim in Section 2.2.

Claim 2.2 *There exists an edge-coloring of edges in $E(G)\backslash(E(C) \cup M')$ with 2 colors such that w.h.p. every pair of vertices in $\{y_1, y_2, \ldots, y_s, u\}$ have a proper path connecting them in G.*

Thus Theorem 1.9 follows from the above arguments. So all we need to do is to prove Claims 2.1 and 2.2.

2.1 Proof of Claim 2.1

We will use arguments similar to those of Cooper et al. [8] and Frieze et al. [10]. The following lemma establishes some structural properties of G, which we will make use of in our proof.

Lemma 2.2 *The following hold in G w.h.p. :*

(1) For any $S \subseteq V$, $|S| \leq \frac{n}{375}$ implies $|E(G[S])| < \frac{|S|np}{250}$.

(2) If $U, W \subseteq V$, $U \cap W = \emptyset$, $|U|, |W| \geq \frac{n}{\log\log n}$, then $e(U, W) > 0$.

(3) There are at most $n^{0.2}$ edges incident with vertices in SMALL.

Proof. (1) The number of edges in an induced subgraph $G[S]$ with $|S| = s$ is a binomial random variable with parameters $\binom{s}{2}$ and p. By Bollobás [2] (see page 14) we have for large deviations of binomial random variables

$$Pr\left[\text{the number of edges in } G[S] \geq \gamma\binom{s}{2}p\right] < \left(\frac{e}{\gamma}\right)^{\gamma\binom{s}{2}p}.$$

Setting $\gamma = \frac{n}{125s}$, we obtain that

$$\begin{aligned}
&\sum_{s=1}^{\frac{n}{375}}\binom{n}{s}\left(\frac{e}{\gamma}\right)^{\gamma\binom{s}{2}p} \\
&\leq \sum_{s=1}^{\frac{n}{375}}\left(\frac{ne}{s}\right)^{s}\left(\frac{125es}{n}\right)^{\frac{n}{125s}\frac{s(s-1)}{2}\frac{\log n}{n}} \\
&\leq \sum_{s=1}^{\frac{n}{375}}\left(\frac{ne}{s}\right)^{s}\left(\frac{125es}{n}\right)^{s\frac{\log n}{500}} \\
&= \sum_{s=1}^{\frac{n}{375}}\left(n^{s-s\frac{\log n}{500}}e^{s+s\frac{\log n}{500}}s^{s\frac{\log n}{500}-s}125^{s\frac{\log n}{500}}\right) \\
&\leq \sum_{s=1}^{\frac{n}{375}}\left(n^{-s\frac{\log n}{501}}e^{s\frac{\log n}{499}}s^{s\frac{\log n}{501}}125^{s\frac{\log n}{500}}\right) \\
&\leq \sum_{s=1}^{\frac{n}{375}}\left(n^{-s\frac{\log n}{501}+\frac{s}{499}}\left(\frac{n}{375}\right)^{s\frac{\log n}{501}}125^{s\frac{\log n}{500}}\right) \\
&= \sum_{s=1}^{\frac{n}{375}}\left(n^{\frac{s}{499}}\left(\frac{1}{375}\right)^{s\frac{\log n}{501}}125^{s\frac{\log n}{500}}\right) \\
&= o(1).
\end{aligned}$$

(2) Let $\mathcal{A}$ denote the event that there exist two subsets $U, W \subseteq V$, $U \cap W = \emptyset$, $|U|, |W| \geq \frac{n}{\log\log n}$ and $e(U, W) = 0$. Then

$$\begin{aligned}
Pr[\mathcal{A}] &\leq \sum_{s \geq \frac{n}{\log\log n}} \sum_{t \geq \frac{n}{\log\log n}} \binom{n}{s}\binom{n-s}{t}(1-p)^{st} \\
&\leq \sum_{s \geq \frac{n}{\log\log n}} \sum_{t \geq \frac{n}{\log\log n}} \left(\frac{ne}{s}\right)^s \left(\frac{ne}{t}\right)^t e^{-pst} \\
&\leq \sum_{s \geq \frac{n}{\log\log n}} \sum_{t \geq \frac{n}{\log\log n}} (ne)^{s+t} \left(\frac{1}{s}\right)^s \left(\frac{1}{t}\right)^t e^{-\frac{\log n}{n} \cdot \frac{n}{\log\log n} \cdot \frac{n}{\log\log n}} \\
&\leq \sum_{s \geq \frac{n}{\log\log n}} \sum_{t \geq \frac{n}{\log\log n}} (ne)^{s+t} \left(\frac{\log\log n}{n}\right)^s \left(\frac{\log\log n}{n}\right)^t e^{-\frac{n \log n}{(\log\log n)^2}} \\
&= \sum_{s \geq \frac{n}{\log\log n}} \sum_{t \geq \frac{n}{\log\log n}} e^{(s+t)(1+\log\log\log n)} e^{-\frac{n \log n}{(\log\log n)^2}} \\
&\leq \sum_{s \geq \frac{n}{\log\log n}} \sum_{t \geq \frac{n}{\log\log n}} e^{n(1+\log\log\log n)} e^{-\frac{n \log n}{(\log\log n)^2}} \\
&\leq n^2 e^{n(1+\log\log\log n)} e^{-\frac{n \log n}{(\log\log n)^2}} \\
&\leq o(n^{-1}).
\end{aligned}$$

(3) Lemma 2.1 implies that SMALL is w.h.p. an independent set, i.e., no edges in the induced subgraph $G[SMALL]$. Since the degree of a small vertex is less than $\frac{\log n}{100}$, we have that the number of edges incident to SMALL is w.h.p. no more than

$$|SMALL| \cdot \frac{\log n}{100} \leq n^{0.1} \cdot \frac{\log n}{100} < n^{0.2}.$$

$\square$

Let $\mathscr{H} = \{G \in G(n,p)$: the conditions of Lemmas 2.1 and 2.2 hold$\}$. The following lemma is an immediate consequence of Lemma 2.2(1).

Lemma 2.3 *Let $G \in \mathscr{H}$, $U \subseteq S \subset V$, $|U| \leq \frac{n}{1500}$, $F \subset E(G[S])$ and $H = (S, F)$. If U is such that the degree of w in H is at least $\frac{\log n}{101}$ for all $w \in U$, then $|N(U,S)| \geq 3|U|$ in H.*

We regard the edges in G as initially colored blue, but with the option of recoloring a set R of the edges red. We require the set R of red edges is "deletable", which is defined as follows.

Definition 2.1 *(1) A set $R \subseteq E(G)$ is deletable if*

(i) R is a matching, and

(ii) no edge of R is incident with a small vertex, and

(iii) $|R| = \lceil n^{0.1} \rceil$.

(2) Let $G_B[V_1]$ denote the subgraph of $G[V_1]$ induced by blue edges.

(3) $N_B(U, V_1)$ denotes the disjoint neighbor set of U in $G_B[V_1]$.

Lemma 2.4 *Let $G \in \mathscr{H}$ and let $U \subseteq V_1$, $|U| \leq \frac{n}{1500}$. Then $|N_B(U, V_1)| \geq 2|U|$.*

Proof. By Lemma 2.2(1), each vertex $w \in U$ has at most one neighbor in SMALL. We have $d_{V_1}(w) \geq \frac{\log n}{100} - 1 - 1 \geq \frac{\log n}{101}$. From Lemma 2.3, we obtain that there are at least $3|U|$ neighbors of U in V_1. Thus the removal of $\min\{|R|, |U|\}$ deletable edges makes $|N_B(U, V_1)| \geq 2|U|$. $\square$

Lemma 2.5 *For $G \in \mathscr{H}$, $G[V_1]$ is connected.*

Proof. If $G[V_1]$ is not connected, then by Lemma 2.4 the smallest component cannot consist of less than $\frac{n}{1500}$ vertices.

On the other hand, by Lemma 2.2(2), any two sets of vertices of size at least $\frac{n}{\log \log n}$ must be connected by an edge. So $G[V_1]$ is connected. $\square$

To prove Claim 2.1, we also need some more definitions and results taken from Pósa [15] and Frieze et al. [10].

Definition 2.2 *Let $\Gamma = (V, E)$ be a non-Hamiltonian graph with a longest path of length ℓ. A pair $\{u, v\} \notin E$ is called a hole if adding $\{u, v\}$ to Γ creates a graph Γ' which is Hamiltonian or contains a path longer than ℓ.*

Definition 2.3 *A graph $\Gamma = (V, E)$ is called a (k, c)-expander if $|N(U)| \geq c|U|$ for every subset $U \subseteq V(G)$ of cardinality $|U| \leq k$.*

Lemma 2.6 *[10] Let Γ be a non-Hamiltonian connected $(k, 2)$-expander. Then Γ has at least $\frac{k^2}{2}$ holes.*

From Lemmas 2.4 and 2.6, we obtain that $G[V_1]$ is a $(\frac{n}{1500}, 2)$-expander, and it has at least $\frac{1}{2}(\frac{n}{1500})^2$ holes depending only on $G_B[V_1]$. We define the set $\mathscr{F}$ to be those $G \in \mathscr{H}$ for which the subgraph $G[V_1]$ is not Hamiltonian. Our aim is to prove the following result.

Lemma 2.7 $\frac{|\mathscr{F}|}{|G(n,p)|} = o(1)$.

Proof. Let R be a set of red edges of G and satisfying the property P such that

(i) R is deletable, and

(ii) $\lambda(G[V_1]) = \lambda(G_B[V_1])$,

where $\lambda(H)$ is the length of a longest path in the graph H.

Let $\mathscr{C}$ be the set of all red-blue colorings of $\mathscr{F}$ which satisfy P. Let $\lambda = \lambda(G[V_1])$, we have $\lambda < |V_1|$. Recall that there are at most $\mu = \lceil n^{0.2} \rceil$ edges incident with small vertices. Set $r = |R|$. Since R is a matching, we can choose it in at least

$$\begin{aligned}
&\frac{1}{r!}(m-\lambda-\mu)(m-\lambda-\mu-2\Delta)\dots(m-\lambda-\mu-2(r-1)\Delta) \\
&\geq \frac{1}{r!}(m-|V_1|-\mu)(m-|V_1|-\mu-2\Delta)\dots(m-|V_1|-\mu-2(r-1)\Delta) \\
&\geq \frac{(m-|V_1|)^r}{r!}(1-o(1))
\end{aligned}$$

ways, where m is the number of edges in G, and Δ is the maximum degree of G. It is known that Δ is w.h.p. at most $3np$ (see e.g. [2]).

Hence,

$$|\mathscr{C}| \geq |\mathscr{F}| \frac{(m-|V_1|)^r}{r!}(1-o(1)).$$

Consider that we fix the blue subgraph. Then, by the definition of holes, we have to avoid replacing at least $\frac{1}{2}(\frac{n}{1500})^2$ edges when adding back the red edges in order to construct a red-blue coloring satisfying property P. Thus

$$|\mathscr{C}| \leq \binom{\binom{n}{2}}{m-r}\binom{\binom{n}{2}-(m-r)-\frac{1}{2}(\frac{n}{1500})^2}{r}.$$

It follows that

$$\frac{|\mathscr{F}|}{|G(n,p)|} \leq \frac{\sum_{m=\frac{1}{100}\binom{n}{2}p}^{\binom{n}{2}} \left[\binom{\binom{n}{2}}{m-r}\binom{\binom{n}{2}-(m-r)-\frac{1}{2}(\frac{n}{1500})^2}{r} \Big/ \frac{(m-|V_1|)^r}{r!}(1-o(1)) \right]}{\binom{\binom{n}{2}}{\binom{n}{2}p}}$$

Note that $\frac{\left[\binom{\binom{n}{2}}{m-r}\binom{\binom{n}{2}-(m-r)-\frac{1}{2}(\frac{n}{1500})^2}{r} \Big/ \frac{(m-|V_1|)^r}{r!}(1-o(1)) \right]}{\binom{\binom{n}{2}}{\binom{n}{2}p}} \leq O(e^{-\frac{r}{1500^2}+\frac{nr}{(n-1)\log n}})$ (see [8]) and $O(e^{-\frac{r}{1500^2}+\frac{nr}{(n-1)\log n}}) = o(n^{-\theta})$ for any constant $\theta > 0$. Thus, for any constant

$\theta > 3$, we have

$$\frac{|\mathscr{F}|}{|G(n,p)|} \leq \sum_{m=\frac{1}{100}\binom{n}{2}p}^{\binom{n}{2}} o(n^{-\theta}) \leq n^2 o(n^{-\theta}) \leq o(n^{-1}).$$

$\square$

2.2 Proof of Claim 2.2

We still assume that $G \in \mathscr{H}$ which was defined in the previous subsection. Recall that a *t-ary tree* with a designated root is a tree whose non-leaf vertices all have exactly t children. For any tree T_w rooted at w and any vertex $x \in T_w \backslash \{w\}$, we use $P_{T_w}(w,x)$ to denote the only path from w to x in T_w. We say that x is at depth k of T_w if $P_{T_w}(w,x)$ is of length k. For any tree T_w, denote by L_w the set of leaves of T_w.

Let $E_1 = E(G[V_1]) \backslash E(C)$ and $H = (V_1, E_1)$ be a subgraph of G. Remember that $x_1, \ldots, x_s$ are the large vertices in M. Let $x_{s+1} = v$ and $y_{s+1} = u$, if $M' = M \cup \{uv\}$. For every $x_i \in \{x_1, x_2, \ldots, x_s, x_{s+1}\}$, we will build vertex-disjoint $\frac{\log n}{101}$-ary trees T_{x_i} of depth $(\frac{1}{2}+\epsilon)D = (\frac{1}{2}+\epsilon)\frac{\log n}{\log\log n}$ in H. Hereafter, let $0 < \epsilon < 1$ be a sufficiently small constant.

Note that if we successfully build such vertex-disjoint trees, then the number of leaves of each tree T_{x_i} is $|L_{x_i}| = (\frac{\log n}{101})^{(\frac{1}{2}+\epsilon)D}$, for $i = 1, 2, \ldots, s+1$. Thus, we have

$$\begin{aligned}
&Pr[\textit{there exist distinct } i,\ j \textit{ such that } e(L_{x_i}, L_{x_j}) = 0] \\
&\leq \binom{s+1}{2}(1-p)^{(\frac{\log n}{101})^{(1+2\epsilon)D}} \\
&\leq n^{0.2} e^{-\frac{\log n}{n}(\frac{\log n}{101})^{(1+2\epsilon)D}} \\
&\leq n^{0.2} \cdot n^{-n^{\epsilon}} \leq n^{-\frac{1}{2}n^{\epsilon}} \\
&= o(1).
\end{aligned}$$

Hence, for every $i \neq j$, there exists a path from x_i to x_j of length $(1+2\epsilon)D + 1$ (these paths are not necessarily vertex-disjoint). Denote that path by P_{ij}. For every tree T_{x_i}, we color the edges between the vertices at depth $2\ell - 1$ to 2ℓ with color 2, and color the edges between the vertices at depth 2ℓ to $2\ell+1$ with color 1, where $\ell = 1, 2, \ldots, \lfloor \frac{(\frac{1}{2}+\epsilon)D}{2} \rfloor$. Color the edges between each L_{x_i} and L_{x_j} $(i \neq j)$ with the color different from the color used in the edges between the vertices at depth $(\frac{1}{2}+\epsilon)D - 1$ to $(\frac{1}{2}+\epsilon)D$. That is, if the edges between the vertices at depth $(\frac{1}{2}+\epsilon)D - 1$ to leaves are colored with color 1, then we color the edges between L_{x_i} and L_{x_j} with color 2;

if the edges between the vertices at depth $(\frac{1}{2}+\epsilon)D-1$ to leaves are colored with color 2, then we color the edges between L_{x_i} and L_{x_j} with color 1. Recalling that we color edges in M' with color 1, then for every $i \neq j$ the path formed by the two edges $\{x_i y_i\}$, $\{x_j y_j\}$ combining with the path P_{ij} is a proper path connecting y_i and y_j. Thus our claim follows.

Now we prove that these $(\frac{\log n}{101})$-ary trees can be constructed successfully w.h.p..

Realize first that every vertex x in H has degree $d_H(x) \geq \frac{\log n}{100} - 2 - 2$, since there are two edges incident with x in C and x can be adjacent to at most one small vertex plus u in G.

For every $i = 1, 2, \ldots, s+1$, we build the tree T_{x_i} level by level from x_i to the leaves. Suppose that we are growing the tree T_{x_j} from vertex w at depth k to vertices at depth $k+1$. Note that the construction halts if we cannot expand by the required amount. That is, we cannot find enough neighbors of w in H to add into the tree T_{x_j}, since w may point to vertices already in T_{x_i}, $i \leq j$. We call such edges as *bad edges emanating from w*. We claim that the number of bad edges emanating from w is small. It is easy to get that at any stage, the number of vertices we used to construct trees is less than

$$\begin{aligned}
&(s+1) \cdot (\frac{1}{2}+\epsilon)\frac{\log n}{\log\log n}\left(\frac{\log n}{101}\right)^{(\frac{1}{2}+\epsilon)\frac{\log n}{\log\log n}} \\
&\leq (\frac{1}{2}+\epsilon)\frac{\log n}{\log\log n} \cdot n^{\frac{1}{2}+\frac{\epsilon}{2}} \cdot n^{0.1} \\
&\leq n^{0.65}.
\end{aligned}$$

For any fixed vertex w, the bad edges from w is stochastically dominated by the random variable $X \sim Bin(n^{0.65}, p)$. Thus,

$$\begin{aligned}
&Pr[there\ are\ at\ least\ 10\ bad\ edges\ emanating\ from\ w] \\
&\leq Pr[X \geq 10] \leq \binom{n^{0.65}}{10} p^{10} \\
&\leq \left(\frac{en^{0.65}}{10} \cdot \frac{\log n + \alpha(n)}{n}\right)^{10} \\
&\leq (n^{-0.34})^{10} \\
&= n^{-3.4}.
\end{aligned}$$

Using the Union Bound taking over all vertices, we have that with probability at least $1 - n^{-2.4}$, any current vertex w has at most 9 bad edges emanating from it. Therefore, there are at least $\frac{\log n}{100} - 4 - 9 - 1 \geq \frac{\log n}{101}$ neighbors of w in H that can be used to continue our construction of T_{x_j}. Hence, w.h.p. we can successfully build such $\frac{\log n}{101}$-ary trees we required. The proof is thus complete. ∎

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References

[1] E. Andrews, E. Laforge, C. Lumduanhom, P. Zhang, On proper-path colorings in graphs, *J. Combin. Math. Combin. Comput.*, To appear.

[2] B. Bollobás, *Random Graphs*, Cambridge University Press, 2001.

[3] A. Blass, F. Harary, Properties of almost all graphs and complexes, *J. Graph Theory* **3**(3) (1979) 225–240.

[4] J.A. Bondy, U.S.R. Murty, *Graph Theory*, GTM 244, Springer, 2008.

[5] V. Borozan, S. Fujita, A. Gerek, C. Magnant, Y. Manoussakis, L. Montero, Z. Tuza, Proper connection of graphs, *Discrete Math.* **312** (2012) 2550–2560.

[6] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, R. Yuster, On rainbow connection, *Electron. J. Comb.* **15** (2008) #R57.

[7] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, Rainbow connection in graphs, *Math. Bohem.* **133** (2008) 85–98.

[8] C. Cooper, A. Frieze, Pancyclic random graphs, Proc. Conf. Random Graphs, Poznan, 1987.

[9] P. Erdös, A. Rényi, On the evolution of random graphs, *Publ. Math. Inst. Hungar. Acad. Sci.* **5** (1960) 17–61.

[10] A. Frieze, M. Krivelevich, On two Hamilton cycle problems in random graphs, *Israel J. Math.* **166** (2008) 221–234.

[11] A. Frieze, C.E. Tsourakakis, Rainbow connection of sparse random graphs, *Electron. J. Comb.* **19** (2012) #P5.

[12] J. He, H. Liang, On rainbow-k-connectivity of random graphs, *Infor. Process. Lett.* **112** (2012) 406–410.

[13] X. Li, Y. Shi, Y. Sun, Rainbow connections of graphs: A survey, *Graphs & Combin.* **29** (2013) 1–38.

[14] X. Li, Y. Sun, *Rainbow Connections of Graphs*, Springer Briefs in Math., Springer, New York, 2012.

[15] L. Pósa, Hamiltonian circuits in random graphs, *Discrete Math.* **14** (1976) 359–364.

[16] R.W. Robinson, N.C. Wormald, Almost all regular graphs are Hamiltonian, *Random Struct. Algor.* **5** (1994) 363–374.