Borderenergetic Graphs*

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Abstract

The energy $E(G)$ of a graph G is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. A graph G of order n is said to be borderenergetic if its energy equals the energy of the complete graph K_n, i.e., if $E(G) = 2(n-1)$. We first show by examples that there exist connected borderenergetic graphs, different from the complete graph K_n. The smallest such graph is of order 7. We then show that for each integer n, $n \geq 7$, there exists borderenergetic graphs of order n, different from K_n, and describe the construction of some of these graphs.

1 Introduction

Throughout this paper, all graphs are assumed to be simple, undirected and finite. Let G be such a graph of order n, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues, i.e., the

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eigenvalues of the adjacency adjacent matrix $\mathbf{A}(G)$ of G. These eigenvalues form the spectrum of the graph G which will be denoted by $Sp(G)$; for details on spectral graph theory see [6].

A graph is said to be integral if all its eigenvalues are integers; for details on integral graphs see [2, 5].

The energy of G, denoted by $E(G)$, is the sum of the absolute values of the eigenvalues of $\mathbf{A}(G)$, that is, $E(G) = \sum_{i=1}^{n} |\lambda_i|$. This notion is related to some applications of graph theory in chemistry and has been studied intensively, see [1, 8, 17, 18, 20, 21, 23, 25–27], the survey papers [9, 11–13, 15], as well as the books [16, 24].

Recall that the complete graph K_n has energy $2(n-1)$, and a graph G of order n is said to be *hyperenergetic* if $E(G) > 2(n-1)$; for details see [10, 14, 19, 22, 26, 28–30]. Consequently, a graph G would be *non-hyperenergetic* if $E(G) < 2(n-1)$. In connection with this, we ask the following question:

Question: Are there graphs of order n, different from the complete graph K_n , whose energy is equal to $2(n-1)$?

For convenience, graphs G of order n, for which $E(G) = 2(n-1)$, will be referred to as *borderenergetic*. A borderenergetic graph is said to be non-complete if it is different from the complete graph K_n .

In this paper, we focus on constructing non-complete borderenergetic graphs. In Section 2, we report the results of our computer search for borderenergetic graphs of small order. In Section 3, we first show how additional disconnected and connected non-complete borderenergetic graphs can be constructed. In particular, we show how such graphs of order n can be constructed for each n , $n \geq 10$.

2 The smallest non-complete borderenergetic graphs

From the Appendix of [6], we established that there are no non-complete borderenergetic graphs of order less than 6. Some non-complete borderenergetic graphs with $n = 7, 8, 9, 10$ can be found in the tables of spectra of integral graphs [2], but a complete list could be obtained only a checking all n-vertex graphs. We have performed such a computer–aided calculation for $n = 7, 8$, and 9 and obtained the following results:

Proposition 1. *The smallest non-complete borderenergetic graph G_0 has $n = 7$ vertices and $m = 17$ edges, and is unique. This graph is depicted in Fig. 1.*

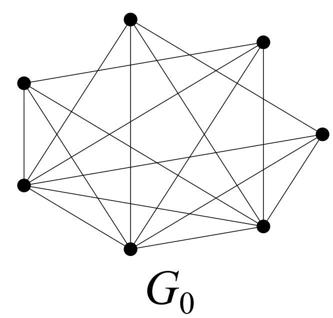

Fig. 1. The smallest non-complete borderenergetic graph. It is an integral graph with spectrum $Sp(G_0) = \{5, 1, -1, -1, -1, -1, -2\}$.

Proposition 2. *There exist (exactly) six non-complete borderenergetic graphs of order 8. These graphs are depicted in Fig. 2.*

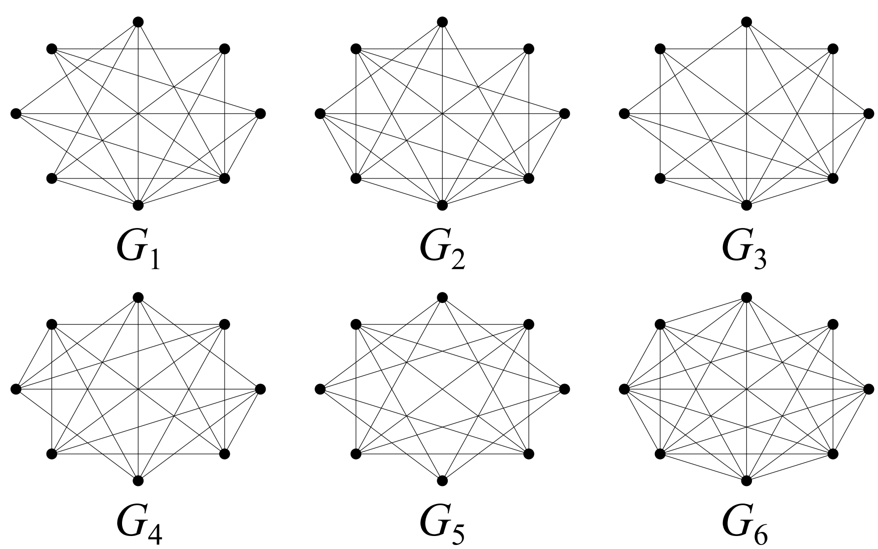

Fig. 2. The six non-complete borderenergetic graphs of order 8. These have different number of edges (between 19 and 25).

The graphs shown in Fig. 2 have the following spectra:

$$
\begin{aligned}
Sp(G_1) &= \{5, 1, 1, -1, -1, -1, -2, -2\} \\
Sp(G_2) &= \{x_1, 1, x_2, -1, -1, -1, -2, -2\} \\
Sp(G_3) &= \{5, 2, -1, -1, -1, -1, -1, -2\} \\
Sp(G_4) &= \{5, 1, 1, -1, -1, -1, -1, -3\} \\
Sp(G_5) &= \{y_1, 1, 1, y_2, -1, -2, -2, -2\} \\
Sp(G_6) &= \{z_1, z_2, -1, -1, -1, -1, -1, -2\}
\end{aligned}
$$

where

$$
\begin{array}{lcl}
x_1 = 3 + \sqrt{6} & ; & x_2 = 3 - \sqrt{6} \\
y_1 = (5 + \sqrt{17})/2 & ; & y_2 = (5 - \sqrt{17})/2 \\
z_1 = (7 + \sqrt{33})/2 & ; & z_2 = (7 - \sqrt{33})/2
\end{array}
$$

We thus see that borderenergetic graphs need not be integral. In fact, the next Proposition suggests that the majority of borderenergetic graphs are non-integral.

Proposition 3. *There exist (exactly) seventeen non-complete borderenergetic graphs of order 9. These graphs are depicted in Fig. 3. Only four among them are integral.*

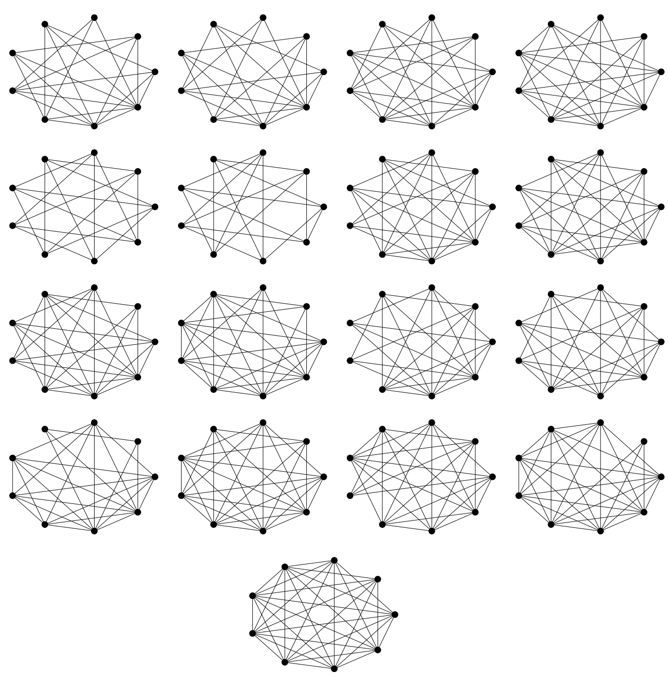

Fig. 3. The seventeen non-complete borderenergetic graphs of order 9. These have different number of edges (between 22 and 34).

3 Non-complete borderenergetic graphs

In this section, we investigate non-complete borderenergetic graphs by using tensor product, line graph, strongly regular graphs, the union of graphs, and complements, respectively.

3.1 Tensor product

The tensor product of two graphs G_1 and G_2 is the graph $G_1 \otimes G_2$ with vertex set $V(G_1) \times V(G_2)$, in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if both $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. For a graph G, denote by $|G|$ and $||G||$ the number of vertices and the number of edges of G, respectively. Based on the definition of the tensor product, we have:

Lemma 4. *Let G_1 and G_2 be two non-empty graphs and $G = G_1 \otimes G_2$. Then*
(i) *$G_1 \otimes G_2$ is not complete.*
(ii) *$G_1 \otimes G_2$ is connected if and only if both G_1 and G_2 are connected, and either G_1 or G_2 contains an odd cycle.*

Proof. From the definition, we have $||G_1 \otimes G_2|| = 2||G_1|| \times ||G_2||$. Then

$$\begin{aligned} ||G_1 \otimes G_2|| &\leq 2 \times \frac{|G_1|(|G_1|-1)}{2} \times \frac{|G_2|(|G_2|-1)}{2} \\ &= \frac{|G_1||G_2|(|G_1|-1)(|G_2|-1)}{2} \\ &< \frac{|G_1||G_2|(|G_1||G_2|-1)}{2} . \end{aligned}$$

Then the claim (i) follows.

For the proof of claim (ii) see Theorem 1 in [4]. □

In [1], Balakrishnan obtained the following result:

Lemma 5. [1, Corollary 5.5] *If G_1 and G_2 are any two graphs, then*

$$E(G_1 \otimes G_2) = E(G_1)\, E(G_2) .$$

Combining Lemmas 4 and 5, we get:

Theorem 1. *Let G be a borderenergetic graph. Suppose that G is obtained from the tensor product of two integral graphs G_1 and G_2. Then both $|G_1|$ and $|G_2|$ are odd numbers.*

Proof. Because the energy of a graph is never an odd integer; see [3], we have that $E(G_i) = 2(|G_i| - k_i)$ for $i = 1, 2$, where k_1, k_2 are integers. Note that since G is borderenergetic, by Lemma 5 we have

$$4(|G_1| - k_1)(|G_2| - k_2) = 2(|G_1||G_2| - 1) \ .$$

That is,

$$|G_1||G_2| - 2k_1|G_2| - 2k_2|G_1| + 2k_1k_2 + 1 = 0 \, ,$$

which implies the result. □

Using Theorem 1, we can construct some borderenergetic graphs of small order. To ensure the connectedness, without loss of generality, suppose that G_1 contains odd cycles. Further, suppose that $G_1 = K_{|G_1|}$. Then $k_1 = 1$, $k_2 = \frac{|G_1||G_2| - 2|G_2| + 1}{2|G_1| - 2}$ and thus

$$E(G_2) = \frac{|G_1||G_2| - 1}{|G_1| - 1} \ . \tag{1}$$

Using the data from the Appendix of [6], we can verify that $K_3 \otimes K_3$, $K_5 \otimes G_1$, and $K_7 \otimes G_2$ are all connected non-complete borderenergetic graphs, where G_1 and G_2 are depicted in Fig. 4.

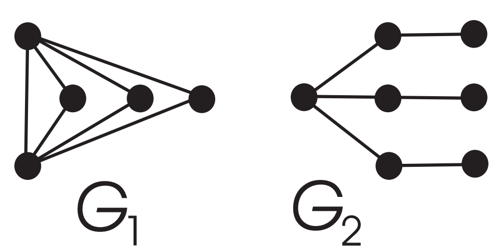

Fig. 4. Two integral graphs: G_1 has energy 6, G_2 has energy 8.

In Eq. (1), $E(G_2) = |G_2| + 1$ if $|G_1| = |G_2|$. Hence, if we can show that for some odd integer n, there exists a connected graph G of order n such that $E(G) = n + 1$, then we can obtain many non-complete borderenergetic graphs. However, this is not an easy task!

3.2 Line graph operation

Let G be a graph with edge set $E(G)$. The line graph of G, denoted by $L(G)$, is the graph whose vertex set is $E(G)$; two vertices of $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. Denote by $\phi(G, x)$ the characteristic polynomial of G. The following result is well known.

Lemma 6. [7, Lemma 8.2.5] *Suppose that G is a k-regular graph of order n. Then*

$$\phi(L(G), x) = (x + 2)^{kn/2-n}\, \phi(G, x - k + 2)\ .$$

Applying Lemma 6, we obtain

Theorem 2. *The line graph of the Petersen graph is a connected non-complete borderenergetic graph.*

Proof. Note that the spectrum of the Petersen graph is

$$\{3, 1, 1, 1, 1, 1, -2, -2, -2, -2\}\ .$$

Then the spectrum of the line graph of the Petersen graph is

$$\{4, -1, -1, -1, -1, 2, 2, 2, 2, 2, -2, -2, -2, -2, -2\}$$

and thus it is borderenergetic. ☐

3.3 Strongly regular graphs

Let G be a regular graph that is neither complete nor empty. Then G is said to be a strongly regular graph with parameters (n,k,a,c) if it is k-regular with order n, every pair of adjacent vertices has a common neighbors, and every pair of distinct non-adjacent vertices has c common neighbors. As known [6, 7], the eigenvalues of a strongly regular graph with parameters (n,k,a,c) are k with multiplicity 1, $\left[(a-c)+\sqrt{\Delta}\right]/2$ with multiplicity m_θ and $\left[(a-c)-\sqrt{\Delta}\right]/2$ with multiplicity m_τ, where

$$\Delta = (a-c)^2 + 4(k-c)$$

and m_θ, m_τ satisfy the equations

$$m_\theta + m_\tau = n-1 \quad , \quad m_\theta\,\theta + m_\tau\,\tau = -k\ . \tag{2}$$

Usually, a strongly regular graph with $m_\theta = m_\tau$ is called a conference graph. Thus we have

Theorem 3. *Let G be a conference graph. If G is integral and non-complete borderenergetic, then G has parameters* $(9,4,1,2)$.

Proof. Note that $m_\theta = m_\tau$. Then from (2) we get $m_\theta = m_\tau = \frac{1}{2}(n-1)$, and thus

$$\theta + \tau = -\frac{2k}{n-1}\,,$$

which implies that $-2k/(n-1)$ is an integer, since G is integral. Note that G is non-complete. Then $k = \frac{1}{2}(n-1)$, $a-c=-1$, and

$$\theta + \tau = -1\,. \tag{3}$$

Eq. (3) implies that the eigenvalue θ is non-negative and τ is negative. Thus

$$E(G) = k + \frac{1}{2}\,(n-1)(1+2\theta) = (n-1)(1+\theta)\ .$$

Recall that G is borderenergetic, that is, $E(G) = 2(n-1)$. Thus $\theta = 1$ and $\tau = -2$. Applying the relation between the sum of all the eigenvalues of the square of $\mathbf{A}(G)$ and its trace, we have

$$\frac{1}{2}\,(n-1)(\theta^2+\tau^2) + k^2 = 2||G|| = \frac{1}{2}\,(n-1)n\ ,$$

from which we find $n = 9$. Recall that $1 = \left[(a-c)+\sqrt{\Delta}\right]/2$ and $\Delta = (a-c)+4(k-c)$, implying $a = 1$ and $c = 2$. The result then follows. □

In addition, from Table 10.1 of [7], one can find that the strongly regular graph with parameters $(16, 5, 0, 2)$ is borderenergetic, since all the eigenvalues of such a graph are 5 with multiplicity 1, 1 with multiplicity 10 and -3 with multiplicity 5.

3.4 The union of graphs

Using the union of graphs, in this subsection we show that for each integer n , $n \geq 13$, there exists a non-complete borderenergetic graph of order n. The union of two vertex-disjoint graphs G_1 and G_2 is denoted by $G_1 \cup G_2$. The union of k vertex-disjoint copies of a graph H is sometimes denoted by $k\,H$.

Theorem 4. *For each integer n , $n \geq 13$, there exists a non-complete borderenergetic graph of order n.*

Proof. Denote by G_i $(i = 0, 1, \ldots, 11)$ the graphs $(K_3 \otimes K_4) \cup K_1$, $(K_3 \otimes K_4) \cup K_2$, $(K_3 \otimes K_4) \cup K_3$, the strongly regular graph with parameters $(16, 5, 0, 2)$, $(K_3 \otimes K_5) \cup 2K_1$, $(K_3 \otimes K_5) \cup K_2 \cup K_1$, $(K_3 \otimes K_5) \cup K_3 \cup K_1$, $(K_3 \otimes K_5) \cup K_3 \cup K_2$, $(K_3 \otimes K_5) \cup 2K_3$, $(K_3 \otimes K_5) \cup K_3 \cup K_4$, $(K_3 \otimes K_5) \cup 2K_4$, and $(K_3 \otimes K_5) \cup K_4 \cup K_5$, respectively. For any given n , $n \geq 13$, there exists a unique pair of integers p, q such that

$$n = 13 + p \times 12 + q$$

where $0 \leq q \leq 11$.

Denote for brevity the graph $K_3 \otimes K_4$ by H. One can easily verify that $E(H) = 24$ and $E(G_i) = 2(|G_i| - 1)$ for $i = 0, 1, \ldots, 11$. Therefore,

$$E(p\,H \cup G_q) = 24p + 2(|G_i| - 1) = 2(n-1)\ .$$

Then the result follows. □

3.5 Complement

Lemma 7. [7, Lemma 8.5.1] *Let G be a k-regular graph of order n with spectrum $Sp(G) = \{k, \lambda_2, \ldots, \lambda_n\}$. Then the spectrum of the complement $\overline{G}$ of G is $Sp(\overline{G}) = \{n-1-k, -1-\lambda_2, \ldots, -1-\lambda_n\}$.*

Theorem 5. *Let p, q, and r be non-negative integers, and let $p+q=2$. Then $\overline{p\,C_4 \cup q\,C_6 \cup r\,C_3}$ is borderenergetic.*

Proof. Note that $Sp(C_4) = \{2, -2, 0, 0\}$, $Sp(C_6) = \{2, 1, 1, -1, -1, -2\}$, and $Sp(C_3) = \{2, -1, -1\}$. Then the order of $\overline{p\,C_4 \cup q\,C_6 \cup r\,C_3}$ is $4p+6q+3r$ and by Lemma 7, the eigenvalues of $\overline{p\,C_4 \cup q\,C_6 \cup r\,C_3}$ are $4p+6q+3r-3$ with multiplicity 1, -3 with multiplicity $p+q+r-1$, 1 with multiplicity $p+q$, 0 with multiplicity $2q+2r$, -1 with multiplicity $2p$, and -2 with multiplicity $2q$. Consequently,

$$E(\overline{p\,C_4 \cup q\,C_6 \cup r\,C_3}) = 10p + 14q + 6r - 6\ .$$

Recall that $p+q=2$. Then

$$E(\overline{p\,C_4 \cup q\,C_6 \cup r\,C_3}) = 2(4p + 6q + 3r - 1)$$

which means that $\overline{p\,C_4 \cup q\,C_6 \cup r\,C_3}$ is borderenergetic. ◻

Corollary 8. *For each integer n , $n \geq 7$, there exists a connected non-complete borderenergetic graph of order n.*

Proof. From Section 2, we know that there are connected non-complete borderenergetic graphs of order 7, 8, and 9. For each integer n , $n \geq 10$, let

$$r = \begin{cases} \left\lfloor \dfrac{n-8}{3} \right\rfloor, & \text{if } n - 3\left\lfloor \dfrac{n-8}{3} \right\rfloor \text{ is even;} \\[2ex] \left\lfloor \dfrac{n-8}{3} \right\rfloor - 1, & \text{otherwise,} \end{cases}$$

$q = (n-3r-8)/2$, and $p = 2-q$. Then p, q, and r satisfy the conditions of Theorem 5 and thus $\overline{p\,C_4 \cup q\,C_6 \cup r\,C_3}$ is borderenergetic. By a well known result of graph theory, since $p\,C_4 \cup q\,C_6 \cup r\,C_3$ is disconnected, its complement must be connected. The result thus follows. ◻

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