More on the colorful monochromatic connectivity*

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Abstract

An edge-coloring of a connected graph is a *monochromatically-connecting coloring* (MC-coloring, for short) if there is a monochromatic path joining any two vertices, which was introduced by Caro and Yuster. Let $mc(G)$ denote the maximum number of colors used in an MC-coloring of a graph G. Note that an MC-coloring does not exist if G is not connected, in which case we simply let $mc(G) = 0$. In this paper, we characterize all connected graphs of size m with $mc(G) = 1, 2, 3, 4$, $m-1$, $m-2$ and $m-3$, respectively. We use $G(n,p)$ to denote the Erdős-Rényi random graph model, in which each of the $\binom{n}{2}$ pairs of vertices appears as an edge with probability p independent from other pairs. For any function $f(n)$ satisfying $1 \leq f(n) < \frac{1}{2}n(n-1)$, we show that if $\ell n \log n \leq f(n) < \frac{1}{2}n(n-1)$, where $\ell \in \mathbb{R}^+$, then $p = \frac{f(n)+n\log\log n}{n^2}$ is a sharp threshold function for the property $mc\left(G\left(n,p\right)\right) \geq f(n)$; if $f(n) = o(n \log n)$, then $p = \frac{\log n}{n}$ is a sharp threshold function for the property $mc\left(G\left(n,p\right)\right) \geq f(n)$.

Keywords: coloring; monochromatic; connectivity; random graphs.
AMS subject classification 2010: 05C15, 05C40, 05C80.

1 Introduction

All graphs in this paper are undirected, finite and simple. We follow [2] for graph theoretical notation and terminology not defined here. Let G be a nontrivial con-

*Supported by NSFC No.11371205 and 11531011, and PCSIRT.

nected graph with an *edge-coloring* $c : E(G) \to \{1, 2, \dots, t\}$, $t \in \mathbb{N}$, where adjacent edges may have the same color. A path of G is said to be a *rainbow path* if no two edges on the path have the same color. An edge-colored graph is *rainbow connected* if there is a rainbow path connecting any two vertices. An edge-coloring of a connected graph is a *rainbow connecting coloring* if it makes the graph rainbow connected. The concept of rainbow connection of graphs was introduced by Chartrand et al. in [8]. The *rainbow connection number* $rc(G)$ of a connected graph G, is the smallest number of colors that are needed in order to make G rainbow connected. The *rainbow k-connectivity* of a connected graph G, denoted by $rc_k(G)$, is defined as the minimum number of colors in an edge-coloring of G such that every two distinct vertices of G are connected by k internally disjoint rainbow paths. These concepts were introduced by Chartrand et al. in [8, 9]. Recently, a lot of relevant results have been published; see [7, 13, 16, 17, 18]. The interested readers can see [14, 15] for a survey on this topic.

In 2011, Caro and Yuster [6] introduced a natural opposite concept of rainbow connecting colorings, which is called the monochromatically-connecting coloring. An edge-coloring of a connected graph is a *monochromatically-connecting coloring* (MC-coloring, for short) if there is a monochromatic path joining any two vertices. Let $mc(G)$ denote the maximum number of colors used in an MC-coloring of a graph G. An important property of an extremal MC-coloring (a coloring that uses $mc(G)$ colors) is that the subgraph induced by edges with the same color forms a tree [6]. For a color c, the color tree T_c is the tree consisting of all the edges of G with color c. A color c is *nontrivial* if T_c has at least two edges; otherwise, c is *trivial*. A nontrivial color tree with t edges is said to waste $t-1$ colors. Every connected graph G has an extremal MC-coloring such that for any two nontrivial colors c and d, the corresponding trees T_c and T_d intersect in at most one vertex [6]. Such an extremal coloring is called *simple*.

Here and throughout we use n and m to denote the number of vertices and edges of a connected graph G, respectively, and denote $diam(G)$ be the diameter of G, $\Delta(G)$ the maximum degree of G, $\delta(G)$ the minimum degree of G. Note that by simply coloring the edges of a spanning tree with one color, and assigning the remaining edges other distinct colors, we obtain an MC-coloring of a graph G. Noting that this MC-coloring provides a straightforward lower bound for $mc(G)$, we summarize that as a theorem below.

Theorem 1.1 *[6]* $mc(G) \geq m - n + 2$.

In particular, $mc(G) = m - n + 2$ whenever G is a tree. Caro and Yuster [6] also showed that there are dense graphs that still meet this lower bound.

Theorem 1.2 *[6] Let G be a connected graph with $n > 3$. If G satisfies any of the following properties, then $mc(G) = m - n + 2$.*

(a) *$\overline{G}$ (the complement of G) is 4-connected.*

(b) *G is triangle-free.*

(c) *$\Delta(G) < n - \frac{2m-3(n-1)}{n-3}$. In particular, this holds if $\Delta(G) \leq (n+1)/2$, and also holds if $\Delta(G) \leq n - 2m/n$.*

(d) *$Diam(G) \geq 3$.*

(e) *G has a cut vertex.*

However, some graphs can be colored with more colors. Indeed, in the extremal case one has $mc(K_n) = m = \binom{n}{2}$, and clearly $G = K_n$ is the only graph having $mc(G) = m$. Moreover, Caro and Yuster [6] gave some results for the bounds of $mc(G)$.

Theorem 1.3 *[6] Let G be a connected graph.*

(a) *$mc(G) \leq m - n + \chi(G)$, where $\chi(G)$ is the vertex chromatic number of G.*

(b) *If G is not k-connected, then $mc(G) \leq m - n + k$. This is sharp for any k.*

(c) *If G is a complete r-partite graph, then $mc(G) = m - n + r$. And if G contains a spanning complete r-partite graph, then $mc(G) \geq m - n + r$.*

Call a graph *s-perfectly-connected* if its vertex set can be partitioned into $s+1$ parts, $\{v\}, V_1, \ldots, V_s$, such that each V_i induces a connected subgraph, any pair V_i, V_j induces a corresponding complete bipartite graph, and v has precisely one neighbor in each V_i. Notice that such a graph has minimum degree s, and v has degree s.

Theorem 1.4 *[6] If $\delta(G) = s$, then $mc(G) \leq m - n + s$, unless G is s-perfectly-connected, in which case $mc(G) = m - n + s + 1$.*

Recently, Cai, Li and Wu [4] considered the following Erdős-Gallai-type problem for $mc(G)$: given two positive integers n and k with $1 \leq k \leq \binom{n}{2}$, compute the minimum integer $f(n,k)$ such that if a connected graph G on n vertices has at least $f(n,k)$ edges, then $mc(G) \geq k$. Also, they investigated another Erdős-Gallai-type problem for $mc(G)$: given two positive integers n and k with $1 \leq k \leq \binom{n}{2}$, compute the maximum integer $g(n,k)$ such that if a connected graph G on n vertices has at most $g(n,k)$ edges, then $mc(G) \leq k$. They completely solved these two problems and gave the exact values for $f(n,k)$ and $g(n,k)$, the details of them are omitted.

In this paper, we characterize all connected graphs with size m for which $mc(G)$ is 1, 2, 3, 4, $m-1$, $m-2$ and $m-3$ respectively, in Section 2. And in Section 3, we study the parameter $mc(G)$ of random graphs. The most frequently occurring probability models of random graphs is the Erdős-Rényi random graph model $G(n,p)$ [10]. The model $G(n,p)$ consists of all graphs with n vertices in which the edges are chosen independently and with probability p. We say an event $\mathcal{A}$ happens *with high probability* if the probability that it happens approaches 1 as $n \to \infty$, i.e., $Pr[\mathcal{A}] = 1 - o_n(1)$. Sometimes, we say *w.h.p.* for short. We will always assume that n is the variable that tends to infinity.

Let G, H be two graphs on n vertices. A property P is said to be *monotone* if whenever $G \subseteq H$ and G satisfies P, then H also satisfies P. For a graph property P, a function $p(n)$ is called a threshold function of P if:

- for every $r(n) = \omega(p(n))$, $G(n, r(n))$ w.h.p. satisfies P; and
- for every $r'(n) = o(p(n))$, $G(n, r'(n))$ w.h.p. does not satisfy P.

Furthermore, $p(n)$ is called a sharp threshold function of P if there exist two positive constants c and C such that:

- for every $r(n) \geq C \cdot p(n)$, $G(n, r(n))$ w.h.p. satisfies P; and
- for every $r'(n) \leq c \cdot p(n)$, $G(n, r'(n))$ w.h.p. does not satisfy P.

In the extensive study of the properties of random graphs, many researchers observed that there are sharp threshold functions for various natural graph properties. It is well known that all monotone graph properties have a sharp threshold function; see [3] and [11]. For the property $rc(G(n,p)) \leq 2$, Caro et al. [5] proved that $p = \sqrt{\log n/n}$ is the sharp threshold function. He and Liang [12] studied further the rainbow connectivity of random graphs. Specifically, they obtained that $(\log n)^{(1/d)}/n^{(d-1)/d}$ is the sharp threshold function for the property $rc(G(n,p)) \leq d$, where d is a constant.

One topic of working on the monochromatic connectivity of a graph is to find as many colors as possible to keep the graph monochromatic connected. Also, it is natural to ask what kind of graphs having large $mc(G)$. That is, we can use a great many colors to make the graph monochromatic connected. Furthermore, what if we require the number of colors relating to the order of the graph? So it is interesting to consider the threshold function of property $mc(G(n,p)) \geq f(n)$, where $f(n)$ is a function of n. For any graph G with n vertices and any function $f(n)$, having

$mc(G) \geq f(n)$ is a monotone graph property (adding edges does not destroy this property), so it has a sharp threshold function. Realize that on the sharp threshold functions for rainbow connectivity of random graphs, the known results all require that the number of colors is independent with the order of the random graph, but our result dose not have that restriction. Our main result is as follows.

Theorem 1.5 *Let $f(n)$ be a function satisfying $1 \leq f(n) < \frac{1}{2}n(n-1)$. Then*

$$p = \begin{cases} \frac{f(n)+n\log\log n}{n^2} & \text{if } \ell n \log n \leq f(n) < \frac{1}{2}n(n-1), \text{ where } \ell \in \mathbb{R}^+, \\ \frac{\log n}{n} & \text{if } f(n) = o(n\log n). \end{cases}$$

is a sharp threshold function for the property $mc\left(G\left(n,p\right)\right) \geq f(n)$.

Remark. Note that $mc\left(G\left(n,p\right)\right) \leq \frac{1}{2}n(n-1)$ for any function $0 \leq p \leq 1$, and $mc\left(G\left(n,p\right)\right) = \frac{1}{2}n(n-1)$ if and only if $G(n,p)$ is isomorphic to the complete graph K_n. Hence we only concentrate on the case $f(n) < \frac{1}{2}n(n-1)$.

2 Characterize graphs having small or large monochromatic connection numbers

Now we characterize all connected graphs G of size m with $mc(G) = 1, 2, 3, 4, m-1, m-2, m-3$, respectively. At first we recall some terminology and notation. For a graph G of order n and size m, the cyclomatic number $c(G)$ of G is defined as $m-n+1$. We call a graph G acyclic (or a tree), unicyclic, bicyclic, or tricyclic if $c(G) = 0$, 1, 2 or 3, respectively.

Theorem 2.1 *Let G be a connected graph. Then $mc(G) = 1$ if and only if G is a tree.*

Proof. Let G be a tree. Then $m = n-1$ and G has a vertex of degree one. By Theorem 1.4 we have $mc(G) = m - n + 2 = 1$. Hence it remains to verify the converse. Let G be a connected graph with $mc(G) = 1$. By Theorem 1.1, we get that $m \leq n-1$. Since G is a connected graph, it follows that $m = n-1$, so G is a tree. $\square$

Theorem 2.2 *Let G be a connected graph. Then $mc(G) = 2$ if and only if G is a unicyclic graph except for K_3.*

Proof. Let G be a unicyclic graph and $G \neq K_3$. Then, $m = n$. If G has a vertex of degree one, then G is 1-perfectly-connected. Thus we have $mc(G) = m - n + 2 = 2$ by Theorem 1.4. Now assume that G has no vertex of degree one, thus G is a cycle. It is easily seen that G is not 2-perfectly-connected. Thus $mc(G) = m-n+2 = 2$ by Theorem 1.1 and 1.4. Conversely, let G be a connected graph with $mc(G) = 2$. We deduce that $m \leq n$ from Theorem 1.1. Moreover, we know that $m = n$ by Theorem 2.1. Since $mc(K_3) = 3$, it follows that G is a unicyclic graph and $G \neq K_3$. □

Theorem 2.3 *Let G be a connected graph. Then $mc(G) = 3$ if and only if G is either K_3 or a bicyclic graph except for $K_4 - e$.*

Proof. If G is K_3, then clearly $mc(K_3) = 3$. Now let G be a bicyclic graph and $G \neq K_4 - e$. Then $m = n + 1$. If G has a cut vertex, then $mc(G) = m - n + 2 = 3$ by Theorem 1.2. If G has no cut vertex, then G is a Θ-graph, i.e., a graph which consists of three internally disjoint paths with common end vertices. Since the only 2-perfectly-connected Θ-graph is K_4-e, we have that G is not 2-perfected-connected, thus $mc(G) = m - n + 2 = 3$ by Theorem 1.1 and 1.4.

For the converse. Let G be a connected graph with $mc(G) = 3$. First, we have $m \leq n + 1$ by Theorem 1.1. In addition, from Theorem 2.1 and 2.2 we have that $m = n + 1$ or $G = K_3$. Since $mc(K_4 - e) = 4$, it implies that G is either K_3 or a bicyclic graph except for $K_4 - e$. □

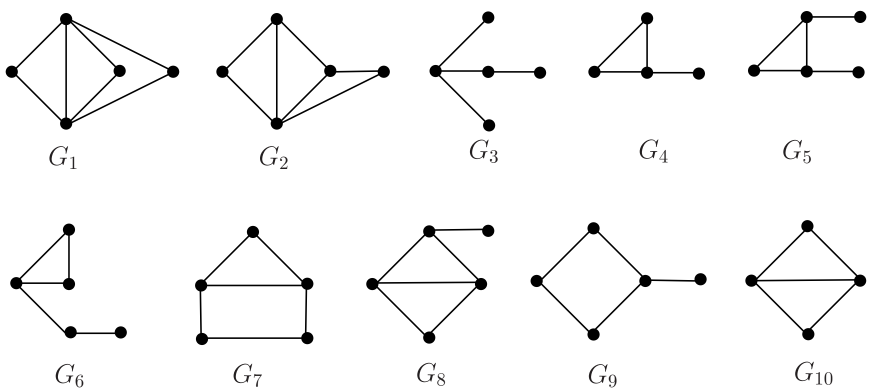

Figure 1: The graphs in Theorem 2.4 and Theorem 2.7

Theorem 2.4 *Let G be a connected graph. Then $mc(G) = 4$ if and only if G is either $K_4 - e$ or a tricyclic graph except for G_1, G_2, K_4, where G_1, G_2 are shown in Figure 1.*

Proof. If G is $K_4 - e$, then clearly $mc(K_4 - e) = 4$. Now let G be a tricyclic graph and $G \notin \{G_1,\ G_2,\ K_4\}$. Then $m = n + 2$. If G has a cut vertex, then $mc(G) = m - n + 2 = 4$ by Theorem 1.2. Now assume that G has no cut vertex. Since $G \neq K_4$, we have that G must contain a vertex with degree two. Since the only two 2-perfectly-connected tricyclic graphs are $G_1,\ G_2$, we further have that G is not 2-perfectly-connected. Thus $mc(G) = m - n + 2 = 4$ by Theorem 1.1 and 1.4.

For the converse. Let G be a connected graph with $mc(G) = 4$. It is easily seen that $m \leq n + 2$ by Theorem 1.1. From Theorem 2.1, 2.2 and 2.3 we get that $m = n + 2$ or $G = K_4 - e$. Since $G_1,\ G_2$ are 2-perfectly-connected, we have $mc(G_1) = mc(G_2) = 5$, and since $mc(K_4) = 6$, it follows that G is either $K_4 - e$ or a tricyclic graph except for $G_1,\ G_2,\ K_4$. □

In order to characterize all connected graphs G of size m having $mc(G) = m - 1, m - 2, m - 3$, we first present some useful results. Let S_n be a star with order n and let tK_2 be t nonadjacent edges of K_n where $t \leq \lfloor \frac{n}{2} \rfloor$. Clearly, $K_n - tK_2 = K_{2,\ldots,2,1,\ldots,1}$, which is a complete $(n - t)$-partite graph. By Theorem 1.3 we have

Corollary 2.1 *Let $G = K_n - tK_2$, where $t \leq \lfloor \frac{n}{2} \rfloor$. Then $mc(G) = m - t$.*

By the definition of s-perfectly-connected graphs, we have

Observation 2.1 *Let G be a connected graph. Then*

(1) *G is $(n - 2)$-perfectly-connected if and only if $G = K_n - e$.*

(2) *G is $(n-3)$-perfectly-connected if and only if $G \in \{K_n - P_3,\ K_n - K_3,\ K_n - P_4\}$.*

(3) *G is $(n-4)$-perfectly-connected if and only if $G \in \{K_n - S_4,\ K_n - K_4,\ K_n - G_i\}$, where i is an integer with $3 \leq i \leq 10$ and G_i is shown in Figure 1.*

Now, we are ready to characterize graphs with large monochromatic connectivity.

Theorem 2.5 *Let G be a connected graph. Then $mc(G) = m - 1$ if and only if $G = K_n - e$.*

Proof. If $G = K_n - e$, then $mc(G) = m - 1$ by Theorem 1.4 and Observation 2.1. Conversely, let G be a graph with $mc(G) = m - 1$. It is easily seen that $G \neq K_n$ and $\delta(G) \leq n - 2$. By Theorem 1.4 we have $mc(G) \leq m - n + \delta(G) + 1 \leq m - 1$, and the equalities holds if and only if G is $(n - 2)$-perfectly-connected. Therefore, $G = K_n - e$. □

Theorem 2.6 *Let G be a connected graph. Then $mc(G) = m - 2$ if and only if $G \in \{K_n - 2K_2,\ K_n - P_3,\ K_n - K_3,\ K_n - P_4\}$.*

Proof. If G is $K_n - 2K_2$, then clearly by Corollary 2.1 we have $mc(K_n - 2K_2) = m - 2$. Now let $G \in \{K_n - P_3,\ K_n - K_3,\ K_n - P_4\}$. From Observation 2.1 we know that G is $(n-3)$-perfectly-connected. Then $mc(G) = m - 2$ by Theorem 1.4. Hence it remains to verify the converse. Let G be a graph with $mc(G) = m - 2$. It is easily seen that $G \neq K_n$ and $\delta(G) \leq n - 2$. If $\delta(G) \leq n - 4$, then $mc(G) \leq m - 3$ by Theorem 1.4, a contradiction. So we divide the proof into the following two cases.

Case 1: $\delta(G) = n - 2$.

Viewing the proof of Theorem 2.5, we just need to consider the graphs which are not $(n-2)$-perfectly-connected. Since $G \neq K_n - e$, we have that G is obtained from K_n by deleting at least 2 nonadjacent edges. Furthermore, by Corollary 3.1, we have $G = K_n - 2K_2$.

Case 2: $\delta(G) = n - 3$.

By Theorem 1.4 we just need to consider all $(n-3)$-perfectly-connected graphs. From Observation 2.1 we know that the $(n-3)$-perfectly-connected graphs are $K_n - P_3,\ K_n - K_3,\ K_n - P_4$. □

Theorem 2.7 *Let G be a connected graph. Then $mc(G) = m - 3$ if and only if $G = F$, where $F \in \{3K_2,\ C_4,\ C_5,\ P_2 \cup P_3,\ P_2 \cup K_3,\ P_5,\ P_2 \cup P_4,\ S_4,\ K_4,\ G_i\}$, here i is an integer with $3 \leq i \leq 10$ and G_i is shown in Figure 1.*

Proof. Let G be a graph with $mc(G) = m - 3$. Obviously, $\delta(G) \leq n - 2$. If $\delta(G) \leq n - 5$, then $mc(G) \leq m - 4$ by Theorem 1.4, a contradiction. If $\delta(G) = n - 2$, then similar to Case 1 of Theorem 2.6, we have $G = K_n - 3K_2$. So we divide the proof into the following two cases.

Case 1: $\delta(G) = n - 3$.

Viewing the proof of Theorem 2.6, we just need to consider the graphs which are not $(n-3)$-perfectly-connected. Let $\overline{G}$ be the complement of G and $\overline{G}^*$ be the edge-induced subgraph of $\overline{G}$. Let $e(\overline{G}^*)$ be the number of edges of $\overline{G}^*$. Since $\delta(G) = n - 3$, we have that $\Delta(\overline{G}^*) = 2$ and $\overline{G}^*$ is the union of paths and cycles.

Let f be a simple extremal MC-coloring of G. Suppose that f consists of k nontrivial color trees, denoted $T_1, \ldots, T_k$, where $t_i = |T_i|$ and $t_1 \geq t_2 \geq \ldots \geq t_k$. Since T_i has $t_i - 1$ edges, we have that it wastes $t_i - 2$ colors. While $mc(G) = m - 3$, they exactly wastes 3 colors in all. Hence $\sum_{i=1}^{k}(t_i - 2) = 3$. Since $t_i \geq 3$, we have $k \leq 3$.

Thus there are the following three cases: (1) $k = 1$ and $t_1 = 5$; (2) $k = 2$ and $t_1 = 4$, $t_2 = 3$; (3) $k = 3$ and $t_1 = t_2 = t_3 = 3$. Since T_i can monochromatically connect at most $\binom{t_i-1}{2}$ pairs of non-adjacent vertices in G, we must have $e(\overline{G}^*) \leq \sum_{i=1}^{k} \binom{t_i-1}{2}$. So for the case (1), $e(\overline{G}^*) \leq 6$; for the case (2), $e(\overline{G}^*) \leq 4$; for the case (3), $e(\overline{G}^*) \leq 3$. Since any two adjacent vertices of $\overline{G}^*$ must have a nontrivial color tree connecting them in G, we have that all vertices of $\overline{G}^*$ must appear in some nontrivial color trees and any two nontrivial color trees intersect in at most one vertex.

Now we consider the order of $\overline{G}^*$. By Observation 2.1 and G is not $(n-3)$-perfectly-connected, we know that $\overline{G}^* \notin \{P_3,\ K_3,\ P_4\}$. So $|V(\overline{G}^*)| \geq 4$. Notice that $\Delta(\overline{G}^*) = 2$. If $|V(\overline{G}^*)| = 4$, then $\overline{G}^* = C_4$. If $G = K_n - C_4$, then G is not $(n-3)$-perfectly-connected and there is a simple extremal MC-coloring of G satisfying the case (1). If $|V(\overline{G}^*)| = 5$, then $\overline{G}^* \in \{C_5,\ P_5,\ P_2 \cup P_3,\ P_2 \cup K_3\}$. If $G \in \{K_n - C_5,\ K_n - P_5\}$, then G is not $(n-3)$-perfectly-connected and there is a simple extremal MC-coloring of G satisfying the case (1). If $G \in \{K_n - (P_2 \cup P_3),\ K_n - (P_2 \cup K_3)\}$, then G is not $(n-3)$-perfectly-connected. We can find a simple extremal MC-coloring of G in which the two nontrivial color trees are P_3 and S_4, and this coloring satisfies the case (2). So $mc(G) = m - 3$. Now let $|V(\overline{G}^*)| \geq 6$. Since all vertices of $\overline{G}^*$ must appear in some nontrivial trees, it follows that G must contain at least two nontrivial trees, otherwise contradicting the case (1). Note that now only the cases (2) and (3) can happen. From the above we know that $e(\overline{G}^*) \leq 4$. Since $\Delta(\overline{G}^*) = 2$, we have $e(\overline{G}^*) \geq (1+1+1+1+1+2)/2 = 3.5$. So we just need to consider the case that $e(\overline{G}^*) = 4$. It is easily seen that $\overline{G}^* \in \{P_2 \cup P_4,\ P_3 \cup P_3,\ P_2 \cup P_2 \cup P_3\}$ and G has exactly two nontrivial trees $T_1,\ T_2$ with $t_1 = 4$ and $t_2 = 3$. If $G = K_n - (P_2 \cup P_4)$, then we can find a simple extremal MC-coloring of G in which the two nontrivial color trees are P_3 and P_4. So $mc(G) = m - 3$. If $G = K_n - (P_3 \cup P_3)$, then set $V(\overline{G}^*) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $E(\overline{G}^*) = \{u_1u_2, u_2u_3, u_4u_5, u_5u_6\}$. Note that u_1, u_2 are in a nontrivial color tree and u_2, u_3 are also in a nontrivial color tree, so either u_2 is the common vertex of two nontrivial color trees or u_1, u_2, u_3 are in the same tree. In either case, we can find that either u_4, u_5 or u_5, u_6 have no monochromatic path connecting them, a contradiction. If $G = K_n - (P_2 \cup P_2 \cup P_3)$, then we can also get a contradiction by an argument similar to the case that $G = K_n - \{P_3 \cup P_3\}$.

Case 2: $\delta(G) = n - 4$.

By Theorem 1.4 we just need to consider all $(n-4)$-perfectly-connected graphs. From Observation 2.1 we know that the $(n-4)$-perfectly-connected graphs are $\{K_n - S_4,\ K_n - K_4,\ K_n - G_i\}$, where i is an integer with $3 \leq i \leq 10$ and G_i is shown in Figure 1.

For the converse. Clearly by Corollary 2.1 we have $mc(K_n - 3K_2) = m - 3$. If $G \in \{K_n - S_4,\ K_n - K_4,\ K_n - G_i\}$, where i is an integer with $3 \leq i \leq 10$, then by Observation 2.1 we know that G is $(n-4)$-perfectly-connected. Thus $mc(G) = m-3$ by Theorem 1.4. Now let $G \in \{K_n - C_4,\ K_n - C_5,\ K_n - (P_2 \cup P_3),\ K_n - (P_2 \cup K_3),\ K_n - P_5,\ K_n - (P_2 \cup P_4)\}$. Since $\delta(G) = n-3$ and from Observation 2.1 we know that G is not $(n-3)$-perfectly-connected, thus $mc(G) \leq m - 3$. On the other hand, we can color the graphs with $m-3$ colors to make the graphs monochromatically connected. Hence $mc(G) = m - 3$. □

Remark: We can also characterize the graphs G with $mc(G) = m-4$. But the proof is similar to the above ones, and very long and tedious, and therefore not written down here.

From the above theorems, we can easily verify the following corollary.

Corollary 2.2 *Let G be a connected graph of order n. Then*

(1) $mc(G) \neq \binom{n}{2} - 1$, $mc(G) \neq \binom{n}{2} - 3$.

(2) $mc(G) = \binom{n}{2} - 2$ *if and only if* $G = K_n - e$.

(3) $mc(G) = \binom{n}{2} - 4$ *if and only if* $G \in \{K_n - 2K_2, K_n - P_3\}$.

3 Colorful monochromatic connectivity of random graphs

In this section, we study the colorful monochromatic connectivity of random graphs. The following version of Chernoff bound is very useful for our proof.

Lemma 3.1 *[1] **(Chernoff Bound)** If X is a binomial random variable with expectation μ, and $0 < \delta < 1$, then*

$$\Pr[X < (1-\delta)\mu] \leq \exp\left(-\frac{\delta^2\mu}{2}\right)$$

and if $\delta > 0$,

$$\Pr[X > (1+\delta)\mu] \leq \exp\left(-\frac{\delta^2\mu}{2+\delta}\right).$$

Throughout the paper "log" denotes the natural logarithm. The following theorem is a classical result on the connection of a random graph.

Theorem 3.1 *[10] Let* $p = (\log n + a)/n$. *Then*

$$Pr[G(n,p) \text{ is connected})] \to \begin{cases} e^{-e^{-a}} & \text{if } |a| = O(1), \\ 0 & a \to -\infty, \\ 1 & a \to +\infty. \end{cases}$$

From Theorem 3.1 and the definition of sharp threshold functions, we can derive the following corollary immediately.

Corollary 3.1 $p = \frac{\log n}{n}$ *is a sharp threshold function for* $G(n,p)$ *being connected.*

Now we prove Theorem 1.5. According to the range of $f(n)$, we have the following two cases.

Case 1. $\ell n \log n \le f(n) < \frac{1}{2}n(n-1)$, where $\ell \in \mathbb{R}^+$.

To establish a sharp threshold function for a graph property, the proof should be two-fold. We first show one direction.

Lemma 3.2 *There exists a constant* C *such that* $mc\left(G\left(n, C\frac{f(n)+n\log\log n}{n^2}\right)\right) \ge f(n)$ *w.h.p. holds.*

Proof. Let

$$C = \begin{cases} 5 & \text{if } n\log n \le f(n) < \frac{1}{2}n(n-1), \\ \frac{5}{\ell} & \text{if } f(n) = \ell n \log n, \text{ where } \ 0 < \ell < 1 \end{cases}$$

and $p = \frac{f(n)+n\log\log n}{n^2}$. By Theorem 3.1, it is easy to check that $G(n, Cp)$ is w.h.p. connected. Let μ_1 be the expectation of the number of edges in $G(n, Cp)$. So

$$\mu_1 = \frac{n(n-1)}{2} \cdot Cp = \frac{C}{2}\left(\frac{n-1}{n}f(n) + (n-1)\log\log n\right).$$

From Lemma 3.1, we have

$$\Pr[|E(G(n,Cp))| < \frac{\mu_1}{2}] \le \exp\left(-\frac{1}{2}\cdot\frac{1}{4}\mu_1\right) = \exp\left(-\frac{1}{8}\mu_1\right) = o(1).$$

Note that if $|E(G(n,Cp))| \ge \frac{\mu_1}{2}$, then by Theorem 1.1, we have that

$$\begin{aligned} mc\left(G\left(n,Cp\right)\right) &\ge |E(G(n,Cp))| - n + 2 \\ &\ge \frac{\mu_1}{2} - n + 2 \\ &= \frac{C}{4}\left(\frac{n-1}{n}f(n) + (n-1)\log\log n\right) - n + 2 \\ &\ge \frac{5}{4}\left(\frac{n-1}{n}f(n) + (n-1)\log\log n\right) - n + 2 \\ &\ge f(n), \end{aligned}$$

for n sufficiently large. Thus, we obtain that with probability at least $1-\exp\left(-\frac{1}{8}\mu_1\right) = 1-o(1)$, $mc\,(G\,(n, Cp)) \geq f(n)$. □

Now we show the other direction.

Lemma 3.3 *$mc\left(G\left(n, \frac{f(n)+n\log\log n}{n^2}\right)\right) < f(n)$ w.h.p. holds.*

Proof. Let $p = \frac{f(n)+n\log\log n}{n^2}$ and μ_2 be the expectation of the number of edges in $G\,(n,p)$. We have

$$\mu_2 = \frac{n(n-1)}{2} \cdot p = \frac{1}{2}\left(\frac{n-1}{n}f(n) + (n-1)\log\log n\right).$$

We obtain that

$$\Pr[|E(G(n,p))| > \frac{3}{2}\mu_2] \leq \exp\left(-\frac{\frac{1}{4}\mu_2}{2+\frac{1}{2}}\right) = \exp\left(-\frac{1}{10}\mu_2\right) = o(1)$$

by Lemma 3.1. If $G(n,p)$ is not connected, then $mc\,(G\,(n,p)) = 0 < f(n)$. If $G(n,p)$ is connected, let d denote the minimum degree of $G(n,p)$, it is obvious that $d < n$. If $|E(G(n,p))| \leq \frac{3}{2}\mu_2$, then from Theorem 1.4, we have that

$$\begin{aligned}
mc\,(G\,(n,p)) &\leq |E(G(n,p))| - n + d + 1\\
&\leq \frac{3}{2}\mu_2 - n + d + 1\\
&= \frac{3}{4}\left(\frac{n-1}{n}f(n) + (n-1)\log\log n\right) - n + d + 1\\
&< \frac{3}{4}\left(\frac{n-1}{n}f(n) + (n-1)\log\log n\right) - n + n + 1\\
&< f(n).
\end{aligned}$$

Hence, we have that with probability at least $1-\exp\left(-\frac{1}{10}\mu_2\right) = 1-o(1)$, $mc\,(G\,(n,p)) < f(n)$ holds. □

Case 2. $f(n) = o(n\log n)$ or $f(n)$ is a constant.

By Corollary 3.1 we have that there exist two positive constants c_1 and c_2 such that: for every $r(n) \geq c_1 \cdot p$, $G(n, r(n))$ is w.h.p. connected; and for every $r'(n) \leq c_2 \cdot p$, $G(n, r'(n))$ is w.h.p. not connected. Moreover, $|E(G(n, c_1 \cdot p))| = O(n\log n)$ by Lemma 3.1. Hence, $mc(G(n, r(n))) \geq |E(G(n, r(n)))| - n + 2 \geq f(n)$, for $r(n) \geq c_1 \cdot p$. On the other hand, since $G(n, r'(n))$ is w.h.p. not connected, for every $r'(n) \leq c_2 \cdot p$, $mc(G(n, r'(n))) = 0 < f(n)$ w.h.p. holds.

Combining Case 1 and Case 2, our result follows. ∎

Acknowledgement: The authors are very grateful to the reviewers for their valuable suggestions and comments, which helped to improve the presentation of the paper.

References

[1] N. Alon, J. Spencer, *The Probabilistic Method*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., third edition, 2008.

[2] J. A. Bondy, U. S. R. Murty, *Graph Theory*, GTM 244, Springer, 2008.

[3] B. Bollobás, A. Thomason, Threshold functions, *Combinatorica* **7**(1986), 35–38.

[4] Q. Cai, X. Li, D. Wu, Erdős-Gallai-type results for colorful monochromatic connectivity of a graph, *J. Combin. Optim.*, in press.

[5] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, R. Yuster, On rainbow connection, *Electron. J. Comb.* **15**(2008) #R57.

[6] Y. Caro, R. Yuster, Colorful monochromatic connectivity, *Discrete Math.* **311**(2011), 1786–1792.

[7] L. Chen, X. Li, K. Yang, Y. Zhao, The 3-rainbow index of a graph, *Discuss. Math. Graph Theory*, **35** (2015) 81–94.

[8] G. Chartrand, G. L. Johns, K. A. McKeon, P. Zhang, Rainbow connection in graphs, *Math. Bohem.* **133** (2008), 85–98.

[9] G. Chartrand, G. Johns, K. McKeon, P. Zhang, The rainbow connectivity of a graph, *Networks* **54**(2)(2009), 75–81.

[10] P. Erdös, A. Rényi, On the evolution of random graphs, *Publ. Math. Inst. Hungar. Acad. Sci.* **5**(1960), 17–61.

[11] E. Friedgut, G. Kalai, Every monotone graph property has a sharp threshold, *Proc. Amer. Math. Soc.* **124**(1996), 2993–3002.

[12] J. He, H. Liang, On rainbow-k-connectivity of random graphs, *Infor. Process. Lett.* **112**(2012), 406–410.

[13] X. Huang, X. Li, Y. Shi, Note on the hardness of rainbow connections for planar and line graphs, *Bull. Malays. Math. Sci. Soc.* **38**(2015), 1235–1241.

[14] X. Li, Y. Shi, Y. Sun, Rainbow connections of graphs: A survey, *Graph & Combin.* 29(2013), 1–38.

[15] X. Li, Y. Sun, *Rainbow Connections of Graphs*, Springer Briefs in Math., Springer, New York, 2012.

[16] X. Li, Y. Sun, Y. Zhao, Characterization of graphs with rainbow connection number $m-2$ and $m-3$, *Australas. J. Combin.* **60**(2014), 306–313.

[17] X. Li, I. Schiermeyer, K. Yang, Y. Zhao, Graphs with 4-rainbow index 3 and $n-1$, *Discuss. Math. Graph Theory* **35**(2015), 1–12.

[18] X. Li, K. Yang, Y. Zhao, I. Schiermeyer, Graphs with 3-rainbow index $n-1$ and $n-2$, *Discuss. Math. Graph Theory* **35**(2015), 105–120.