

Nordhaus-Gaddum-type results for the generalized edge-connectivity of graphs*

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Abstract

For a graph G and a set S of vertices of G , let $\lambda(S)$ denote the maximum number ℓ of pairwise edge-disjoint Steiner trees T_1, T_2, \dots, T_ℓ in G such that $S \subseteq V(T_i)$ for every $1 \leq i \leq \ell$. For an integer k with $2 \leq k \leq n$, where n is the order of G , the generalized k -edge-connectivity $\lambda_k(G)$ of G is defined as $\lambda_k(G) = \min\{\lambda(S) \mid S \subseteq V(G) \text{ and } |S| = k\}$. In this paper, we consider the Nordhaus-Gaddum-type results for the parameter $\lambda_k(G)$. We obtain sharp upper and lower bounds of $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$ for a graph G of order n , as well as a graph G of order n and size m . Some graph classes attaining these bounds are also given.

Keywords: edge-connectivity; Steiner tree; edge-disjoint Steiner trees; generalized edge-connectivity; packing; complementary graph; Nordhaus-Gaddum-type result.

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1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to the book [4] for graph theoretical notation and terminology not described here. For a graph $G(V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, an S -Steiner tree or a Steiner tree connecting S (a Steiner tree for short) is a subgraph $T(V', E')$ of G that is a tree with $S \subseteq V'$. Two Steiner trees T and T' connecting S are *edge-disjoint* if $E(T) \cap E(T') = \emptyset$. The *Steiner Tree Packing Problem* for a given graph $G(V, E)$ and $S \subseteq V(G)$ asks to find a set of maximum number of edge-disjoint S -Steiner trees in G . This problem has obtained wide attention and many results have been obtained, see [7, 8, 10, 11, 24, 26]. The problem for $S = V(G)$ is called the *Spanning Tree Packing Problem*. For any graph G of order n , the *spanning tree packing number* or *STP number*, is the maximum number of edge-disjoint spanning trees contained in G . For the *STP number*, we refer the reader to Palmer's survey [23].

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Recently, we introduced the concept of the generalized edge-connectivity of a graph G in [21]. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local edge-connectivity* $\lambda(S)$ is the maximum number of edge-disjoint Steiner trees connecting S in G . Note that when $|S| = 2$ a minimum Steiner tree connecting S is just a path connecting S . For an integer k with $2 \leq k \leq n$, where n is the order of G , the *generalized k -edge-connectivity* $\lambda_k(G)$ of a graph G is defined as $\lambda_k(G) = \min\{\lambda(S) : S \subseteq V(G) \text{ and } |S| = k\}$. Clearly, when $|S| = 2$, $\lambda_2(G)$ is nothing new but the edge-connectivity $\lambda(G)$ of G , that is, $\lambda_2(G) = \lambda(G)$, which is the reason why we address $\lambda_k(G)$ as the generalized k -edge-connectivity of G . Obviously, the *STP* number of a graph G is just $\lambda_n(G)$. By convention, for a connected graph G with less than k vertices, we set $\lambda_k(G) = 1$, and set $\lambda_k(G) = 0$ when G is disconnected. $\lambda_k(G)$ is called the generalized k -edge-connectivity also because it is a natural counterpart of the concept of the generalized (vertex) connectivity, introduced by Chartrand et al. [5] in 1984. Results on the generalized connectivity can be seen in [12, 13, 14, 15, 17, 18, 19, 20, 21].

Let $\mathcal{G}(n)$ denote the class of simple graphs of order n ($n \geq 2$) and $\mathcal{G}(n, m)$ the subclass of $\mathcal{G}(n)$ in which every graph has n vertices and m edges. Give a graph parameter $f(G)$ and a positive integer n , the *Nordhaus-Gaddum(N-G) Problem* is to determine sharp bounds for (1) $f(G) + f(\overline{G})$ and (2) $f(G) \cdot f(\overline{G})$, as G ranges over the class $\mathcal{G}(n)$, and characterize the extremal graphs, i.e., graphs that achieve the bounds. The Nordhaus-Gaddum type relations have received wide attention; see a recent survey paper [3] by Aouchiche and Hansen.

In this paper, we study the above problem on the generalized edge-connectivity. The paper is organized as follows. In Section 2, we study $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$ for the parameter $\lambda_k(G)$ where $G \in \mathcal{G}(n)$, and get the following result.

Theorem 1. *Let $G \in \mathcal{G}(n)$ and let k be an integer with $3 \leq k \leq n$. Then*

- (1) $1 \leq \lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil k/2 \rceil$;
- (2) $0 \leq \lambda_k(G) \cdot \lambda_k(\overline{G}) \leq \lfloor \frac{n - \lceil k/2 \rceil}{2} \rfloor^2$.

Moreover, the upper and lower bounds are sharp.

In Section 3, we focus our attention on the graph class $\mathcal{G}(n, m)$ and obtain the sharp bounds of $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$.

Theorem 2. *Let $G \in \mathcal{G}(n, m)$ and let k be an integer with $3 \leq k \leq n$. For $n \geq 6$, we have*

- (1) $L(n, m) \leq \lambda_k(G) + \lambda_k(\overline{G}) \leq M(n, m)$;
- (2) $0 \leq \lambda_k(G) \cdot \lambda_k(\overline{G}) \leq N(n, m)$,

where $L(n, m)$, $M(n, m)$, $N(n, m)$ are defined in Lemmas 8 and 9.

Moreover, the upper and lower bounds are sharp.

The following theorem and corollary will be used in Section 3 and Section 2, respectively.

Theorem 3. *(Nash-Williams [22], Tutte [25]) A multigraph G contains a system of ℓ edge-disjoint spanning trees if and only if*

$$\|G/\mathcal{P}\| \geq \ell(|\mathcal{P}| - 1)$$

holds for every partition \mathcal{P} of $V(G)$, where $\|G/\mathcal{P}\|$ denotes the number of crossing edges in G , i.e., edges between distinct parts of \mathcal{P} .

Corollary 1. *Every 2ℓ -edge-connected graph contains a system of ℓ edge-disjoint spanning trees.*

2 Nordhaus-Gaddum-type results in $\mathcal{G}(n)$

All graphs considered in this section are of order n . The following observation is obvious.

Observation 1. *Let G be a graph of order n , and let k be an integer with $3 \leq k \leq n$.*

- (1) *If G is a connected graph, then $1 \leq \lambda_k(G) \leq \lambda(G) \leq \delta(G)$.*
- (2) *If H is a spanning subgraph of G , then $\lambda_k(H) \leq \lambda_k(G)$.*
- (3) *Let G be a connected graph with minimum degree δ . If G has two adjacent vertices of degree δ , then $\lambda_k(G) \leq \delta - 1$.*

Alavi and Mitchem in [2] considered Nordhaus-Gaddum-type results for the connectivity and edge-connectivity parameters. We are concerned with analogous inequalities involving the generalized k -edge-connectivity.

To start with, let us recall the definition of Harary graph $H_{n,d}$:

Case 1. d even. Let $d = 2r$. Then $H_{n,2r}$ is constructed as follows. It has vertices $0, 1, \dots, n-1$ and two vertices i and j are jointed if $i - r \leq j \leq i + r$ (where addition is taken modulo n).

Case 2. d odd, n even. Let $d = 2r + 1$. Then $H_{n,2r+1}$ is constructed by first drawing $H_{n,2r}$ and then adding edges joining vertex i to vertex $i + \frac{n}{2}$ for $1 \leq i \leq \frac{n}{2}$.

Case 3. d odd, n odd. Let $d = 2r + 1$. Then $H_{n,2r+1}$ is constructed by first drawing $H_{n,2r}$ and then adding edges joining vertex 0 to vertices $\frac{n-1}{2}$ and $\frac{n+1}{2}$ and i to vertex $i + \frac{n+1}{2}$ for $1 \leq i \leq \frac{n-1}{2}$.

Observe that the Harary graph $H_{n,d}$ is constructed by arranging the n vertices in a circular order and spreading the d edges around the boundary in a nice way, keeping the chords as short as possible. They have the maximum connectivity for their size and $\kappa(H_{n,d}) = \lambda(H_{n,d}) = \delta(H_{n,d}) = d$. Palmer [23] gave the *STP* number of some special graph classes.

Lemma 1. [23] (1) *The *STP* number of a complete bipartite graph $K_{a,b}$ is $\lfloor \frac{ab}{a+b-1} \rfloor$.*

(2) *The *STP* number of a Harary graph $H_{n,d}$ is $\lfloor d/2 \rfloor$.*

According to (1) of Observation 1, we can obtain a sharp lower bound for the generalized k -edge-connectivity by Corollary 1. Actually, a λ -edge-connected graph G contains $\lfloor \frac{1}{2}\lambda(G) \rfloor$ edge-disjoint spanning trees, each of which is also a Steiner tree connecting S . So the following proposition is immediate.

Proposition 1. *For a connected graph G of order n and $3 \leq k \leq n$, $\lambda_k(G) \geq \lfloor \frac{1}{2}\lambda(G) \rfloor$. Moreover, the lower bound is sharp.*

For the sharpness of this lower bound when $k = n$, we consider the Harary graph $H_{n,2r}$. Clearly, $\lambda(G) = 2r$. From (2) of Lemma 1, $H_{n,2r}$ contains exactly r spanning trees, that is, $\lambda_n(H_{n,2r}) = r$. So $\lambda_n(H_{n,2r}) = \lfloor \frac{1}{2}\lambda(G) \rfloor$. For a general k ($3 \leq k \leq n$), one can check that the cycle C_n can attain the lower bound since $\frac{1}{2}\lambda(C_n) = 1 = \lambda_k(C_n)$.

The following proposition indicates that the monotone properties of λ_k , that is, $\lambda_n \leq \lambda_{n-1} \leq \dots \leq \lambda_4 \leq \lambda_3 \leq \lambda$, is true for $2 \leq k \leq n$.

Proposition 2. *For two integers k and n with $2 \leq k \leq n - 1$, and a connected graph G , $\lambda_{k+1}(G) \leq \lambda_k(G)$.*

Proof. Assume $3 \leq k \leq n - 1$. Set $\lambda_{k+1}(G) = \ell$. For each $S \subseteq V(G)$ with $|S| = k$, we let $S' = S \cup \{u\}$, where $u \in V(G)$ but $u \notin S$. Since $\lambda_{k+1}(G) = \ell$, there exist ℓ edge-disjoint trees connecting S' . These trees are also ℓ edge-disjoint trees connecting S . So $\lambda_k(G) \geq \ell$ and $\lambda_{k+1}(G) \leq \lambda_k(G)$. Combining this with (1) of Observation 1, we get that $\lambda_{k+1}(G) \leq \lambda_k(G)$ for $2 \leq k \leq n - 1$. \square

Now we give the lower bounds of $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$.

Lemma 2. *Let $G \in \mathcal{G}(n)$ and let k be an integer with $3 \leq k \leq n$. Then*

- (1) $\lambda_k(G) + \lambda_k(\overline{G}) \geq 1$;
- (2) $\lambda_k(G) \cdot \lambda_k(\overline{G}) \geq 0$.

Moreover, the two lower bounds are sharp.

Proof. (1) If $\lambda_k(G) + \lambda_k(\overline{G}) = 0$, then $\lambda_k(G) = \lambda_k(\overline{G}) = 0$, that is, both G and \overline{G} are disconnected, which is impossible, and so $\lambda_k(G) + \lambda_k(\overline{G}) \geq 1$.

(2) By definition, $\lambda_k(G) \geq 0$ and $\lambda_k(\overline{G}) \geq 0$, and so $\lambda_k(G) \cdot \lambda_k(\overline{G}) \geq 0$. \square

The following observation gives the graphs attaining the lower bound of (2) in Lemma 2.

Observation 2. $\lambda_k(G) \cdot \lambda_k(\overline{G}) = 0$ if and only if G or \overline{G} is disconnected.

In [21] we obtained the exact value of the generalized k -edge-connectivity of a complete graph K_n .

Lemma 3. [21] *For two integers n and k with $2 \leq k \leq n$, $\lambda_k(K_n) = n - \lceil k/2 \rceil$.*

For a connected graph G of order n , we know that $1 \leq \lambda_k(G) \leq \lambda_k(K_n) = n - \lceil k/2 \rceil$. In [21] we characterized the graphs attaining the upper bound.

Lemma 4. [21] *For a connected graph G of order n with $3 \leq k \leq n$, $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$ if and only if $G = K_n$ for k even; $G = K_n \setminus M$ for k odd, where M is an edge set such that $0 \leq |M| \leq \frac{k-1}{2}$.*

Now we want to characterize the graphs that attain the lower bound 1 of $\lambda_k(G) + \lambda_k(\overline{G})$. Before doing so, we give some graph classes (each graph of the classes has order n).

For $n \geq 5$, \mathcal{G}_n^1 is a graph class as shown in Figure 1 (a), each graph G of which satisfies that $\lambda(G) = 1$ and $d_G(v_1) = n - 1$, where $v_1 \in V(G)$; \mathcal{G}_n^2 is a graph class as shown in

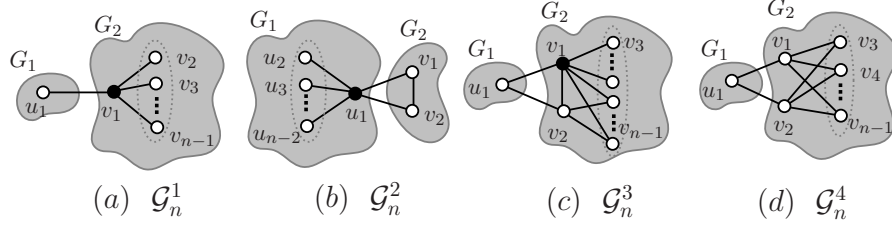


Figure 1. Graphs for Proposition 3 (The degree of a black vertex is $n - 1$).

Figure 1 (b), each graph G of which satisfies that $\lambda(G) = 2$ and $d_G(u_1) = n - 1$, where $u_1 \in V(G)$; \mathcal{G}_n^3 is a graph class as shown in Figure 1 (c), each graph G of which satisfies that $\lambda(G) = 2$ and $d_G(v_1) = n - 1$, where $v_1 \in V(G)$; \mathcal{G}_n^4 is a graph class as shown in Figure 1 (d), each graph G of which satisfies $\lambda(G) = 2$.

The following observation and lemma are preparations for Proposition 3.

For $n \geq 5$, let $K_{2,n-2}^+$ be the graph obtained from the complete bipartite graph $K_{2,n-2}$ by adding one edge on the part having $n - 2$ vertices and let $K_{2,n-2}^{++}$ denote any of the two graphs which are obtained from $K_{2,n-2}$ by adding two edges on the part having $n - 2$ vertices.

Observation 3. *Let n be an integer with $n \geq 5$. Then*

- (1) $\lambda_n(K_{2,n-2}^{++}) \geq 2$;
- (2) $\lambda_{n-1}(K_{2,n-2}^+) \geq 2$, $\lambda_n(K_{2,n-2}^+) = 1$;
- (3) $\lambda_{n-2}(K_{2,n-2}) \geq 2$, $\lambda_n(K_{2,n-2}) = \lambda_{n-1}(K_{2,n-2}) = 1$.

Proof. (1) As shown in Figure 2 (a), we have $\lambda_n(K_{2,n-2}^{++}) \geq 2$.

(2) As shown in Figure 2 (b), we have $\lambda_{n-1}(K_{2,n-2}^+) \geq 2$. Since $|E(K_{2,n-2}^+)| = 2(n - 2) + 1$ and $\lambda_n(K_{2,n-2}^+) \leq \lfloor \frac{2(n-2)+1}{n-1} \rfloor$, then $\lambda_n(K_{2,n-2}^+) \leq 1$. Since $K_{2,n-2}^+$ is connected, then $\lambda_n(K_{2,n-2}^+) = 1$.

(3) As shown in Figure 2 (c), it follows that $\lambda_{n-2}(K_{2,n-2}) \geq 2$. Let $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, \dots, w_{n-2}\}$ be the two parts of the complete bipartite graph $K_{2,n-2}$. Choose $S = \{u_1, u_2, w_1, w_2, \dots, w_{n-3}\}$. If there exists an S -tree containing the vertex w_{n-2} , then this tree will use $n - 1$ edges of $E(K_{2,n-2})$, which implies that $\lambda_{n-1}(K_{2,n-2}) \leq 1$ since $|E(K_{2,n-2})| = 2(n - 2)$. Suppose that any S -tree does not contain the vertex w_{n-2} . Pick up such a tree, say T . Then there exists a vertex with degree 2 in T , which implies that there is no other S -tree in $K_{2,n-2}$. So $\lambda_{n-1}(K_{2,n-2}) \leq 1$. Since $K_{2,n-2}$ is connected, $\lambda_{n-1}(K_{2,n-2}) = 1$. From Proposition 2, $\lambda_n(K_{2,n-2}) = 1$. \square

Lemma 5. *Let G be a connected graph of order n , and let k be an integer with $3 \leq k \leq n$. If $\lambda(G) = 3$ and there exists a vertex $u \in V(G)$ such that $d_G(u) = n - 1$, then $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$.*

Proof. Let G_1, \dots, G_r be the connected components of $G \setminus u$. Since $\lambda(G) = 3$, it follows that $\delta(G_i) \geq 2$ ($1 \leq i \leq r$). Let $|V(G_i)| = n_i$ ($1 \leq i \leq r$) and $V(G_i) = \{v_{i,1}, v_{i,2}, \dots, v_{i,n_i}\}$. Then there exists an edge, without loss of generality, say $e_i = v_{i,1}v_{i,2} \in E(G_i)$ such

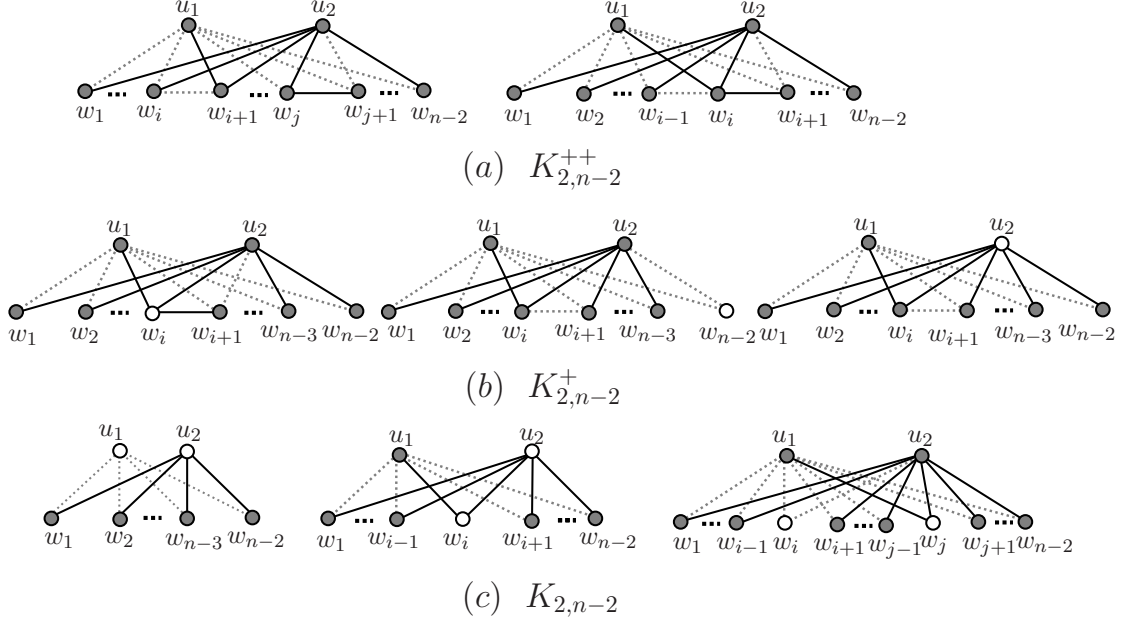


Figure 2. Graphs for Observation 3.

that $G_i \setminus e_i$ is connected for $1 \leq i \leq r$. Thus $G_i \setminus e_i$ contains a spanning tree, say T_i ($1 \leq i \leq r$). The tree T induced by the edges in $\{uv_{1,1}, uv_{2,1}, \dots, uv_{r,1}\} \cup E(T_1) \cup E(T_2) \cup \dots \cup E(T_r)$ and the tree T' induced by the edges in $\{v_{1,1}v_{1,2}, uv_{1,2}, \dots, uv_{1,n_1}\} \cup \{v_{2,1}v_{2,2}, uv_{2,2}, \dots, uv_{2,n_2}\} \cup \dots \cup \{v_{r,1}v_{r,2}, uv_{r,2}, \dots, uv_{r,n_r}\}$ are two spanning trees of G , and hence $\lambda_n(G) \geq 2$. Combining this with Proposition 2, we get $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$. \square

Proposition 3. *Let G be a graph of order n , and let k be an integer with $3 \leq k \leq n$. $\lambda_k(G) + \lambda_k(\overline{G}) = 1$ if and only if G (symmetrically, \overline{G}) satisfies one of the following conditions:*

- (1) $G \in \mathcal{G}_n^1$ or $G \in \mathcal{G}_n^2$;
- (2) $G \in \mathcal{G}_n^3$ and there exists a component G_i of $G \setminus v_1$ such that G_i is a tree and $|V(G_i)| < k$;
- (3) $G \in \{K_{2,n-2}^+, K_{2,n-2}\}$ for $k = n$ and $n \geq 5$, or $G \in \{P_3, C_3\}$ for $k = n = 3$, or $G \in \{C_4, K_4 \setminus e\}$ for $k = n = 4$, or $G = K_{3,3}$ for $k = n = 6$, or $G = K_{2,n-2}$ for $k = n - 1$ and $n \geq 5$, or $G = C_4$ for $k = n - 1 = 3$.

Proof. Sufficiency. Let G be a graph satisfying one of the conditions of (1), (2) and (3). One can see that G is connected and its complement \overline{G} is disconnected. Thus $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(G)$ and $\lambda_k(G) \geq 1$. We only need to show that $\lambda_k(G) \leq 1$ for each graph G satisfying one of the conditions of (1), (2) and (3). For $G \in \mathcal{G}_n^1$, since $\delta(G) = 1$ we have $\lambda_k(G) \leq 1$ by (1) of Observation 1. For $G \in \mathcal{G}_n^2$, it follows that $\lambda_k(G) \leq \delta(G) - 1 = 1$ by (3) of Observation 1 since $d_G(v_1) = d_G(v_2) = \delta(G) = 2$. Suppose that $G \in \mathcal{G}_n^3$ and there exists a connected component G_i of $G \setminus v_1$ such that G_i is a tree and $|V(G_i)| < k$. Set $V(G_i) = \{v_{i,1}, v_{i,2}, \dots, v_{i,n_i}\}$. We choose $S \subseteq V(G)$ such that $V(G_i) \cup \{v_1\} = S' \subseteq S$. Then $|E(G[S'])| = 2n_i - 1$. Since every spanning tree of $G[S']$ uses n_i edges of $E(G[S'])$,

there exists at most one spanning tree in $G[S']$, which implies that there is at most one tree connecting S in G . So $\lambda_k(G) \leq 1$. For $G = K_{2,n-2}^+$, $\lambda_n(G) = 1$ by (2) of Observation 3. For $G = K_{2,n-2}$, by (3) of Observation 3, we have $\lambda_n(K_{2,n-2}) = \lambda_{n-1}(K_{2,n-2}) = 1$. For $G = K_{3,3}$, $\lambda_n(G) \leq \lfloor \frac{|E(G)|}{n-1} \rfloor = \lfloor \frac{9}{5} \rfloor = 1$. For $G \in \{P_3, C_3, C_4, K_4 \setminus e\}$, one can check that $\lambda_k(G) \leq 1$ for $k = n$ or $k = n - 1$. From these together with $\lambda_k(G) \geq 1$, we have $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(G) = 1$.

Necessity. Suppose $\lambda_k(G) + \lambda_k(\overline{G}) = 1$. Then $\lambda_k(G) = 1$ and $\lambda_k(\overline{G}) = 0$, or $\lambda_k(\overline{G}) = 1$ and $\lambda_k(G) = 0$. By symmetry, without loss of generality, let $\lambda_k(G) = 1$ and $\lambda_k(\overline{G}) = 0$. From these together with Proposition 1, $\lambda(\overline{G}) = 0$ and $1 \leq \lambda(G) \leq 3$. So we have the following three cases to consider.

Case 1. $\lambda(G) = 1$.

For $n = 3$, one can check that $G = P_3$ satisfies $\lambda(G) = 1$ but $\lambda(\overline{G}) = 0$. Now we assume $n \geq 4$. Since $\lambda(G) = 1$, there exists a cut edge in G , say $e = u_1v_1$. Let G_1 and G_2 be the two connected components of $G \setminus e$ such that $u_1 \in V(G_1)$ and $v_1 \in V(G_2)$. Set $V(G_1) = \{u_1, u_2, \dots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \dots, v_{n_2}\}$, where $n_1 + n_2 = n$. Suppose $n_i \geq 2$ ($i = 1, 2$). For any $u_i, u_j \in V(G_1)$, u_i and u_j are connected in \overline{G} since there exists a path $u_i v_2 u_j$ in \overline{G} ; for any $v_i, v_j \in V(G_2)$, v_i and v_j are connected in \overline{G} since there exists a path $v_i u_2 v_j$ in \overline{G} ; for any $u_i \in V(G_1)$ and $v_j \in V(G_2)$ ($i \neq 1$ or $j \neq 1$), $v_i v_j \in E(\overline{G})$. Clearly, the path $u_1 v_2 u_2 v_1$ connects u_1 and v_1 in \overline{G} . So \overline{G} is connected, a contradiction. Thus $n_1 = 1$ or $n_2 = 1$. Without loss of generality, let $n_1 = 1$. Then $V(G_1) = \{u_1\}$ and $V(G_2) = \{v_1, v_2, \dots, v_{n-1}\}$. Clearly, G is a graph obtained from G_2 by attaching the edge $e = u_1v_1$. Since $u_1v_j \notin E(G)$ ($1 < j \leq n - 1$), $u_1v_j \in E(\overline{G})$. If $d_G(v_1) \leq n - 2$, then there exists a vertex v_j such that $v_1v_j \in E(\overline{G})$, which results in $\lambda(\overline{G}) \geq 1$, a contradiction. So $d_G(v_1) = n - 1$ and $G \in \mathcal{G}_n^1$; see Figure 1 (a).

Case 2. $\lambda(G) = 2$.

For $n = 3, 4$, the graph $G \in \{C_3, C_4, K_4 \setminus e\}$ satisfies that $\lambda(G) = 2$ and $\lambda(\overline{G}) = 0$. Since $\lambda_3(C_3) = 1$, $\lambda_3(C_4) = 1$, $\lambda_4(C_4) = 1$, $\lambda_3(K_4 \setminus e) = 2$ and $\lambda_4(K_4 \setminus e) = 1$, we have $G = C_3$ for $k = n = 3$; $G \in \{C_4, K_4 \setminus e\}$ for $k = n = 4$; $G = C_4$ for $k = n - 1 = 3$. Now we assume $n \geq 5$. Since $\lambda(G) = 2$, there exists an edge cut M such that $|M| = 2$. Let G_1 and G_2 be the two connected components of $G \setminus M$, $V(G_1) = \{u_1, \dots, u_{n_1}\}$ and $V(G_2) = \{v_1, \dots, v_{n_2}\}$, where $n_1 + n_2 = n$. Clearly, $G[M] = 2K_2$ or $G[M] = P_3$.

At first, we consider the case $G[M] = 2K_2$. Without loss of generality, let $M = \{u_1v_1, u_2v_2\}$. Since $n \geq 5$, $n_1 \geq 3$ or $n_2 \geq 3$. Without loss of generality, let $n_1 \geq 3$. Clearly, any two vertices $v_i, v_j \in V(G_2)$ are connected in \overline{G} since there exists a path $v_i u_3 v_j$ in \overline{G} . Furthermore, for any $u_i \in V(G_1)$, $u_i v_1 \in E(\overline{G})$ or $u_i v_2 \in E(\overline{G})$. So \overline{G} is connected and $\lambda(\overline{G}) \geq 1$, a contradiction.

Next, we consider the case $G[M] = P_3$. Without loss of generality, let $P = v_1u_1v_2$ be the path of order 3. Since $n \geq 5$, there exist at least two vertices in $G \setminus \{u_1, v_1, v_2\}$. If $n_1 \geq 2$ and $n_2 \geq 3$, then we can check that \overline{G} is connected, a contradiction. So we assume $n_1 = 1$ or $n_2 = 2$, that is, $V(G_2) = \{v_1, v_2\}$ or $V(G_1) = \{u_1\}$.

For the former, $V(G_1) = \{u_1, u_2, \dots, u_{n-2}\}$. Since $\lambda(G) = 2$, $v_1v_2 \in E(G)$. Clearly, $v_1u_j, v_2u_j \notin E(G)$ ($2 \leq j \leq n - 2$), which implies that $v_1u_j, v_2u_j \in E(\overline{G})$. Therefore, $u_1u_j \notin E(\overline{G})$ ($2 \leq j \leq n - 2$) since \overline{G} is disconnected. Thus $u_1u_j \in E(G)$ for each

j ($2 \leq j \leq n-2$). So $d_G(u_1) = n-1$ and $G \in \mathcal{G}_n^2$; see Figure 1 (b).

For the latter, let $V(G_2) = \{v_1, v_2, \dots, v_{n-1}\}$. First we consider the case $v_1v_2 \in E(G)$. Since $u_1v_j \notin E(G)$ ($3 \leq j \leq n-1$), we have $u_1v_j \in E(\overline{G})$. If $3 \leq d_G(v_1) \leq n-2$ and $3 \leq d_G(v_2) \leq n-2$, then there exist two vertices v_i and v_j such that $v_1v_i, v_2v_j \in E(\overline{G})$ ($3 \leq i, j \leq n-1$), which implies that \overline{G} is connected, a contradiction. So $d_G(v_1) = n-1$ or $d_G(v_2) = n-1$. Without loss of generality, let $d_G(v_1) = n-1$. Thus $G \in \mathcal{G}_n^3$; see Figure 1 (c).

Now we focus on the graph $G \setminus v_1$. Let G_1, G_2, \dots, G_r be the connected components of $G \setminus v_1$ and $V(G_i) = \{v_{i,1}, v_{i,2}, \dots, v_{i,n_i}\}$ ($1 \leq i \leq r$), where $\sum_{i=1}^r n_i = n-1$. If there exists some connected component G_i such that $G_i = K_2$, then $G \in \mathcal{G}_n^2$; see Figure 1 (b). So we assume $n_i \geq 3$. Then we show the following claim and get a contradiction.

Claim 1. For each connected component G_i of $G \setminus v_1$, if $n_i \geq k$, or $n_i \leq k-1$ and $|E(G_i)| \geq n_i$, then $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$.

Proof of Claim 1. For an arbitrary $S \subseteq V(G)$ with $|S| = k$, we only prove that $\lambda(S) \geq 2$ for $v_1 \notin S$. The case $v_1 \in S$ can be proved similarly. If there exists some connected component G_i such that $S = V(G_i)$, then $n_i = k$ and G_i has a spanning tree, say T_i . It is also a Steiner tree connecting S . Since the tree T'_i induced by the edges in $\{v_1v_{i,1}, v_1v_{i,2}, \dots, v_1v_{i,n_i}\}$ is another Steiner tree connecting S and T_i, T'_i are two edge-disjoint trees, it follows that $\lambda(S) \geq 2$. Assume now $S \neq V(G_i)$ for $n_i \geq k$ ($1 \leq i \leq r$). Let $S_i = S \cap V(G_i)$ ($1 \leq i \leq r$) and $|S_i| = k_i$. It is clear that $\bigcup_{i=1}^r S_i = S$ and $\sum_{i=1}^r k_i = k$. Thus $S_i \subset V(G_i)$ for each connected component G_i such that $n_i \geq k$, and $S_j \subseteq V(G_j)$ for each connected component G_j such that $n_j \leq k-1$ and $|E(G_j)| \geq n_j$. We will show that there are two edge-disjoint Steiner trees connecting $S_i \cup \{v_1\}$ in $G[S_i \cup \{v_1\}]$ for each i ($1 \leq i \leq r$) so that we can combine these trees to form two edge-disjoint Steiner trees connecting S in G . Suppose that G_i is a connected component such that $n_i \geq k$. Note that $V(G_i) = \{v_{i,1}, v_{i,2}, \dots, v_{i,n_i}\}$. Since $S_i \subset V(G_i)$, there exists a vertex, without loss of generality, say $v_{i,1}$, such that $v_{i,1} \notin S_i$. Clearly, G_i contains a spanning tree, say $T'_{i,1}$. Thus $T_{i,1} = v_1v_{i,1} \cup T'_{i,1}$ is a Steiner tree connecting $S_i \cup \{v_1\}$ in $G[G_i \cup \{v_1\}]$. Since the tree $T_{i,2}$ induced by the edges in $\{v_1v_{i,2}, v_1v_{i,3}, \dots, v_1v_{i,n_i}\}$ is another Steiner tree connecting $S_i \cup \{v_1\}$. Clearly, $T_{i,1}$ and $T_{i,2}$ are edge-disjoint. Assume that G_j is a connected component such that $n_j \leq k-1$ and $|E(G_j)| \geq n_j$. Note that $V(G_j) = \{v_{j,1}, v_{j,2}, \dots, v_{j,n_j}\}$. Then there exists an edge, without loss of generality, say $e_j = v_{j,1}v_{j,2} \in E(G_j)$ such that $G_j \setminus e_j$ contains a spanning tree of G_j , say $T'_{j,1}$. Thus the tree $T_{j,1}$ induced by the edges in $\{v_1v_{j,1}\} \cup E(T'_{j,1})$ and the tree $T_{j,2}$ induced by the edges in $\{v_{j,1}v_{j,2}, v_1v_{j,2}, \dots, v_1v_{j,n_j}\}$ are two edge-disjoint Steiner trees connecting $S_j \cup \{v_1\}$. Now we combine these small trees connecting $S_i \cup \{v_1\}$ ($1 \leq i \leq r$) by the vertex v_1 to form two big trees connecting S . It is clear that the tree T_1 induced by the edges in $E(T_{1,1}) \cup E(T_{2,1}) \cup \dots \cup E(T_{r,1})$ and the tree T_2 induced by the edges in $E(T_{1,2}) \cup E(T_{2,2}) \cup \dots \cup E(T_{r,2})$ are our desired trees, and hence $\lambda(S) \geq 2$. From the arbitrariness of S , we have $\lambda_k(G) \geq 2$. \square

By Claim 1, we know that $G \in \mathcal{G}_n^3$ and there exists a connected component G_i of $G \setminus \{v_1\}$ such that $n_i \leq k-1$ and G_i is a tree.

We next consider the case $v_1v_2 \notin E(G)$; see Figure 1 (d). Thus $v_1v_2 \in E(\overline{G})$. Since $u_1v_j \notin E(G)$ ($3 \leq j \leq n-1$), $u_1v_j \in E(\overline{G})$, which results in $v_1v_j, v_2v_j \notin E(\overline{G})$ since \overline{G} is

disconnected. Thus $v_1v_j, v_2v_j \in E(G)$ for each j ($3 \leq j \leq n-1$). Let $R = \{v_j | 3 \leq j \leq n-1\}$. If $|E(G[R])| \geq 2$, then G contains a subgraph $K_{2,n-2}^{++}$, which implies that $\lambda_n(G) \geq 2$ by (1) of Observation 3. Combining this with Proposition 2, $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$, a contradiction. If $|E(G[R])| < 2$, then $G = K_{2,n-2}$ and $K_{2,n-2}^+$. From Observation 3 and Proposition 2, we have $\lambda_k(K_{2,n-2}^+) \geq 2$ for $3 \leq k \leq n-1$ and $\lambda_k(K_{2,n-2}) \geq 2$ for $3 \leq k \leq n-2$, a contradiction. So $G = K_{2,n-2}^+$ for $k = n$, or $G = K_{2,n-2}$ for $k = n$, or $G = K_{2,n-2}^+$ for $k = n-1$.

Case 3. $\lambda(G) = 3$.

For $n = 4$, $G = K_4$, $\lambda_3(G) = \lambda_4(G) = 2$ by Lemma 3, that is, $\lambda_k(G) \geq 2$, a contradiction. Assume $n \geq 5$. Since $\lambda(G) = 3$, there exists an edge cut M such that $|M| = 3$. Let

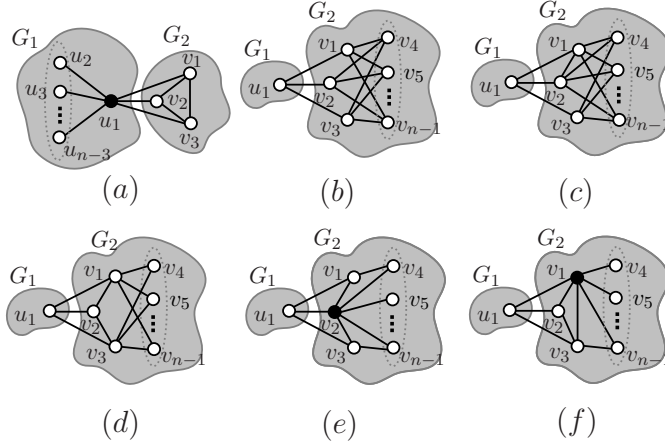


Figure 3. Graphs for Case 3 of Proposition 3.

G_1 and G_2 be the two connected components of $G \setminus M$, $V(G_1) = \{u_1, u_2, \dots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \dots, v_{n_2}\}$, where $n_1 + n_2 = n$. Clearly, $G[M] = P_4$ or $G[M] = P_3 \cup K_2$ or $G[M] = 3K_2$ or $G[M] = K_{1,3}$. For the former three cases, $n_i \geq 3$ ($i = 1, 2$) and $n \geq 6$ since $\lambda(G) = 3$. To shorten the discussion, we only show $\lambda(\overline{G}) \geq 1$ for $G[M] = P_4$ and get a contradiction among the former three cases. Without loss of generality, let $G[M] = P_4 = u_1v_1u_2v_2$. For any $u_i, u_j \in V(G_1)$ ($1 \leq i \leq n_1$), u_i and u_j are connected in \overline{G} since there exists a path $u_iv_3u_j$ in \overline{G} ; for any $v_i, v_j \in V(G_2)$ ($1 \leq i \leq n_2$), v_i and v_j are connected in \overline{G} since there exists a path $v_iv_3v_j$ in \overline{G} ; for any $u_i \in V(G_1)$ and $v_j \in V(G_2)$ ($i \neq 3$ and $j \neq 3$), u_i and v_j are connected in \overline{G} since there exists a path $u_iv_3u_3v_j$ in \overline{G} . Since $u_3v_j \in E(\overline{G})$ ($1 \leq j \leq n_2$) and $v_3u_i \in E(\overline{G})$ ($1 \leq i \leq n_1$), \overline{G} is connected, as desired.

Now we consider the graph G such that $G[M] = K_{1,3}$. Assume $n_1 \geq 2$. If $n_2 \geq 4$, then we can check that \overline{G} is connected and get a contradiction. Therefore, $n_2 = 3$, $V(G_2) = \{v_1, v_2, v_3\}$ and $V(G_1) = \{u_1, u_2, \dots, u_{n-3}\}$. Since $\lambda(G) = 3$, it follows that $v_1v_2, v_2v_3, v_1v_3 \in E(G)$. Since $v_iv_j \notin E(G)$ ($1 \leq i \leq 3, 2 \leq j \leq n-3$), we have $v_iv_j \in E(\overline{G})$. If there exists some vertex u_j ($2 \leq j \leq n-3$) such that $u_1u_j \in E(\overline{G})$, then \overline{G} is connected, a contradiction. So $u_1u_j \in E(G)$ for $2 \leq j \leq n-3$. Thus $d_G(u_1) = n-1$ (See Figure 3 (a)). From Lemma 5, $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$ since $\lambda(G) = 3$, a contradiction.

Now assume $n_1 = 1$. Then $V(G_1) = \{u_1\}$ and $V(G_2) = \{v_1, v_2, \dots, v_{n-1}\}$. If

$G[\{v_1, v_2, v_3\}] = 3K_1$ or $G[\{v_1, v_2, v_3\}] = K_1 \cup K_2$, then we have $u_1v_j \in E(\overline{G})$ since $u_1v_j \notin E(G)$ ($4 \leq j \leq n-1$). From this together with the fact that \overline{G} is disconnected and $v_1v_3, v_2v_3 \in E(\overline{G})$, $v_iv_j \notin E(\overline{G})$ ($1 \leq i \leq 3, 4 \leq j \leq n-1$), we have $v_iv_j \in E(G)$ ($1 \leq i \leq 3, 4 \leq j \leq n-1$). Thus G contains a complete bipartite graph $K_{3, n-3}$ as its subgraph; see Figure 3 (b) and (c). From (1) of Lemma 1, $\lambda_n(G) = \lfloor \frac{3(n-3)}{n-1} \rfloor \geq 2$ for $n \geq 7$, which implies that $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$ and $n \geq 7$. Since $\lambda(G) = 3$, $n \geq 6$. So we only need to consider the case $n = 6$. Thus $G = H_i$ ($1 \leq i \leq 4$) (See Figure 4). If $G = H_i$ ($2 \leq i \leq 4$), then $\lambda_n(G) \geq 2$ for $k = n = 6$; see Figure 4 (b), (c) and (d). Therefore $\lambda_k(G) \geq 2$ for $3 \leq k \leq 6$. If $G = H_1$, then $\lambda_n(G) \leq \lfloor \frac{|E(G)|}{n-1} \rfloor = \lfloor \frac{9}{5} \rfloor = 1$ for $k = n = 6$. For $k = 5$, we can check that $\lambda_3(G) \geq \lambda_4(G) \geq \lambda_5(G) \geq 2$; see Figure 4 (e). So $G = K_{3,3}$ for $k = n = 6$.

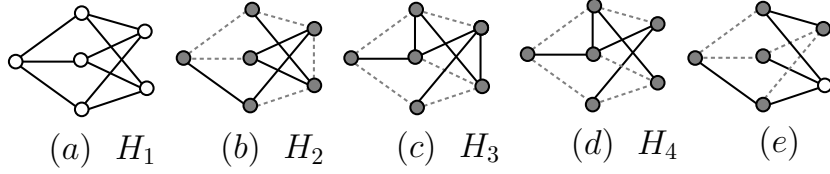


Figure 4. Graphs for Case 3 of Proposition 3.

Suppose $G[\{v_1, v_2, v_3\}] = P_3$. Without loss of generality, let $v_1v_2, v_2v_3 \in E(G)$. If $3 \leq d_G(v_2) \leq n-2$ (see Figure 3 (d)), then there exists at least one vertex v_j such that $v_2v_j \in E(\overline{G})$, which results in $v_1v_j, v_3v_j \notin E(\overline{G})$ ($4 \leq j \leq n-1$) since $u_1v_j \in E(\overline{G})$ ($4 \leq j \leq n-1$), $v_1v_3 \in E(\overline{G})$ and \overline{G} is disconnected. Thus $v_1v_t, v_3v_t \in E(G)$ for each t ($4 \leq t \leq n-1$). Since $d(v_4) \geq \delta(G) \geq \lambda(G) = 3$, we have $v_4v_2 \in E(G)$ or there exists some vertex v_j ($5 \leq j \leq n-1$) such that $v_4v_j \in E(G)$, which implies that G contains a subgraph $K_{2, n-2}^{++}$ and so $\lambda_n(G) \geq 2$ by (1) of Observation 3. From Proposition 2, $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$, a contradiction. If $d_G(v_2) = n-1$ (See Figure 3 (e)), then $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$ by Lemma 5 since $\lambda(G) = 3$, a contradiction.

Assume that $G[\{v_1, v_2, v_3\}] = K_3$. Without loss of generality, let $v_1v_2, v_1v_3, v_2v_3 \in E(G)$. If $d_G(v_1) = n-1$ or $d_G(v_2) = n-1$ or $d_G(v_3) = n-1$ (see Figure 3 (f)), then by Lemma 5 $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$ since $\lambda(G) = 3$, a contradiction. If $3 \leq d_G(v_i) \leq n-2$ ($1 \leq i \leq 3$), then \overline{G} is connected, another contradiction. \square

Now we turn to studying the upper bounds of $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$.

Lemma 6. *Let $G \in \mathcal{G}(n)$, and let k be an integer with $3 \leq k \leq n$. Then*

- (1) $\lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil k/2 \rceil$.
- (2) $\lambda_k(G) \cdot \lambda_k(\overline{G}) \leq \lceil \frac{n - \lceil k/2 \rceil}{2} \rceil^2$.

Moreover, the two upper bounds are sharp.

Proof. (1) Since $G \cup \overline{G} = K_n$, $\lambda_k(G) + \lambda_k(\overline{G}) \leq \lambda_k(K_n)$. Combining this with Lemma 3, $\lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil \frac{k}{2} \rceil$.

- (2) The conclusion holds by (1). \square

Consider (1) of Lemma 6. If one of G and \overline{G} is disconnected, we can characterize the graphs attaining the upper bound by Lemma 4.

Proposition 4. *Let G be a graph of order n , and let k be an integer with $3 \leq k \leq n$. If G is disconnected, then $\lambda_k(G) + \lambda_k(\overline{G}) = n - \lceil \frac{k}{2} \rceil$ if and only if $\overline{G} = K_n$ for k even; $\overline{G} = K_n \setminus M$ for k odd, where M is an edge set such that $0 \leq |M| \leq \frac{k-1}{2}$.*

If both G and \overline{G} are connected, we can obtain a property of the graphs attaining the upper bound.

Proposition 5. *Let G be a graph of order n , and let k be an integer with $3 \leq k \leq n$. If $\lambda_k(G) + \lambda_k(\overline{G}) = n - \lceil \frac{k}{2} \rceil$, then $\Delta(G) - \delta(G) \leq \lceil \frac{k}{2} \rceil - 1$.*

Proof. Assume that $\Delta(G) - \delta(G) \geq \lceil \frac{k}{2} \rceil$. Since $\lambda_k(\overline{G}) \leq \delta(\overline{G}) = n - 1 - \Delta(G)$, $\lambda_k(G) + \lambda_k(\overline{G}) \leq \delta(G) + n - 1 - \Delta(G) \leq n - 1 - \lceil \frac{k}{2} \rceil$, a contradiction. \square

The next example shows that for $k = n$ the two upper bounds in Lemma 6 are sharp.

Example 1. Let n, r be two positive integers such that $n = 4r + 1$. From (1) of Lemma 1, we know that the *STP* number of the complete bipartite graph $K_{2r, 2r+1}$ is $\lfloor \frac{2r(2r+1)}{2r+(2r+1)-1} \rfloor = r$, that is, $\lambda_n(K_{2r, 2r+1}) = r$. Let \mathcal{E} be the set of the edges of these r spanning trees in $K_{2r, 2r+1}$. Then there remain $2r(2r+1) - 4r^2 = 2r$ edges in $K_{2r, 2r+1}$ except the edges in \mathcal{E} . Let M be the set of these $2r$ edges. Set $G = K_{2r, 2r+1} \setminus M$. Then $\lambda_n(G) = r$, $M \subseteq E(\overline{G})$ and \overline{G} is a graph obtained from two cliques K_{2r} and K_{2r+1} by adding $2r$ edges of M between them, that is, one endpoint of each edge belongs to K_{2r} and the other endpoint belongs to K_{2r+1} . Note that $E(\overline{G}) = E(K_{2r}) \cup M \cup E(K_{2r+1})$. Now we show that $\lambda_n(\overline{G}) \geq r$. As we know, K_{2r} contains r Hamiltonian paths, say P_1, P_2, \dots, P_r , and so does K_{2r+1} , say P'_1, P'_2, \dots, P'_r . Pick up r edges from M , say e_1, e_2, \dots, e_r , and let T_i be the tree induced by the edges in $E(P_i) \cup E(P'_i) \cup \{e_i\}$ ($1 \leq i \leq r$). Then T_1, T_2, \dots, T_r are r spanning trees in \overline{G} , thus, $\lambda_n(\overline{G}) \geq r$. Since $|E(\overline{G})| = \binom{2r}{2} + \binom{2r+1}{2} + 2r = 4r^2 + 2r$ and each spanning tree uses $4r$ edges, these edges can form at most $\lfloor \frac{4r^2+2r}{4r} \rfloor = r$ spanning trees, and hence $\lambda_n(\overline{G}) \leq r$. So $\lambda_n(\overline{G}) = r$. Clearly, $\lambda_n(G) + \lambda_n(\overline{G}) = 2r = \frac{n-1}{2} = n - \lceil \frac{n}{2} \rceil$ and $\lambda_n(G) \cdot \lambda_n(\overline{G}) = r^2 = \lfloor \frac{n - \lceil n/2 \rceil}{2} \rfloor^2$, which implies that the upper bounds of Lemma 6 are sharp.

Combining Lemmas 2 and 6, we complete the proof of Theorem 1.

3 Nordhaus-Gaddum-type results in $\mathcal{G}(n, m)$

Achthan et al. [1] restricted their attention to the subclass of $\mathcal{G}(n, m)$ consisting of graphs with n vertices and m edges. They investigated the edge-connectivity, diameter and chromatic number parameters. For the edge-connectivity $\lambda(G)$, they showed that $\lambda(G) + \lambda(\overline{G}) \geq \max\{1, n - 1 - m\}$. In this section, we consider a similar problem on the generalized edge-connectivity.

Lemma 7. *If $M \subseteq E(K_n)$ such that $0 \leq m = |M| \leq \lfloor \frac{n}{3} \rfloor$, then $G = K_n \setminus M$ contains ℓ edge-disjoint spanning trees, where $\ell = \min\{n - 2m - 1, \lfloor \frac{n}{2} - \frac{2m}{n-1} \rfloor\}$.*

Proof. Let $\mathcal{P} = \bigcup_{i=1}^p V_i$ be a partition of $V(G)$ with $|V_i| = n_i$ ($1 \leq i \leq p$), and \mathcal{E}_p be the set of edges between distinct parts of \mathcal{P} in G . It suffices to show that $|\mathcal{E}_p| \geq \ell(|\mathcal{P}| - 1)$ so that we can use the Nash-Williams-Tutte Theorem.

The case $p = 1$ is trivial, and thus we assume $2 \leq p \leq n$. Then $|\mathcal{E}_p| \geq \binom{n}{2} - \sum_{i=1}^p \binom{n_i}{2} - |M| \geq \binom{n}{2} - \sum_{i=1}^p \binom{n_i}{2} - m$. We will show that $\binom{n}{2} - \sum_{i=1}^p \binom{n_i}{2} - m \geq \ell(p - 1)$, that is, $\frac{n(n-1)}{2} - m - \ell(p - 1) \geq \sum_{i=1}^p \binom{n_i}{2}$. We only need to prove that $\frac{n(n-1)}{2} - m - \ell(p - 1) \geq \max\{\sum_{i=1}^p \binom{n_i}{2}\}$. Since $f(n_1, n_2, \dots, n_p) = \sum_{i=1}^p \binom{n_i}{2}$ achieves its maximum value when $n_1 = n_2 = \dots = n_{p-1} = 1$ and $n_p = n - p + 1$, we need the inequality $\frac{n(n-1)}{2} - m - \ell(p - 1) \geq \binom{1}{2}(p - 1) + \binom{n-p+1}{2}$, that is, $\frac{n(n-1)}{2} - m - \frac{(n-p+1)(n-p)}{2} \geq \ell(p - 1)$. Actually, $\ell \leq \frac{n(n-1) - (n-p+1)(n-p) - 2m}{2(p-1)}$ is our required inequality, namely, $\ell \leq n - \frac{1}{2} - (\frac{p-1}{2} + \frac{2m}{p-1})$. Since $f(x) = \frac{x}{2} + \frac{2m}{x}$ achieves its maximum value $\max\{2m + \frac{1}{2}, \frac{n-1}{2} + \frac{2m}{n-1}\}$ when $1 \leq x \leq n - 1$, we need $\ell \leq \min\{n - 2m - 1, \frac{n}{2} - \frac{2m}{n-1}\}$. Since this inequality holds for $0 \leq m \leq \lfloor \frac{n}{3} \rfloor$, we have $|\mathcal{E}_p| \geq \binom{n}{2} - \sum_{i=1}^p \binom{n_i}{2} - |M| \geq \ell(p - 1)$. From Theorem 1, we know that G has ℓ edge-disjoint spanning trees. \square

Lemma 8. *Let $G \in \mathcal{G}(n, m)$, and let k be an integer with $3 \leq k \leq n$. For $n \geq 6$, we have*

$$(1) \lambda_k(G) + \lambda_k(\overline{G}) \geq L(n, m), \text{ where}$$

$$L(n, m) = \begin{cases} \max\{1, \lfloor \frac{1}{2}(n - 2 - m) \rfloor\}, & \text{if } \lfloor \frac{n}{3} \rfloor + 1 \leq m \leq \binom{n}{2}; \\ \min\{n - 2m - 1, \lfloor \frac{n}{2} - \frac{2m}{n-1} \rfloor\}, & \text{if } 0 \leq m \leq \lfloor \frac{n}{3} \rfloor. \end{cases}$$

$$(2) \lambda_k(G) \cdot \lambda_k(\overline{G}) \geq 0.$$

Moreover, the above lower bounds are sharp.

Proof. (1) Since at least one of G and \overline{G} must be connected, we have $\lambda_k(G) + \lambda_k(\overline{G}) \geq 1$. For $m < n - 1$, $\lambda_k(G) + \lambda_k(\overline{G}) \geq \lfloor \frac{1}{2}\lambda(G) \rfloor + \lfloor \frac{1}{2}\lambda(\overline{G}) \rfloor \geq \lfloor \frac{1}{2}(\lambda(G) + \lambda(\overline{G}) - 1) \rfloor \geq \lfloor \frac{1}{2}(\max\{1, n - 1 - m\} - 1) \rfloor \geq \lfloor \frac{1}{2}(n - 2 - m) \rfloor$ by Proposition 1. So $\lambda_k(G) + \lambda_k(\overline{G}) \geq \max\{1, \lfloor \frac{1}{2}(n - 2 - m) \rfloor\}$. In particular, for $0 \leq m \leq \lfloor \frac{n}{3} \rfloor$, we can give a better lower bound of $\lambda_k(G) + \lambda_k(\overline{G})$ by Lemma 7, that is, $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(\overline{G}) \geq \lambda_n(\overline{G}) \geq \min\{n - 2m - 1, \lfloor \frac{n}{2} - \frac{2m}{n-1} \rfloor\}$.

To show the sharpness of the above lower bound for $\lfloor \frac{n}{3} \rfloor + 1 \leq m \leq \binom{n}{2}$, we consider the graph $G = K_{1, n-2} \cup K_1$. Then $m = n - 2$ and \overline{G} is a graph obtained from a complete graph K_{n-1} by attaching a pendant edge. Clearly, $\lambda_k(G) = 0$ and $\lambda_k(\overline{G}) = 1$. So $\lambda_k(G) + \lambda_k(\overline{G}) = 1 = \max\{1, \lfloor \frac{1}{2}(n - 2 - m) \rfloor\}$. To show the sharpness of the above lower bound for $0 \leq m \leq \lfloor \frac{n}{3} \rfloor$, we consider the graph $G = nK_1$. Thus $m = 0$ and $\overline{G} = K_n$. Since $\lambda_n(G) + \lambda_n(\overline{G}) = 0 + \lfloor \frac{n}{2} \rfloor = \min\{n - 2 \cdot 0 - 1, \lfloor \frac{n}{2} - \frac{2 \cdot 0}{n-1} \rfloor\}$, that is, the lower bound is sharp for $k = n$.

(2) The inequality follows from Theorem 1.

To show the sharpness of the above lower bound for $0 \leq m \leq \binom{n-1}{2}$, we consider the graph $G = G' \cup K_1$, where G' is a graph of order $n - 1$ and size m . Observe that G is disconnected. Thus, $\lambda_k(G) = 0$ and hence $\lambda_k(G) \cdot \lambda_k(\overline{G}) = 0$. To show the sharpness of the above lower bound for $\binom{n-1}{2} + 1 \leq m \leq \binom{n}{2}$, we consider a graph G of order $n - 1$ and size m . Note that $|E(\overline{G})| \leq \binom{n}{2} - \binom{n-1}{2} - 1 = n - 2$. Therefore, $\lambda_k(\overline{G}) = 0$ and hence $\lambda_k(G) \cdot \lambda_k(\overline{G}) = 0$. \square

It was pointed out by Harary [9] that given the number of vertices and edges of a graph, the largest connectivity possible can also be read out of the inequality $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Theorem 4. [9] For each n, m with $0 \leq n - 1 \leq m \leq \binom{n}{2}$,

$$\kappa(G) \leq \lambda(G) \leq \left\lfloor \frac{2m}{n} \right\rfloor,$$

where the maximum is taken over all graphs $G \in \mathcal{G}(n, m)$.

Corollary 2. For any graph $G \in \mathcal{G}(n, m)$ and $3 \leq k \leq n$, $\lambda_k(G) \leq \lfloor \frac{2m}{n} \rfloor$ for $m \geq n - 1$. Moreover, the upper bound is sharp.

Proof. Since $m \geq n - 1$, $\lambda_k(G) \leq \lambda(G) \leq \lfloor \frac{2m}{n} \rfloor$ by (1) of Observation 1 and Theorem 4. One can check that the complete bipartite graph $G = K_{r, r+1}$ satisfies that $\lambda_3(G) = r$, $m = e(G) = r(r+1)$ and $\lfloor \frac{2m}{n} \rfloor = \lfloor \frac{2r(r+1)}{2r+1} \rfloor = \lfloor r + \frac{r}{2r+1} \rfloor = r$. Thus $\lambda_3(G) = r = \lfloor \frac{2m}{n} \rfloor$ and so the upper bound is sharp. \square

Although the above bound of $\lambda_k(G)$ is the same as $\lambda(G)$, the graphs attaining the upper bound seem to be very rare. Actually, we can obtain some properties of these graphs.

Proposition 6. For any $G \in \mathcal{G}(n, m)$ and $3 \leq k \leq n$, if $\lambda_k(G) = \lfloor \frac{2m}{n} \rfloor$ for $m \geq n - 1$, then

- (1) $\frac{2m}{n}$ is not an integer;
- (2) $\delta(G) = \lfloor \frac{2m}{n} \rfloor$;
- (3) for $u, v \in V(G)$ such that $d_G(u) = d_G(v) = \lfloor \frac{2m}{n} \rfloor$, $uv \notin E(G)$.

Proof. One can check that the conclusion holds for the case $m = n - 1$. Assume $m \geq n$. We claim that $\frac{2m}{n}$ is not an integer; otherwise, let $r = \frac{2m}{n}$ be an integer. We will show that $\lambda_k(G) \leq r - 1 = \frac{2m}{n} - 1$ and get a contradiction. If G has at least one vertex v_i such that $d(v_i) > r$, then, since the average degree of G is exactly r , there must be a vertex v_j whose degree $d(v_j) < r$. From (1) of Observation 1, we have $\lambda_k(G) \leq \delta(G) \leq d(v_j) < r$, that is, $\lambda_k(G) \leq r - 1$. If, on the other hand, G is a regular graph, then by (3) of Observation 1, $\lambda_k(G) \leq \delta(G) - 1 = r - 1$. So (1) holds.

For a graph G such that $\frac{2m}{n}$ is not an integer, $\lfloor \frac{2m}{n} \rfloor = \lambda_k(G) \leq \delta(G) \leq \lfloor \frac{2m}{n} \rfloor$, that is, $\delta(G) = \lfloor \frac{2m}{n} \rfloor$. So (2) holds.

For $u, v \in V(G)$ such that $d_G(u) = d_G(v) = \lfloor \frac{2m}{n} \rfloor$, we claim that $uv \notin E(G)$; otherwise, $uv \in E(G)$. Since $d_G(u) = d_G(v) = \delta(G) = \lfloor \frac{2m}{n} \rfloor$, $\lambda_k(G) \leq \delta(G) - 1 = \lfloor \frac{2m}{n} \rfloor - 1$ by (3) of Observation 1, a contradiction. So (3) holds. \square

Corollary 3. For any graph G with n vertices and m edges, if $\frac{2m}{n}$ is an integer, then $\lambda_k(G) \leq \frac{2m}{n} - 1$.

Lemma 9. Let $G \in \mathcal{G}(n, m)$, and let k be an integer with $3 \leq k \leq n$. Then

- (1) $\lambda_k(G) + \lambda_k(\overline{G}) \leq M(n, m)$, where

$$M(n, m) = \begin{cases} n - \lceil \frac{k}{2} \rceil, & \text{if } m \geq n - 1, \\ & \text{or } k \text{ is even and } m = 0, \\ & \text{or } k \text{ is odd and } 0 \leq m \leq \frac{k-1}{2}; \\ n - \lceil \frac{k}{2} \rceil - 1, & \text{if } k \text{ is even and } 1 \leq m < n - 1, \\ & \text{or } k \text{ is odd and } \frac{k+1}{2} \leq m < n - 1. \end{cases}$$

(2) $\lambda_k(G) \cdot \lambda_k(\overline{G}) \leq N(n, m)$, where

$$N(n, m) = \begin{cases} 0, & \text{if } 0 \leq m \leq n - 2; \\ (\frac{2m}{n} - 1)(n - 2 - \frac{2m}{n}), & \text{if } m \geq n - 1 \text{ and } 2m \equiv 0 \pmod{n}; \\ \lfloor \frac{2m}{n} \rfloor (n - 2 - \lfloor \frac{2m}{n} \rfloor), & \text{otherwise.} \end{cases}$$

Moreover, these upper bounds are sharp.

Proof. From Theorem 1, (1) holds for $m \geq n - 1$. We have given Example 1 to show that the upper bound is sharp. From Proposition 4, $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(\overline{G}) = n - \lceil \frac{k}{2} \rceil$ for k even and $m = 0$, or k odd and $0 \leq m \leq \frac{k-1}{2}$. So for k even and $1 \leq m < n - 1$, or k odd and $\frac{k+1}{2} \leq m < n - 1$, $\lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil \frac{k}{2} \rceil - 1$.

To prove the sharpness of the bound for k odd and $\frac{k+1}{2} \leq m < n - 1$, we consider the graph $G = K_{1, \frac{k+1}{2}} \cup (n - \frac{k+3}{2})K_1$. Clearly, \overline{G} is a graph obtained from the complete graph K_n by deleting all the edges of a star $K_{1, \frac{k+1}{2}}$. On one hand, by Lemma 4, it follows that $\lambda_k(\overline{G}) \leq n - \frac{k+1}{2} - 1$. On the other hand, by Lemma 4, we have $\lambda_k(\overline{G} + e) = n - \frac{k+1}{2}$ for any $e \notin E(\overline{G})$, which implies that $\lambda_k(\overline{G}) \geq n - \frac{k+1}{2} - 1$ (note that $\lambda_k(H \setminus e) \geq \lambda_k(H) - 1$ for a connected graph H , where $e \in E(H)$). So $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(\overline{G}) = n - \frac{k+1}{2} - 1$. By the same reason, for k even and $1 \leq m < n - 1$ one can check that the graph $G = K_2 \cup (n - 2)K_1$ satisfies that $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(\overline{G}) = n - \frac{k}{2} - 1$.

(2) First, if $0 \leq m \leq n - 2$, then $G \in \mathcal{G}(n, m)$ is disconnected. So $\lambda_k(G) \cdot \lambda_k(\overline{G}) = 0$. Next, if $m \geq n - 1$ and $\frac{2m}{n} = r$ is an integer, then $\frac{2e(\overline{G})}{n} = n - 1 - r$ is also an integer. From Corollary 3, we have $\lambda_k(G) \leq r - 1$ and $\lambda_k(\overline{G}) \leq n - 2 - r$. So $\lambda_k(G) \cdot \lambda_k(\overline{G}) \leq (r - 1)(n - 2 - r) = (\frac{2m}{n} - 1)(n - 2 - \frac{2m}{n})$. Finally, if $2m = nr + \ell$ where $1 \leq \ell \leq n - 1$, then $\Delta(G) \geq r + 1$. By (1) of Observation 1, $\lambda_k(\overline{G}) \leq \delta(\overline{G}) = n - 1 - \Delta(G) \leq n - 2 - r$. So $\lambda_k(G) \cdot \lambda_k(\overline{G}) \leq r(n - 2 - r) = \lfloor \frac{2m}{n} \rfloor (n - 2 - \lfloor \frac{2m}{n} \rfloor)$.

To show the sharpness of the upper bound for $0 \leq m \leq n - 2$, we consider the graph G of size m . Clearly, $\lambda_k(G) = 0$ and hence $\lambda_k(G) \cdot \lambda_k(\overline{G}) = 0$. For $m \geq n - 1$ and $\frac{2m}{n} = r + \ell$ ($1 \leq \ell \leq n - 1$), we let $G = P_4$. Then $\lambda_3(G) = 1 = \lfloor \frac{6}{4} \rfloor = \lfloor \frac{2m}{n} \rfloor$ and $\lambda_3(\overline{G}) = \lambda_3(P_4) = 1 = 4 - 2 - \lfloor \frac{6}{4} \rfloor = n - 2 - \lfloor \frac{2m}{n} \rfloor$. So $\lambda_3(G) \cdot \lambda_3(\overline{G}) = \lfloor \frac{2m}{n} \rfloor (n - 2 - \lfloor \frac{2m}{n} \rfloor)$. \square

To show the sharpness of the upper bound for $m \geq n - 1$ and $2m \equiv 0 \pmod{n}$, we consider the following example.

Example 2. Let G be a cycle $C_n = w_1 w_2 \cdots w_n w_1$ ($n \geq 9$). Clearly, $\lambda_3(G) = 1 = \frac{2m}{n} - 1$. Since $\frac{2m}{n} = 2$ is an integer, it suffices to show that $\lambda_3(\overline{G}) = n - 2 - \frac{2m}{n} = n - 4$. First we show that $\lambda_3(\overline{G}) \geq n - 4$. For arbitrary $S = \{x, y, z\} \subseteq V(G) = V(C_n)$.

By the definition of $\lambda_3(\overline{G})$, we need to show that $\lambda(S) \geq n - 4$. If $d_{C_n}(x, y) = 1$ and $d_{C_n}(y, z) = 1$, without loss of generality, let $N_{C_n}(x) = \{x_1, y\}$ and $N_{C_n}(z) = \{y, z_2\}$, then the trees T_i induced by the edges in $\{xw_i, yw_i, zw_i\}$ together with the tree T_1 induced by the edges in $\{xz, zx_1, x_1y\}$ form $n - 4$ edge-disjoint S -trees in \overline{G} (see Figure 5 (a)) and hence $\lambda(S) \geq n - 4$, where $\{w_1, w_2, \dots, w_{n-5}\} = V(G) \setminus \{x, y, z, x_1, z_2\}$. If $d_{C_n}(x, y) = 2$ and $d_{C_n}(y, z) = 1$, without loss of generality, let $N_{C_n}(x) = \{x_1, y_1\}$ and $N_{C_n}(y) = \{y_1, z\}$ and $N_{C_n}(z) = \{y, z_2\}$, then the trees T_i induced by the edges in $\{xw_i, yw_i, zw_i\}$ together with the tree T_1 induced by the edges in $\{xy, xz\}$ and the tree T_2 induced by the edges in $\{z_2x, z_2y, z_2y_1, y_1z\}$ form $n - 4$ edge-disjoint S -trees in \overline{G} (see Figure 5 (b)) and hence $\lambda(S) \geq n - 4$, where $\{w_1, w_2, \dots, w_{n-6}\} = V(G) \setminus \{x, y, z, x_1, y_1, z_2\}$. If $d_{C_n}(x, y) \geq 3$ and $d_{C_n}(y, z) = 1$, without loss of generality, let $N_{C_n}(x) = \{x_1, x_2\}$ and $N_{C_n}(z) = \{y_1, z\}$ and $N_{C_n}(y) = \{y, z_2\}$, then the trees T_i induced by the edges in $\{xw_i, yw_i, zw_i\}$ together with the tree T_1 induced by the edges in $\{xy, xz\}$ and the tree T_2 induced by the edges in $\{z_2x, z_2y, z_2y_1, y_1z\}$ and the tree T_3 induced by the edges in $\{xy_1, y_1x_1, x_1y, x_1z\}$ form $n - 4$ edge-disjoint S -trees in \overline{G} (see Figure 5 (c)), and hence $\lambda(S) \geq n - 4$, where $\{w_1, w_2, \dots, w_{n-7}\} = V(G) \setminus \{x, y, z, x_1, x_2, y_1, z_2\}$. If $d_{C_n}(x, y) = 2$ and $d_{C_n}(y, z) = 2$, without loss of generality, let $N_{C_n}(x) = \{x_1, y_1\}$ and $N_{C_n}(y) = \{y_1, z_1\}$ and $N_{C_n}(z) = \{z_1, z_2\}$, then the trees T_i induced by the edges in $\{xw_i, yw_i, zw_i\}$ together with the tree T_1 induced by the edges in $\{xz, xy\}$ and the tree T_2 induced by the edges in $\{xz_2, yz_2, yz\}$ and the tree T_3 induced by the edges in $\{x_1y, x_1z, x_1z_1, xz_1\}$ form $n - 4$ edge-disjoint S -trees in \overline{G} (see Figure 5 (d)), and hence $\lambda(S) \geq n - 4$, where $\{w_1, w_2, \dots, w_{n-7}\} = V(G) \setminus \{x, y, z, x_1, y_1, z_1, z_2\}$. If $d_{C_n}(x, y) \geq 3$ and $d_{C_n}(y, z) = 2$, without loss of generality, let $N_{C_n}(x) = \{x_1, x_2\}$ and $N_{C_n}(y) = \{y_1, z_1\}$ and $N_{C_n}(z) = \{z_1, z_2\}$, then the trees T_i induced by the edges in $\{xw_i, yw_i, zw_i\}$ together with the tree T_1 induced by the edges in $\{xz, xy\}$ and the tree T_2 induced by the edges in $\{xz_2, z_2y, yz\}$ and the tree T_3 induced by the edges in $\{x_1y, x_1z, x_1y_1, x_1x\}$ and the tree T_4 induced by the edges in $\{x_2y, x_2z, x_2z_1, z_1x\}$ form $n - 4$ edge-disjoint S -trees in \overline{G} (see Figure 5 (e)), and thus $\lambda(S) \geq n - 4$, where $\{w_1, w_2, \dots, w_{n-8}\} = V(G) \setminus \{x, y, z, x_1, x_2, y_1, y_2, z_2\}$. Suppose that $d_{C_n}(x, y) \geq 3$ and $d_{C_n}(y, z) \geq 3$, without loss of generality, let $N_{C_n}(x) = \{x_1, x_2\}$ and $N_{C_n}(y) = \{y_1, y_2\}$ and $N_{C_n}(z) = \{z_1, z_2\}$. Then the trees T_i induced by the edges in $\{xw_i, yw_i, zw_i\}$ together with the tree T_1 induced by the edges in $\{xz, xy\}$ and the tree T_2 induced by the edges in $\{xz_2, yz_2, yz\}$ and the tree T_3 induced by the edges in $\{x_1y, x_1z, x_1y_1, y_1x\}$ and the tree T_4 induced by the edges in $\{x_2y, x_2z, x_2y_2, y_2x\}$ and the tree T_5 induced by the edges in $\{x_1z_1, yz_1, y_2z_1, y_2z\}$ and the tree T_6 induced by the edges in $\{x_2y, x_2z, x_2y_2, y_2x\}$ form $n - 4$ edge-disjoint S -trees in \overline{G} (see Figure 5 (f)), and hence $\lambda(S) \geq n - 4$, where $\{w_1, w_2, \dots, w_{n-9}\} = V(G) \setminus \{x, y, z, x_1, x_2, y_1, y_2, z_1, z_2\}$. From the arbitrariness of S , we know that $\lambda_3(\overline{G}) \geq n - 4$. We now prove that $\lambda_3(\overline{G}) \leq n - 4$ for $\overline{G} = \overline{C_n}$. Choose $S = \{w_1, w_2, w_3\} \subseteq V(G) = V(C_n)$. Then $w_1w_n \in E(C_n)$ and $w_3w_4 \in E(C_n)$. Thus $|E(\overline{G}[S])| = 1$ and $|E_{\overline{G}}[S, \overline{S}]| = 3(n - 3) - 2$, which implies that $|E(\overline{G}[S]) \cup E_{\overline{G}}[S, \overline{S}]| = 3(n - 3) - 1$ (see Figure 5 (g)). One can see that each tree connecting S in \overline{G} uses at least 3 edges from $E(\overline{G}[S]) \cup E_{\overline{G}}[S, \overline{S}]$. Therefore $\lambda_3(\overline{G}) \leq \frac{3(n-3)-1}{3} = n - 3 - \frac{1}{3}$, which results in $\lambda_3(\overline{G}) \leq n - 4$ since $\lambda_3(\overline{G})$ is an integer. So $\lambda_3(\overline{G}) = n - 4$ and $\lambda_3(G) \cdot \lambda_3(\overline{G}) = \lambda_3(C_n) \cdot \lambda_3(\overline{C_n}) = 1 \cdot (n - 4) = \binom{2m}{n} - 1(n - 2 - \frac{2m}{n})$. The upper bound is sharp.

Combining with Lemmas 8 and 9, we complete the proof of Theorem 2.

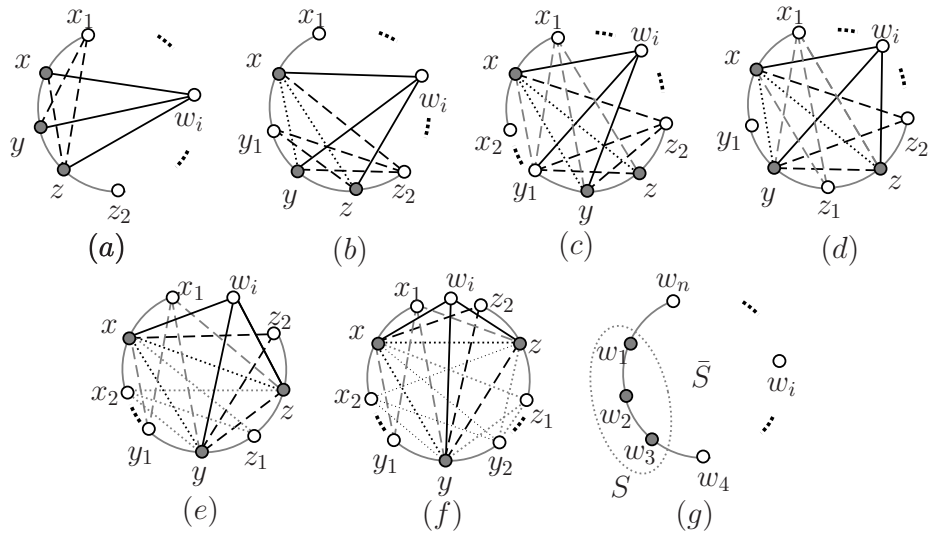


Figure 5. Graphs for Example 3.

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