

THE EGZ-CONSTANT AND SHORT ZERO-SUM SEQUENCES OVER FINITE ABELIAN GROUPS

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ABSTRACT. Let G be an additive finite abelian group with exponent $\exp(G)$. Let $\eta(G)$ be the smallest integer t such that every sequence of length t has a nonempty zero-sum subsequence of length at most $\exp(G)$. Let $\mathfrak{s}(G)$ be the EGZ-constant of G , which is defined as the smallest integer t such that every sequence of length t has a zero-sum subsequence of length $\exp(G)$. Let p be an odd prime. We determine $\eta(G)$ for some groups G with $\mathbf{D}(G) \leq 2\exp(G) - 1$, including the p -groups of rank three and the p -groups $G = C_{\exp(G)} \oplus C_p^r$. We also determine $\mathfrak{s}(G)$ for the groups G above with more larger exponent than $\mathbf{D}(G)$, which confirms a conjecture by Schmid and Zhuang from 2010, where $\mathbf{D}(G)$ denotes the Davenport constant of G .

1. INTRODUCTION

Throughout this paper, let p denote a prime. Let G be an additive finite abelian group with exponent $\exp(G)$. Let $S = g_1 \cdot \dots \cdot g_k$ be a sequence over G . We call S a zero-sum sequence if $0 = \sum_{i=1}^k g_i$. The Davenport's constant, denoted by $\mathbf{D}(G)$, is the minimal integer t such that every sequence S over G of length $|S| \geq t$ has a nonempty zero-sum subsequence. Let $\eta(G)$ be the minimal integer t such that every sequence of length t has a nonempty zero-sum subsequence of length less than or equal to $\exp(G)$. Let $\mathfrak{s}(G)$ be the minimal integer t such that every sequence of length t has a zero-sum subsequence of length $\exp(G)$.

These are classical invariants in combinatorial number theory and have received a lot of attention (see [17], [18], [8], [2], [11]). For G is cyclic, we have $\eta(G) = |G|$, and $\mathfrak{s}(G) = 2|G| - 1$ by the well known Erdős-Ginzburg-Ziv theorem [5]. For the case that G is of rank two, the key step of determine $\eta(G)$ (resp. $\mathfrak{s}(G)$) is to determine $\eta(C_p^2)$ (resp. $\mathfrak{s}(C_p^2)$). In 1969, Olson [17] proved $\eta(C_p^2) = 3p - 2$. While the determining of $\mathfrak{s}(C_p^2)$ is very complicated. In 1983, Kemnitz [15] conjectured that $\mathfrak{s}(C_p^2) = 4p - 3$ and it was confirmed by C. Reiher [18] in 2007. The precise values of $\eta(G)$ and $\mathfrak{s}(G)$ for groups with rank at most two has been summarized in ([13, Theorem 5.8.3]) as follows.

If $G = C_m \oplus C_n$ with $1 \leq m|n$, then $\mathfrak{s}(G) = \eta(G) + n - 1 = 2m + 2n - 3$.

The situation is very different for groups of higher rank. Even for the group $G = C_p^3$ with p being a prime, the precise value of the $\eta(G)$, $\mathfrak{s}(G)$ is unknown (for general p). Fan, Gao, Wang, and Zhong [8] determined the $\eta(G)$ and $\mathfrak{s}(G)$ for a special type groups with rank three. When $G = C_3^r$, the precise value of $\eta(G)$ and $\mathfrak{s}(G)$ has been determined for $r \leq 6$ (see [4]). Apart the results mentioned above, Schmid and Zhuang [19] proved that if G is a finite abelian p -group with $\mathbf{D}(G) = 2\exp(G) - 1$, then $2\mathbf{D}(G) - 1 = \eta(G) + \exp(G) - 1 = \mathfrak{s}(G)$, which has

been generalized recently by Geroldinger, Gryniewicz and Schmid [12, Theorem 4.2]. Schmid and Zhuang further conjectured the following.

Conjecture 1.1. ([19]) *Let G be a finite abelian p -group with $D(G) \leq 2 \exp(G) - 1$. Then*

$$2D(G) - 1 = \eta(G) + \exp(G) - 1 = s(G).$$

In this paper we verify this conjecture for some p -groups with $D(G) < 2 \exp(G) - 1$ and our main results are the following.

Theorem 1.2. *Let a, n be positive integers, let H be a finite abelian p -group, and let $G = C_{ap^n} \oplus H$. Suppose that $D(C_{p^n} \oplus H) \leq 2p^n - 1$. If $p > 2r(H)$ then*

$$\eta(G) = 2D(G) - ap^n = ap^n + 2D(H) - 2$$

provided that H satisfies one of the following conditions:

- (1) $D(H) \leq 2 \exp(H)$.
- (2) $\lceil (k + \frac{1}{2}) \exp(H) \rceil < D(H) \leq (k + 1) \exp(H)$ for some integer $k \geq 2$.

Theorem 1.3. *Let H be a finite abelian p -group with $\exp(H) = p^m$, and let $G = C_{ap^n} \oplus H$. If $p > 2r(H)$, $p^n \geq D(H)$ and $a > |H|p^{2m-n}$, then*

$$s(G) = \eta(G) + ap^n - 1 = 2ap^n + 2D(H) - 3$$

provided that H satisfies one of the following conditions:

- (1) $D(H) \leq 2 \exp(H)$.
- (2) $\lceil (k + \frac{1}{2}) \exp(H) \rceil < D(H) \leq (k + 1) \exp(H)$ for some integer $k \geq 2$.

It is easy to see that the conditions of Theorem 1.2 are fulfilled by the following groups H and G .

- $r(H) = 2$ and $D(C_{p^n} \oplus H) \leq 2p^n - 1$.
- $D(C_{p^n} \oplus H) \leq 2p^n - 1$, $H = C_{p^m}^r$ and $p \geq 2r + 1$.

It is easy to see that the conditions of Theorem 1.3 are fulfilled by the following groups H and G .

- $r(H) = 2$, $D(C_{p^n} \oplus H) \leq 2p^n - 1$ and $a > |H|p^{2m-n}$.
- $D(C_{p^n} \oplus H) \leq 2p^n - 1$, $H = C_{p^m}^r$, $p \geq 2r + 1$ and $a > |H|p^{2m-n}$

2. PRELIMINARIES

Let \mathbb{N} denote the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a real number x , we denote by $\lfloor x \rfloor$ the largest integer that is less than or equals to x , and denote by $\lceil x \rceil$ the smallest integer that is greater than or equals to x .

Throughout, all abelian groups will be written additively. By the Fundamental Theorem of Finite Abelian Groups we have

$$G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}$$

where $r = r(G) \in \mathbb{N}_0$ is the rank of G , $n_1, \dots, n_r \in \mathbb{N}$ are integers with $1 < n_1 | \dots | n_r$, moreover, n_1, \dots, n_r are uniquely determined by G , and $n_r = \exp(G)$ is the *exponent* of G . Let

$$D^*(G) = 1 + \sum_{i=1}^r (n_i - 1).$$

For $g_1, \dots, g_l \in G$ (repetition allowed), we call $S = g_1 \cdot \dots \cdot g_l$ a *sequence* over G . We write sequences S in the form

$$S = \prod_{g \in G} g^{\mathbf{v}_g(S)} \text{ with } \mathbf{v}_g(S) \in \mathbb{N}_0 \text{ for all } g \in G.$$

We call $\mathbf{v}_g(S)$ the *multiplicity* of g in S .

For $S = g_1 \cdot \dots \cdot g_l = \prod_{g \in G} g^{\mathbf{v}_g(S)}$, we call

- $|S| = l = \sum_{g \in G} \mathbf{v}_g(S) \in \mathbb{N}_0$ the *length* of S .
- $\sigma(S) = \sum_{i=1}^l g_i = \sum_{g \in G} \mathbf{v}_g(S)g \in G$ the *sum* of S .
- S is a *zero-sum sequence* if $\sigma(S) = 0$.
- S is a *short zero-sum sequence* if it is a zero-sum sequence of length $|S| \in [1, \exp(G)]$

Let $S = g_1 \cdot \dots \cdot g_l$ be a sequence over G of length $|S| = l \in \mathbb{N}_0$ and let $g \in G$. For every $k \in \mathbb{N}_0$ let

$$N_g^k(S) = |\{I \subset [1, l] \mid \sum_{i \in I} g_i = g, |I| = k\}|$$

denote the number of subsequences T of S having sum $\sigma(T) = g$ and length $|T| = k$ (counted with the multiplicity of their appearance in S).

For convenience, let $N^k(S)$ denote $N_0^k(S)$.

Lemma 2.1. ([4, Lemma 3.2]) *Let H be a finite abelian group, and let $G = C_n \oplus H$. If $\exp(H) \mid n$ then $\eta(G) \leq n + 2\mathbf{D}(H) - 2$.*

Lemma 2.2. ([16]) *Let G be a finite abelian p -group. Then*

$$\mathbf{D}(G) = \mathbf{D}^*(G).$$

Moreover, if S is a sequence over G with $|S| = l \geq \mathbf{D}^*(G)$, then

$$1 - N^1(S) + N^2(S) + \dots + (-1)^l N^l(S) \equiv 0 \pmod{p}.$$

Lemma 2.3. *Let m be a positive integer, let G be a finite abelian p -group, and let S be a sequence over G of length $|S| \geq \mathbf{D}(G) + p^m - 1$. Let $t = \lfloor \frac{|S|}{p^m} \rfloor$. Then*

$$1 + \sum_{j=1}^t (-1)^j N^{jp^m}(S) \equiv 0 \pmod{p}.$$

Proof. Let $G \oplus C_{p^m} = G \oplus \langle e \rangle$ with $\langle e \rangle = C_{p^m}$. Let $\varphi : G \rightarrow G \oplus C_{p^m}$ be defined by $\varphi(g) = g + e$ for every $g \in G$. Let $S = g_1 \cdot \dots \cdot g_l$. Then $\varphi(S) = (g_1 + e) \cdot \dots \cdot (g_l + e)$ is a sequence over $G \oplus C_{p^m}$. Thus let $\varphi(T)$ be a subsequence of $\varphi(S)$ over $G \oplus C_{p^m}$, $\sigma(\varphi(T)) = 0$ if and only if $\sigma(T) = 0$ and $|T| \equiv 0 \pmod{p^m}$.

Apply lemma 2.2 to the sequence $\varphi(S)$, we get

$$1 + \sum_{j=1}^t (-1)^j N^{jp^m}(\varphi(S)) \equiv 0 \pmod{p},$$

hence

$$1 + \sum_{j=1}^t (-1)^j N^{jp^m}(S) \equiv 0 \pmod{p}.$$

This completes the proof. □

The following congruence is first used by Lucas [14], we give a proof for the convenience of the reader.

Lemma 2.4. *Let a, b be positive integers with $a = a_n p^n + \cdots + a_1 p + a_0$ and $b = b_n p^n + \cdots + b_1 p + b_0$ be the p -adic expansions, where p is a prime, define $\binom{k}{0} = 1$ for $k \geq 0$. Then*

$$\begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} a_n \\ b_n \end{pmatrix} \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} \cdots \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \pmod{p}.$$

Proof. We have

$$\begin{aligned} (1+x)^a &= (1+x)^{a_n p^n + \cdots + a_1 p + a_0} \\ &= (1+x^{p^n})^{a_n} \cdots (1+x^p)^{a_1} (1+x)^{a_0} \pmod{p} \end{aligned}$$

Since $0 \leq a_i \leq p-1$, comparing the coefficient of x^b , we get the desired result. \square

Lemma 2.5. *Let n and k be positive integers with $1 \leq 2k \leq n$, and let $A = \left(\binom{n-j}{i} \right)_{0 \leq i, j \leq k}$, that is*

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \binom{n}{1} & \binom{n-1}{1} & \cdots & \binom{n-k}{1} \\ \binom{n}{2} & \binom{n-1}{2} & \cdots & \binom{n-k}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n}{k} & \binom{n-1}{k} & \cdots & \binom{n-k}{k} \end{pmatrix}_{(k+1) \times (k+1)}.$$

Then we have

$$\det(A) = \frac{1}{\prod_{1 \leq t \leq k} t!} \prod_{1 \leq i < j \leq k} (j-i).$$

Proof. Let

$$B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ n & n-1 & \cdots & n-k \\ n(n-1) & (n-1)(n-2) & \cdots & (n-k)(n-k-1) \\ \vdots & \vdots & \ddots & \vdots \\ n \cdots (n-k+1) & (n-1) \cdots (n-k) & \cdots & (n-k) \cdots (n-2k+1) \end{pmatrix}.$$

In what follows, we denote the i th row of B by $Row_B(i)$.

Firstly, replace $Row_B(3)$ by $Row_B(3) + Row_B(2)$, and we get the following matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ n & n-1 & \cdots & n-k \\ n^2 & (n-1)^2 & \cdots & (n-k)^2 \\ \vdots & \vdots & \ddots & \vdots \\ n \cdots (n-k+1) & (n-1) \cdots (n-k) & \cdots & (n-k) \cdots (n-2k+1) \end{pmatrix}.$$

by abuse of language we also denote the corresponding new matrix by B .

Similarly, let $f_i(x) = x(x-1) \cdots (x-i+2) = x^{i-1} + a_{i-2}x^{i-2} + \cdots + a_1x$. Then replace $Row_B(i)$ by $Row_B(i) - a_{i-2}Row_B(i-1) - \cdots - a_1Row_B(2)$ successively for

$4 \leq i \leq k + 1$ and after each step also denote the corresponding new matrix by B , we can get the following matrix

$$C = \begin{pmatrix} 1 & 1 & \dots & 1 \\ n & n-1 & \dots & n-k \\ n^2 & (n-1)^2 & \dots & (n-k)^2 \\ \vdots & \vdots & \ddots & \vdots \\ n^k & (n-1)^k & \dots & (n-k)^k \end{pmatrix}.$$

It is well known that $\det(C)$ is a Vandermonde determinant, then clearly

$$\det(A) = \frac{1}{\prod_{1 \leq t \leq k} t!} \det(B) = \frac{1}{\prod_{1 \leq t \leq k} t!} \det(C) = \frac{1}{\prod_{1 \leq t \leq k} t!} \prod_{1 \leq i < j \leq k} (j - i).$$

□

Lemma 2.6. [8, Theorem 1.2] *Let H be an arbitrary finite abelian group with $\exp(H) = u \geq 2$, and let $G = C_{vu} \oplus H$. If $v \geq \max\{u|H| + 1, 4|H| + 2u\}$, then $s(G) = \eta(G) + \exp(G) - 1$.*

Lemma 2.7. ([12, Page 7, (4.1)]) *Let K be subgroup of a finite abelian group G . Then, $\eta(G) \leq \exp(G/K)(\eta(K) - 1) + \eta(G/K)$.*

Lemma 2.8. ([3]) *Let a, n be a positive integer, let H be a finite abelian p -group with $D(H) \leq p^n - 1$, and let $G = C_{ap^n} \oplus H$. Then, $D(G) = ap^n + D(H) - 1$.*

3. PROOF OF THE MAIN THEOREMS

Let $G = C_{p^n} \oplus H$ be a finite abelian p -group with $\exp(H) = p^m$. Let k be the integer with

$$kp^m \leq D(H) - 1 < (k + 1)p^m,$$

and let

$$v = (k + 1)p^m - D(H).$$

The following technical result is crucial in the proof of Theorem 1.2.

Lemma 3.1. *Let $G = C_{p^n} \oplus H$ be a finite p -group with $\exp(H) = p^m$, and let S be a sequence over G of length $s = |S| = p^n + 2D(H) - 2$. Suppose that S has no short zero-sum subsequence. If $p^n \geq 2D(H) - 2$, then we have the following congruences:*

$$(3.1) \quad 1 + \sum_{u=0}^h \binom{h}{u} \sum_{j=1}^k (-1)^{j-1} N^{p^n + jp^m - u}(S) \equiv 0 \pmod{p}$$

holds for every $h \in [0, v]$, and

$$(3.2) \quad \sum_{j=1}^k (-1)^{j-1} N^{p^n + jp^m - h}(S) \equiv 0 \pmod{p}$$

for every $h \in [1, v]$, and

$$(3.3) \quad \binom{|S|}{ip^m} + \sum_{j=1}^k (-1)^{j-1} \sum_{u=0}^v \binom{|S| - p^n - jp^m + u}{ip^m} \binom{v}{u} N^{p^n + jp^m - u}(S) \equiv 0 \pmod{p}$$

for every $i \in [0, k]$.

Proof. We first have the following claim.

Claim. $N^i(S) = 0$ for every $i \in [1, p^n] \cup [p^n + D(H), |S|]$.

Since S has no short zero-sum subsequence, we obtain $N^i(S) = 0$ for every $i \in [1, p^n]$. Assume that $N^i(S) \neq 0$ for some $i \in [p^n + D(H), |S|]$, then S has a zero-sum subsequence W of length $|W| = i \geq p^n + D(H) = D(G) + 1$, which implies W can be divided into two nonempty zero-sum subsequences $W = W_1W_2$ with $|W_1| \leq |W_2|$. Since $2D(H) - 2 \leq p^n$, we have

$$|W_1| \leq \frac{|W|}{2} \leq \frac{|S|}{2} = \frac{p^n + 2D(H) - 2}{2} \leq p^n = \exp(G),$$

it is a contradiction completing the proof of the Claim.

Consider the following homomorphism

$$\varphi: G = C_{p^n} \oplus H \rightarrow C_{p^n} \oplus H \oplus C_{p^m} = G \oplus \langle e \rangle$$

with $\varphi(g) = g + e$ for every $g \in G$, where $\langle e \rangle = C_{p^m}$. Let $S = g_1 \cdots g_s$. Then $\varphi(S) = (g_1 + e) \cdots (g_s + e)$ is a sequence over $G \oplus C_{p^m}$.

For $i \in [0, k]$, let T be an arbitrary subsequence of S of length

$$|T| = |S| - ip^m.$$

Note that

$$\begin{aligned} |T0^h| &= |T| + h = p^n + 2D(H) - 2 - ip^m + h \geq p^n + D(H) - 1 + p^m \\ &= D(G) + p^m - 1 \end{aligned}$$

holds for $i \in [0, k-1]$ and $h \in [0, v]$, or $i = k$ and $h = v$. Applying Lemma 2.3 to the sequence $\varphi(T0^h)$ with $i \in [0, k-1]$ and $h \in [0, v]$, or $i = k$ and $h = v$, we get

$$(3.4) \quad 1 + \sum_{j=1}^t (-1)^j N^{jp^m}(\varphi(T0^h)) \equiv 0 \pmod{p}.$$

where $t = \lfloor \frac{|T0^h|}{p^m} \rfloor$. Therefore,

$$(3.5) \quad 1 + \sum_{j=1}^t (-1)^j \left(\sum_{u=0}^h \binom{h}{u} \right) N^{jp^m - u}(T) \equiv 0 \pmod{p},$$

since $N^{jp^m}(\varphi(T0^h)) = \sum_{u=0}^h \binom{h}{u} N^{jp^m - u}(T)$ for every $j \in [1, t]$.

Note that $N^i(T) \leq N^i(S)$. Applying the claim above we obtain, $N^i(T) = 0$ for every $i \in [1, p^n] \cup [p^n + D(H), |T|]$. Since $p^n + D(H) = p^n + (k+1)p^m - v$, by (3.5), we obtain

$$(3.6) \quad 1 + \sum_{u=0}^h \binom{h}{u} \sum_{j=1}^k (-1)^{j-1} N^{p^n + jp^m - u}(T) \equiv 0 \pmod{p}$$

holds for every pair of (h, i) with $h \in [0, v]$ and $i \in [0, k-1]$, or $h = v$ and $i = k$.

Taking $T = S$ in (3.6) we obtain (3.1).

Let $F(h) = 1 + \sum_{u=0}^h \binom{h}{u} \sum_{j=1}^k (-1)^{j-1} N^{p^n + jp^m - u}(S)$. By (3.1), we obtain that $F(h+1) - F(h) \equiv 0 \pmod{p}$. That is,

$$(3.7) \quad \begin{aligned} &\sum_{j=1}^k (-1)^{j-1} N^{p^n + jp^m - (h+1)}(S) = \\ &-\sum_{u=0}^h \left(\binom{h+1}{u} - \binom{h}{u} \right) \sum_{j=1}^k (-1)^{j-1} N^{p^n + jp^m - u}(S) \pmod{p}. \end{aligned}$$

Taking $h = 0$ in (3.7), we obtain

$$\sum_{j=1}^k (-1)^{j-1} N^{p^n + jp^m - 1}(S) \equiv 0 \pmod{p}.$$

This proves (3.2) for $h = 1$. Suppose that (3.2) is true for all $h < \ell$ ($\leq v$). Again by (3.7) taking $h = \ell - 1$, we obtain $\sum_{j=1}^k (-1)^{j-1} N^{p^n + jp^m - \ell}(S) \equiv 0 \pmod{p}$ completing the proof of (3.2). Now it remains to prove (3.3).

By (3.6) we have

$$(3.8) \quad \sum_{|T|=|S|-ip^m} (1 + \sum_{u=0}^v \binom{v}{u}) \sum_{j=1}^k (-1)^{j-1} N^{p^n + jp^m - u}(T) \equiv 0 \pmod{p},$$

where the sum is taken over all $T|S$ of length $|T| = |S| - ip^m$.

Note that each subsequence W of S of length $|W| \leq |S| - ip^m$ can be extended to a subsequence T of length $|T| = |S| - ip^m$ in $\binom{|S|-|W|}{|T|-|W|} = \binom{|S|-|W|}{|S|-|T|} = \binom{|S|-|W|}{ip^m}$ way. Therefore, the left side of (3.8) equals

$$\binom{|S|}{ip^m} + \sum_{j=1}^k (-1)^{j-1} \sum_{u=0}^v \binom{|S| - p^n - jp^m + u}{ip^m} \binom{v}{u} N^{p^n + jp^m - u}(S).$$

Now (3.3) follows. □

Remark 3.2. Note that v could be 0 and the list of (3.2) is empty.

Proposition 3.3. *Let H be a finite abelian p -group with $2r(H) < p$, and let $G = C_{p^n} \oplus H$ with $D(G) \leq 2p^n - 1$. Let $\exp(H) = p^m$, and let $D(H) - 1 = kp^m + t$ with k a integer and $t \in [0, p^m - 1]$. If $k = 1$ or $2t \geq p^m$, then*

$$\eta(G) = 2D(G) - p^n = p^n + 2D(H) - 2.$$

Proof. By Lemma 2.1, it suffices to prove that $\eta(G) \leq p^n + 2D(H) - 2$.

Let S be a sequence over G of length $s = |S| = p^n + 2D(H) - 2 = p^n + 2kp^m + 2t$. We need to show S has a short zero-sum subsequence. Assume to the contrary that S has no short zero-sum sequence.

Case 1 $k = 1$.

Since $p > 2r(H)$, we know that p is a odd prime and $D(H) = D^*(H)$ is odd. Therefore,

$$v = 2p^m - D(H) \geq 1.$$

In this case, (3.2) becomes

$$(3.9) \quad N^{p^n + p^m - h}(S) \equiv 0 \pmod{p}$$

for every $h \in [1, v]$.

By (3.3) taking $i = 1$, we obtain

$$\binom{p^n + 2p^m + 2t}{p^m} + \sum_{u=0}^v \binom{p^m + 2t + u}{p^m} N^{p^n + p^m - u}(S) \equiv 0 \pmod{p}.$$

This together with (3.9) gives that

$$\binom{p^n + 2p^m + 2t}{p^m} + \binom{p^m + 2t}{p^m} N^{p^n + p^m}(S) \equiv 0 \pmod{p}.$$

It follows from Lemma 2.4 that

$$(3.10) \quad \binom{2p^m + 2t}{p^m} + \binom{p^m + 2t}{p^m} N^{p^n + p^m}(S) \equiv 0 \pmod{p}.$$

Again by Lemma 2.4 and according to $2t < p^m$ or not, from (3.10) we know that either

$$2 + N^{p^n+p^m}(S) \equiv 0 \pmod{p}$$

or

$$3 + 2N^{p^n+p^m}(S) \equiv 0 \pmod{p}.$$

Both contradict to $1 + N^{p^n+p^m}(S) \equiv 0 \pmod{p}$ by (3.1).

Case 2: $2t \geq p^m$

From the assumption that $2r(H) < p$ we infer that $kp^m + t = D(H) - 1 = D^*(H) - 1 < r(H)p^m < \frac{p}{2} \times p^m$. Therefore,

$$2k < p.$$

Since

$$|S| - p^n - jp^m = p^n + 2kp^m + 2t - p^n - jp^m = (2k - j)p^m + 2t$$

and

$$\begin{aligned} |S| - p^n - jp^m + v &= p^n + 2kp^m + 2t - p^n - jp^m + (k+1)p^m - (kp^m + t + 1) \\ &= (2k - j)p^m + p^m + t - 1. \end{aligned}$$

By Lemma 2.4, $\binom{|S|-p^n-jp^m+u}{ip^m} = \binom{|S|-p^n-jp^m}{ip^m}$ for every $u \in [0, v]$. Thus in (3.3), we can treat $\sum_{u=0}^v \binom{v}{u} N^{p^n+jp^m-u}(S)$ as one variable.

Set $i = 0, \dots, k$ respectively in (3.3), we get a group of linear equations in variables $1, \sum_{u=0}^v \binom{v}{u} N^{p^n+p^m-u}(S), \dots, (-1)^{k-1} \sum_{u=0}^v \binom{v}{u} N^{p^n+kp^m-u}(S)$. That is

$$(3.11) \quad \begin{aligned} &\binom{2kp^m+2t}{ip^m} + \sum_{j=1}^k (-1)^{j-1} \binom{(2k-j)p^m+2t}{ip^m} \sum_{u=0}^v \binom{v}{u} N^{p^n+p^m-u}(S) \\ &\equiv 0 \pmod{p} \end{aligned}$$

for every $i \in [0, k]$.

Let $2t = p^m + d$ and $l = 2k + 1$, where $0 \leq d \leq p^m - 1$. Note that $2k < p$. By Lemma 2.4 we have

$$\binom{(2k-j)p^m+2t}{ip^m} = \binom{2k+1-j}{i} = \binom{l-j}{i}$$

for all $i \in [0, k]$ and $j \in [0, k]$.

So, the coefficient matrix of the group of linear equations of (3.11) is $A = \left(\binom{l-j}{i} \right)_{0 \leq i, j \leq k}$, that is

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \binom{l}{1} & \binom{l-1}{1} & \cdots & \binom{l-k}{1} \\ \binom{l}{2} & \binom{l-1}{2} & \cdots & \binom{l-k}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{l}{k} & \binom{l-1}{k} & \cdots & \binom{l-k}{k} \end{pmatrix}_{(k+1) \times (k+1)}.$$

By lemma 2.5 we have

$$\det(A) = \frac{1}{\prod_{1 \leq t \leq k} t!} \prod_{1 \leq i < j \leq k} (j - i) \not\equiv 0 \pmod{p}.$$

Therefore the linear equations above has only trivial solution

$$\begin{aligned} 1 &\equiv \sum_{u=0}^v \binom{v}{u} N^{p^n+p^m-u}(S) \equiv \dots \equiv (-1)^{k-1} \sum_{u=0}^v \binom{v}{u} N^{p^n+kp^m-u}(S) \\ &\equiv 0 \pmod{p}, \end{aligned}$$

a contradiction. \square

Proof of Theorem 1.2. By Lemma 2.1 and Lemma 2.8, we only need to prove that $\eta(G) \leq ap^n + 2D(H) - 2$. Note that if $\eta(G) \leq ap^n + 2D(H) - 2$ for $a = 1$, then by Lemma 2.7 taking $K \simeq C_a$ as a subgroup of G , we obtain that $\eta(G) \leq \exp(G/K)(\eta(K) - 1) + \eta(G/K) = p^n(a - 1) + p^n + 2D(H) - 2 = ap^n + 2D(H) - 2$. So, it suffices to prove the theorem for $a = 1$ which we now assume.

(1): $D(H) \leq 2 \exp(H)$.

Since $D(H) - 1 \leq 2 \exp(H) - 1$, we have

$$k = 1.$$

Hence by Proposition 3.3, we get the desired result.

(2): $\lceil (k + \frac{1}{2}) \exp(H) \rceil < D(H) \leq (k + 1) \exp(H)$ for some integer $k \geq 2$.

Let $D(H) - 1 = kp^m + t$. Since $\lceil (k + \frac{1}{2}) \exp(H) \rceil < D(H) \leq (k + 1) \exp(H)$, we have

$$2t \geq p^m.$$

Hence by Proposition 3.3, we get the desired result. \square

Proof of Theorem 1.3. If H is cyclic, then G is of rank at most two and the result is true as mentioned in the introduction. Now we assume that $r(H) \geq 2$. Since $p > 2r(H)$, we have $p \geq 5$ and $|H| \geq 25$ follows. Let $u = p^m$ and $v = ap^{n-m}$. Then $uv = ap^n$. By $a > p^{2m-n}|H|$ we obtain $v \geq m|H| + 1 = \max\{m|H| + 1, 4|H| + 2m\}$. It follows from Lemma 2.6 that $s(G) = \eta(G) + \exp(G) - 1$. Now the result follows from Theorem 1.2. \square

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